Synchronization for complex dynamical Lurie networks*

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(Received 31 October 2012; revised manuscript received 20 March 2013)

This paper investigates the synchronization problem for two different complex dynamical Lurie networks. The first one is with constant coupling and the second one is with constant coupling and discrete-delay coupling. Based on contraction theory and matrix measure properties, some new delay-independent synchronization conditions depending on coupling strength and network topology are proposed. Finally, simulation results are presented to support the theoretical results.

Keywords: contraction theory, virtual system, partial contraction, uniformly negative definite, hybrid coupling

PACS: 05.45.Gg, 05.45.Jn, 05.45.Xt **DOI:** 10.1088/1674-1056/22/10/100503

1. Introduction

Synchronization is a universal phenomenon in nature, such as pendulum clocks motion, fireflies glowing, and so on, and aroused many experts who came from different fields to pay attention on it. As is known, synchronization is an important branch in chaotic systems and has many applications in different fields. For example, synchronization can be found in biology, secure communication, cryptography, and other areas. In previous works, many methods like back-stepping,[1,2] adaptive control,[3,4] and feedback control[5–7] are utilized to achieve synchronization. In fact, the essence of these methods is based on Lyapunov theory. Recently, the appearance of contraction theory has provided another way to study synchronization problem. In contraction theory, there is no need to know the accurate motion of the error system between the drive system and the response system, while we only need to focus on the motion between neighboring trajectories. In other words, stability of the error system can be analyzed by judging whether all trajectories will converge to each other or not, rather than constructing an available Lyapunov function. Specially, the partial contraction theory has provided a more straightforward way to consider the synchronization problem.

On the other hand, with the emergence of complex networks, many natural systems can be expressed by a complex network which includes some interconnected nodes or vertices. In the network, some special properties are not explained by the individual dynamics of each node. Therefore, many results about synchronization or convergence of complex networks have been proposed. DeLellis *et al.*[8] reviewed all approaches which had been used to consider the synchronization problem and focused on the contraction analysis method. Arenas *et al.*[9] mainly used the master stability function (MSF) to solve synchronization problem. Cao *et al.*[10] deliberated global synchronization for a general array model of coupled delayed neural networks which had hybrid coupling. The definition of hybrid couplin,g which consists of constant coupling, discrete-delay coupling, and distributed-delay coupling, will be used to support our results. Slotine and Wang[11,12] employed contraction analysis to deal with the synchronization problem. Wang[13] discussed the synchronization problem for some special complex networks which included regular, small-world, and scale-free networks. At the same time, the Lurie system has been widely studied for many years, due to its special property, which means the nonlinear function satisfies sector and slope restrictions (more precisely, the nonlinear function can be linearized). Just as mentioned in Ref. [14], some chaotic systems can be categorized as Lurie systems, where the most representative example is Chua's system. No matter which type of synchronization problem about which the Lurie system has been discussed, Chua's system can be employed to illustrate the effectiveness of the conclusion. Suykens *et al.*[15] declared that under a large parameter mismatch between Lurie systems, synchronization can still be kept. He *et al.*[16] studied the L_r synchronization and adaptive synchronization for the Lurie system under perturbations. Liu *et al.*[17] exploited Kalman–Yakubovich–Popov (KYP) lemma and Schur complement theorem to consider the synchronization problem of a general Lurie system with time-delay feedback control. With the development of complex networks, many researchers have combined the Lurie system and complex networks to consider together. Xu and Yang[14] analyzed the synchronization problem of complex dynamical Lurie networks which only possess the constant coupling. Li *et al.*[18] presented the conditions for global synchronization of coupled Lurie systems which had

*Project supported by the National Natural Science Foundation of China (Grant Nos. 61174021 and 61104155), the Fundamental Research Funds for the Central Universities (Grant No. JUDCF12033), the Jiangsu Innovation Program for Graduates (Grant No. CXZZ12_0742), the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD), the Fundamental Research Funds for the Central Universities (Grant No. JUSRP51322B), and the 111 Project (Grant No. B12018).

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time-delay and hybrid coupling. Wang *et al.*[19] designed a pinning controller to deal with the cluster synchronization of delayed Lurie dynamical networks. Li *et al.*[20] used absolute stability theory to solve the synchronization problem of Lurie complex networks. Li *et al.*[21] employed a decomposition approach to convert the synchronization problem of high-dimensional Lurie networks into the test of a set of matrix inequalities. Note that all these works about synchronization problems are transformed into the analysis of the stability of the error system. In this paper, we use a new method to deal with the synchronization problem of complex Lurie networks.

Based on contraction theory and matrix measure properties, some time-delay-independent conditions for synchronization are given. Moreover, the constant time delay case and time-varying delay case are investigated, respectively. Comparing with the existing results, several contributions are included in this paper. Firstly, we use the contraction analysis to consider the synchronization problem of complex dynamical Lurie networks which contain time delay, constant coupling, and discrete coupling. Secondly, we have shown that the delay-independent synchronization is depended on coupling strength and network topology. The paper is organized as follows. In Section 2, some preliminaries about contraction theory and matrix theory are summarized. Meanwhile, model description about the complex Lurie networks with constant coupling is also introduced. In Section 3, the main results of this paper are presented. In Section 4, three examples are provided to verify the effectiveness of the results. Conclusions are drawn in Section 5.

2. Preliminaries and model description

2.1. Preliminaries

2.1.1. Contraction analysis

To get the main results of this paper, the definitions and some important conclusions about contraction theory are briefly summarized in this section, details of which can be found in Refs. [11], [12], and [22].

Consider a nonlinear system

$$\dot{x} = f(x,t), \tag{1}$$

where $x(t) \in R^n$ is the state vector and f is assumed to be an $n \times 1$ continuously differentiable vector function. Define δx as a virtual displacement between two different neighboring trajectories, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x^{\mathrm{T}}\delta x) \leq 2\delta x^{\mathrm{T}}\delta\dot{x} = 2\delta x^{\mathrm{T}}\frac{\partial f}{\partial x}\delta x \leq 2\lambda_{\max}(x,t)\delta x^{\mathrm{T}}\delta x,$$

where $\lambda_{\max}$ is the largest eigenvalue of the symmetric part of the Jacobian $J = \partial f/\partial x$. Note that if $\lambda_{\max}(x,t)$ is uniformly strictly negative, any infinitesimal length $\|\delta x\|$ will converge exponentially to zero. By path integration at fixed time, this implies that all solutions of system (1) will converge exponentially to a single trajectory, independent of the initial conditions.[12]

Definition 1[12] Uniformly negative definiteness of Jacobian $\partial f(x,t)/\partial x$ means that there exists a scalar $\alpha > 0$, $\forall x$, $\forall t \geq 0$ s.t. $\partial f/\partial x \leq -\alpha I < 0$. As all matrix inequalities will refer to the symmetric part of the square matrix involved, so one can further write

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^{\mathrm{T}}\right) \leq -\alpha I < 0.$$

For a more general case, the related results can refer to Refs. [11] and [12]. Here we omit them.

2.1.2. Partial contraction analysis

Partial contraction theory is a significant tool for studying the synchronization problem. It makes the results of contraction analysis more practical and more direct. The results include the convergence of dynamical systems behaviors or specific properties such as equality of state components or convergence.[23] We now introduce the concept of partial contraction, which forms the basic results of this paper.

Lemma 1[12] Consider a nonlinear system of the form

$$\dot{x} = f(x,x,t)$$

and assume that the auxiliary system

$$\dot{y} = f(y,x,t)$$

is contracting with respect to y. If a particular solution of the auxiliary y system verifies a smooth specific property, then all trajectories of the original x system verify this property exponentially. The original system is said to be partially contracting.

Proof The virtual, observer-like y system has two particular solutions, namely $y(t) = x(t)$ for all $t \geq 0$ and the solution with the specific property. Since all trajectories of the y system converge exponentially to a single trajectory, this implies that $x(t)$ verifies the specific property exponentially.

2.1.3. Matrix theory

Definition 2[24] The matrix measure of a real square matrix $A = (a_{ij})_{n\times n}$ is defined as follows:

$$\mu(A) = \lim_{\varepsilon\to 0^+}\frac{\|I+\varepsilon A\|-1}{\varepsilon},$$

where I is the identity matrix. Furthermore, when some conventional matrix norms are defined as follows:

$$\|A\|_1 = \max_j \sum_{i=1}^{n}|a_{ij}|, \quad \|A\|_2 = [\lambda_{\max}(A^{\mathrm{T}}A)]^{1/2},$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n}|a_{ij}|, \quad \|A\|_\omega = \max_j \sum_{i=1}^{n}\frac{\omega_i}{\omega_j}|a_{ij}|,$$

we have the corresponding matrix measure

$$\mu_1(A) = \max_j\left\{a_{jj} + \sum_{i=1,i\neq j}^{n}|a_{ij}|\right\},$$

$$\mu_2(A)=\frac{1}{2}\lambda_{\max}(A^{\mathrm{T}}+A),$$
$$\mu_\infty(A)=\max_i\left\{a_{ii}+\sum_{j=1,j\neq i}^{n}|a_{ij}|\right\},$$
$$\mu_\omega(A)=\max_j\left\{a_{jj}+\sum_{i=1,i\neq j}^{n}\frac{\omega_i}{\omega_j}|a_{ij}|\right\},$$

respectively, where $\omega_i>0$.

2.2. Model description

Suppose a dynamical network is composed of N linearly and diffusively coupled nodes, in which each node is identical and has the form of an n-dimensional nonlinear Lurie dynamical system with time varying delay. The dynamical system of the the i-th node is described as follows:

$$\begin{cases}\dot{x}_i(t)=Ax_i(t)+Bx_i(t-\tau(t))\\ \qquad\quad +Df(Cx_i(t))+Fg(Ex_i(t-\tau(t))),\\ x_i(\theta)=\varphi_i(\theta),\theta\in[-\tau_m,0],i=1,2,\ldots,N,\end{cases}\tag{2}$$

where $0\le\tau_0\le\tau(t)\le\tau_m,\dot{\tau}(t)\le\gamma<+\infty$, state vectors $x_i=[x_{i1},x_{i2},\ldots,x_{in}]^{\mathrm{T}}\in R^n$, and A, $B\in R^{n\times n}$, D, $F\in R^{n\times m}$, $C=[c_1,\cdots,c_m]^{\mathrm{T}}$, $E=[e_1,\cdots,e_m]^{\mathrm{T}}\in R^{m\times n}$ are known constant matrices. $\varphi_i(\theta)$ is the initial vector function defined on the interval $[-\tau_m,0]$. The nonlinear functions $f(Cx_i(t))$ and $g(Ex_i(t-\tau(t)))$ are defined as

$$f(Cx_i(t))=[f_1(c_1^{\mathrm{T}}x_i(t)),\ldots,f_m(c_m^{\mathrm{T}}x_i(t))]^{\mathrm{T}},$$
$$g(Ex_i(t-\tau(t)))=[g_1(e_1^{\mathrm{T}}x_i(t-\tau(t))),\ldots,g_m(e_m^{\mathrm{T}}x_i(t-\tau(t)))]^{\mathrm{T}}.$$

In which, $f_l(\cdot)$, $g_k(\cdot)$ are time-varying piecewise continuous functions satisfying the following conditions

$$\text{(i)}\ \ f_l(0)=g_k(0)=0,$$
$$\text{(ii)}\ \ k_l^-\le\frac{\mathrm{d}f_l(\sigma_1)}{\mathrm{d}\sigma_1}\le k_l^+,\tag{3}$$
$$\widetilde{k}_l^-\le\frac{\mathrm{d}g_k(\widetilde{\sigma}_k)}{\mathrm{d}\widetilde{\sigma}_k}\le\widetilde{k}_l^+,\ \ l,k=1,2,\ldots,m,\tag{4}$$

where $\sigma_l=C_l^{\mathrm{T}}x_i(t)$, $\sigma=Cx_i(t)=(\sigma_1,\sigma_2,\ldots,\sigma_m)^{\mathrm{T}}$, $\widetilde{\sigma}_k=e_k^{\mathrm{T}}x_i(t-\tau(t))$, $\widetilde{\sigma}=Ex_i(t-\tau(t))=(\widetilde{\sigma}_1,\widetilde{\sigma}_2,\ldots,\widetilde{\sigma}_m)^{\mathrm{T}}$.

Firstly we consider that the nodes are coupled with state $x_i(t)$ $(i=1,\ldots,N)$ in the Lurie system form with constant time delay. The network model can be expressed as

$$\dot{x}_i(t)=Ax_i(t)+Bx_i(t-\tau)+Df(Cx_i(t))+Fg(Ex_i(t-\tau))+c\sum_{j=1}^{N}l_{ij}Gx_j(t),\tag{5}$$

where the constant $c>0$ represents the coupling strength of the network, $\tau>0$ is a constant time delay, $G=\mathrm{diag}(g_1,g_2,\ldots,g_n)\in R^{n\times n}$ is the inner-coupling matrix in which every $g_i\ge0$, $L=[l_{ij}]_{N\times N}$ associated with the coupling configuration of the network is defined as $l_{ij}=l_{ji}=1$ if there is a connection between node i and j $(i\neq j)$, otherwise $l_{ij}=l_{ji}=0$ $(i\neq j)$. Moreover, $\sum_{j=1,j\neq i}^{N}l_{ij}=-l_{ii}$ is satisfied.

Remark 1 According to the definition of hybrid coupling in Ref. [10], we say that the system (5) only has the constant coupling. System (2) with constant and discrete-delay coupling will be discussed in the next section.

From the unified viewpoint and based on the definition of consensus in Ref. [25], we give out the following definition.

Definition 3 The system described in Eq. (2) is said to achieve global asymptotic synchronization if for any initial state $x_0(t)$, the states of nodes satisfy $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0$ for all $i,j\in N$. Furthermore, if there exists $x^*\in R^n$ such that $\lim_{t\to\infty}\|x_i(t)-x^*\|=0$ for all $i\in R^n$, then we call x^* as the synchronization state of the system (2).

Lemma 2[26] Given a system $\dot{x}(t)=Ax(t)+A_dx(t-\tau)$, the stability of which implies that of the system $\dot{y}(t)=(A+zA_d)y(t),\forall|z|=1$, and *vice versa*, where $z=\exp(\mathrm{j}\theta)$, $\theta\in[0,2\pi]$, $\mathrm{j}=\sqrt{-1}$, and $z\in C$.

Lemma 3[27] For the matrices A and B with compatible dimensions, the following inequality is satisfied

$$\|\exp[(A+zB)t]\|\le\exp[\mu(A+zB)t],t>0,|z=1|.$$

Lemma 4[28] For the matrix measure, the following inequality is satisfied

$$\exp[\mu(A+zB)t]\le\exp[(\mu(A)+\|B\|)t],t>0,|z=1|.$$

3. Main results

In this section, the results about the synchronization problem are given in two different cases: case I: constant time delay with constant coupling, case II: time-varying delay with constant and discrete-delay coupling.

3.1. Synchronization of Lurie network with constant time delay

On the basis of contraction analysis, to ensure the synchronization of system (5), we should construct a suitable virtual system. Then the discussion of the stability of the error system is replaced by considering the contracting behaviors of the virtual system. Now choosing a virtual system as follows:

$$\dot{y}_i=Ay_i(t)+By_i(t-\tau)+Df(Cy_i(t))+Fg(Ey_i(t-\tau))+cl_{ii}y_i(t)+c\sum_{j=1,j\neq i}^{N}l_{ij}Gx_j(t).\tag{6}$$

The virtual displacement of the virtual system (6) is defined as $\delta y_i(t)$, then we have

$$\delta\dot{y}_i(t)=A\delta y_i(t)+B\delta y_i(t-\tau)+D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\delta y_i(t)+F\frac{\partial g(Ey_i(t-\tau))}{\partial y_i(t-\tau)}\delta y_i(t-\tau)+cl_{ii}G\delta y_i(t).\tag{7}$$

According to Lemma 2, the stability of the system (7) is equivalent to the stability of the following system:

$$\begin{aligned}\delta\dot{y}_i(t) &= \left\{A+z_1B+D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\right.\\ &\quad\left.+z_2F\frac{\partial g(Ey_i(t-\tau))}{\partial y_i(t-\tau)}+cl_{ii}G\right\}\delta y_i(t), \qquad (8)\end{aligned}$$

where $z_k=\exp(,\mathrm{j}\theta)$, $|z_k|=1$, $\theta\in[0,2\pi]$, $\mathrm{j}=\sqrt{-1}$, and $z_k\in C$, $(k=1,2)$. So from Lemma 1, the virtual system (6) is contracting if and only if the system (8) is globally exponentially stable.

Theorem 1 If there exists the matrix measure

$$\begin{aligned}\beta &= \mu(A+cl_{ii}G)+\|B\|+\left\|D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\right\|\\ &\quad+\left\|F\frac{\partial g(Ey_i(t-\tau))}{\partial y_i(t-\tau)}\right\|\end{aligned}$$

satisfies $\beta<0$ $(\forall t_0\geq 0)$, then the virtual system (6) is contracting, implying that the system (5) is globally asymptotically synchronized.

Proof From the system (8), when $t\in[t_0,\infty)$, we have that

$$\begin{aligned}\delta y_i(t) &= \exp\left\{\int_{t_0}^{t}\left(A+z_1B+D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\right.\right.\\ &\quad\left.\left.+z_2F\frac{\partial g(Ey_i(t-\tau))}{\partial y_i(t-\tau)}+cl_{ii}G\right)\mathrm{d}s\right\}\delta y_i(t_0). \qquad (9)\end{aligned}$$

Taking the norm on both sides and using Lemmas 3 and 4, the following result can be obtained:

$$\begin{aligned}\|\delta y_i(t)\| &\leq \left\|\exp\left\{\int_{t_0}^{t}\left(A+z_1B+D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\right.\right.\right.\\ &\quad\left.\left.\left.+z_2F\frac{\partial g(Ey_i(t-\tau))}{\partial y_i(t-\tau)}+cl_{ii}G\right)\mathrm{d}s\right\}\right\|\|\delta y_i(t_0)\|\\ &\leq \exp\left\{\int_{t_0}^{t}\left[\mu(A+cl_{ii}G)+\|B\|+\left\|D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\right\|\right.\right.\\ &\quad\left.\left.+\left\|F\frac{\partial g(Ey_i(t-\tau))}{\partial y_i(t-\tau)}\right\|\right]\mathrm{d}s\right\}\|\delta y_i(t_0)\|. \qquad (10)\end{aligned}$$

Define

$$\begin{aligned}\beta &= \mu(A+cl_{ii}G)+\|B\|+\left\|D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\right\|\\ &\quad+\left\|F\frac{\partial g(Ey_i(t-\tau))}{\partial y_i(t-\tau)}\right\|,\end{aligned}$$

expression (10) is equivalent to

$$\begin{aligned}\|\delta y_i(t)\| &\leq \exp\left[\int_{t_0}^{t}\beta\,\mathrm{d}s\right]\|\delta y_i(t_0)\|\\ &= \exp[\beta(t-t_0)]\|\delta y_i(t_0)\|. \qquad (11)\end{aligned}$$

It follows that if $\beta<0$, system (8) will be globally exponentially stable. So system (7) is also globally exponentially stable. That means the virtual system (6) is contracting. From Lemma 1, the coupled system (5) is globally asymptotically synchronized.

Remark 2 Generally speaking, the coefficient matrices of system (5), the inner coupling matrix, and the configuration of the network are known, so in this paper we mainly select suitable c to make sure $\beta<0$. The value of c determines the synchronization region of the system (5).

3.2. Synchronization of the Lurie network with time-varying delay

We now extend the above result to a general case. Consider a time-varying delay complex network of the Lurie type with constant coupling and discrete-delay coupling:

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t)+Bx_i(t-\tau(t))+Df(Cx_i(t))\\ &\quad+Fg(Ex_i(t-\tau(t)))+c\sum_{j=1}^{N}l_{ij}G_1x_j(t)\\ &\quad+c\sum_{j=1}^{N}l_{ij}G_2x_j(t-\tau(t)), \qquad (12)\end{aligned}$$

where G_1, $G_2\in R^{n\times n}$ represents the linking matrix, the discrete-delay linking matrix respectively of the network, and other parameters have the similar definitions in system (2).

Just as mentioned above, to consider the synchronization region of the system (12), it is necessary to find an available virtual system. So we firstly choose an auxiliary system of Eq. (12) given by

$$\begin{aligned}\dot{y}_i(t) &= Ay_i(t)+By_i(t-\tau(t))+Df(Cy_i(t))\\ &\quad+Fg(Ey_i(t-\tau(t))+c\sum_{j=1,i\neq j}^{N}l_{ij}G_1x_j(t)\\ &\quad+cl_{ii}G_1y_i(t)+c\sum_{j=1}^{N}l_{ij}G_2y_j(t-\tau(t)). \qquad (13)\end{aligned}$$

Defining the virtual displacement of the auxiliary system (13) as $\delta y_i(t)$, we have:

$$\begin{aligned}\delta\dot{y}_i(t) &= A\delta y_i(t)+B\delta y_i(t-\tau(t))+D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\delta y_i(t)\\ &\quad+F\frac{\partial g(Ey_i(t-\tau(t)))}{\partial y_i(t-\tau(t))}\delta y_i(t-\tau(t))+cl_{ii}G_1\delta y_i(t)\\ &\quad+c\sum_{j=1}^{N}l_{ij}G_2\delta y_j(t-\tau(t)). \qquad (14)\end{aligned}$$

Therefore the virtual system can be expressed as the following form:

$$\begin{aligned}\delta\dot{y}(t) &= \widetilde{A}\delta y_(t)+\widetilde{B}\delta y_(t-\tau(t))+\tilde{D}\delta y(t)+\widetilde{F}\delta y(t-\tau(t))\\ &\quad+c\widetilde{G}_1\delta y(t)+c\widetilde{G}_2\delta y(t-\tau(t)), \qquad (15)\end{aligned}$$

where

$$\begin{aligned}\widetilde{A} &= I_N\otimes A,\widetilde{B}=I_N\otimes B,\\ \tilde{D} &= \mathrm{diag}\left(D\frac{\partial f(Cy_1(t))}{\partial y_1(t)},D\frac{\partial f(Cy_2(t))}{\partial y_2(t)},\ldots,\right.\\ &\quad\left.D\frac{\partial f(Cy_N(t))}{\partial y_N(t)}\right),\end{aligned}$$

$$\widetilde{F} = \mathrm{diag}\left(F\frac{\partial g(Ey_1(t-\tau(t)))}{\partial y_1(t-\tau(t))}, F\frac{\partial g(Ey_2(t-\tau(t)))}{\partial y_2(t-\tau(t))}, \ldots, F\frac{\partial g(Ey_N(t-\tau(t)))}{\partial y_N(t-\tau(t))}\right),$$
$$\widetilde{G}_1 = \mathrm{diag}\,(l_{11}G_1, l_{22}G_1, \ldots, l_{NN}G_1),$$
$$\widetilde{G}_2 = L\otimes G_2, L = [l_{ij}]_{N\times N}.$$

From Lemma 1, the virtual system (15) is contracting if and only if the system

$$\delta\dot{y}(t) = \left(\widetilde{A} + z_1\widetilde{B} + \tilde{D} + z_2\widetilde{F} + c\widetilde{G}_1 + cz_3\widetilde{G}_2\right)\delta y(t) \quad (16)$$

for all $|z_m| = 1$ is globally exponentially stable, where $z_m = \exp(\mathrm{j}\theta)$, $\theta \in [0, 2\pi]$, $\mathrm{j} = \sqrt{-1}$, and $z_m \in C$ $(m = 1, 2, 3)$.

Theorem 2 If there exists the matrix measure

$$\widetilde{\beta} = \mu(\widetilde{A} + c\widetilde{G}_1) + \|c\widetilde{G}_2\| + \|\widetilde{B}\| + \|\tilde{D}\| + \|\widetilde{F}\|$$

satisfies $\widetilde{\beta} < 0$ $(\forall t_0 \geq 0)$, then the virtual system (15) is contracting implying that the system (12) is globally asymptotically synchronized.

Proof According to Eq. (16), when $t \in [0, \infty)$, we have that

$$\delta y(t) = \exp\left\{\int_{t_0}^{t}\left(\widetilde{A} + z_1\widetilde{B} + \tilde{D} + z_2\widetilde{F} + c\widetilde{G}_1 + cz_3\widetilde{G}_2\right)\mathrm{d}s\right\}\delta y(t_0). \quad (17)$$

Just as the proof in Theorem 1, we also take the norm on both sides of Eq. (17) and use Lemmas 3 and 4, the following results can be obtained:

$$\begin{aligned}\|\delta y(t)\| &\leq \left\|\exp\left\{\int_{t_0}^{t}\left(\widetilde{A} + z_1\widetilde{B} + \tilde{D} + z_2\widetilde{F} + c\widetilde{G}_1 + cz_3\widetilde{G}_2\right)\mathrm{d}s\right\}\right\|\|\delta y(t_0)\| \\ &\leq \exp\left\{\int_{t_0}^{t}\left[\mu(\widetilde{A} + c\widetilde{G}_1) + \|c\widetilde{G}_2\| + \|\widetilde{B}\| + \|\tilde{D}\| + \|\widetilde{F}\|\right]\mathrm{d}s\right\}\|\delta y(t_0)\|. \quad (18)\end{aligned}$$

Defining $\widetilde{\beta} = \mu(\widetilde{A} + c\widetilde{G}_1) + \|c\widetilde{G}_2\| + \|\widetilde{B}\| + \|\tilde{D}\| + \|\widetilde{F}\|$, then expression (18) can be converted into the following inequality:

$$\begin{aligned}\|\delta y(t)\| &\leq \exp\left[\int_{t_0}^{t}\widetilde{\beta}\,\mathrm{d}s\right]\|\delta y(t_0)\| \\ &= \exp[\widetilde{\beta}(t - t_0)]\|\delta y(t_0)\|. \quad (19)\end{aligned}$$

So when $\widetilde{\beta} < 0$, system (16) will be globally exponentially stable. Then the virtual system (15) is contracting. From Lemma 1, the coupled system (12) can realize global synchronization. This finishes the proof.

Remark 3 From the viewpoint of the contraction theory, we can know that the final behaviors of the system are independent of the initial state. So the synchronization region of the system (5) or system (12) is not related to the initial conditions. More precisely, regardless what the initial state is, we only need to ensure $\beta < 0$ or $\widetilde{\beta} < 0$, then the synchronization of the system can be achieved.

4. Numerical examples

In this section, three representative examples are presented to verify the results derived in Section 3.

Example 1 Firstly a special case of the system (5) is considered. Here the function $g(Ex_i(t-\tau)) = 0$, then the complex Lurie network is described by the following parameters

$$A = \begin{pmatrix}-2 & 0\\ -1 & -2\end{pmatrix},\ B = \begin{pmatrix}-0.2 & -0.5\\ 0.5 & -0.2\end{pmatrix},\ C = \begin{pmatrix}0.6\\ 0.8\end{pmatrix}^{\mathrm{T}},$$
$$D = \begin{pmatrix}-0.2\\ -0.3\end{pmatrix},\ G = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
$$f(y_i(t)) = |y_i(t) + 1| - |y_i(t) - 1|.$$

It is easy to verify that $0 \leq \mathrm{d}f_l/\mathrm{d}\sigma_l \leq 2$ satisfying the demand of expressions (3) with $k_l^- = 0$, $k_l^+ = 2$. Furthermore the coupling matrix of the network with identical nodes $(N = 5)$ is described by the following form, which is usually called a star-like network and also has been studied in Refs. [13] and [14]

$$L = \begin{pmatrix}-4 & 1 & 1 & 1 & 1\\ 1 & -1 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0\\ 1 & 0 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & -1\end{pmatrix}.$$

After calculating, we can conclude that $\|B\| = 0.5385$,

$$A + cl_{ii}G = \begin{pmatrix}-2 + cl_{ii} & 0\\ -1 & -2 + cl_{ii}\end{pmatrix},$$

so $\mu(A + cl_{ii}G) = -3/2 + cl_{ii}$. Let $M_1 = \partial f/\partial\sigma$, then we have

$$\begin{aligned}\left\|D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\right\| &= \|DCM_1\| \\ &= \left\|\begin{pmatrix}-0.12M_1 & -0.16M_1\\ -0.18M_1 & -0.24M_1\end{pmatrix}\right\| \\ &\approx 0.36M_1.\end{aligned}$$

In addition, because $0 \leq M_1 \leq 2$, then

$$\left\|D\frac{\partial f(Cy_i(t))}{\partial y_i(t)}\right\| = 0.72$$

and $\beta = -3/2 + cl_{ii} + 0.72 + 0.54$ are obtained. Now by using Theorem 1, when $\beta < 0$, system (5) can achieve synchronization. From $l_{ii} < 0$, we get $c > 0.24/l_{ii}$. Generally we choose $l_{ii} = \min\{l_{ii}\} = -4$, $(i = 1, 2, \ldots, 5)$, then $c > -0.06$. However, according to the definition of c which stands for the coupling strength, so we only choose $c > 0$. Then the coupled system (5) can realize global synchronization. It follows from Remark 3 that, the synchronization region is not relevant to the initial state, so the initial states of the system (5) are taken randomly. Without loss of generality, in this simulation, we choose them as in Ref. [14]. Now let $x_1 = (0.6, -0.5)^{\mathrm{T}}$, $x_2 = (0.2, 0.2)^{\mathrm{T}}$, $x_3 = (-0.5, 0.5)^{\mathrm{T}}$, $x_4 = (0.8, -1)^{\mathrm{T}}$, $x_5 = (-0.3, 0.6)^{\mathrm{T}}$, $c = 1(> 0)$, and $\tau = 2$, the state behaviors of the system are shown in Figs. 1 and 2. The error trajectories are shown in Fig. 3. From Figs. 1–3, we can know that the system (5) is globally asymptotically synchronized.

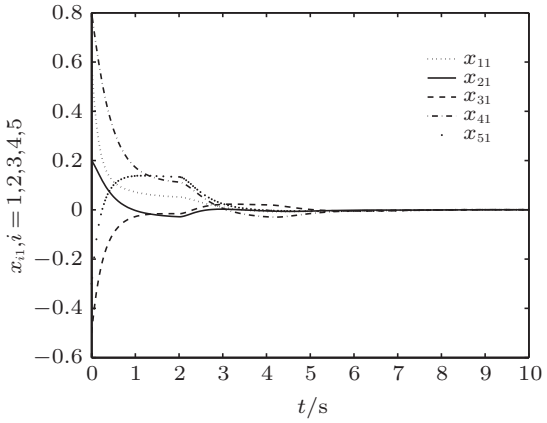


Fig. 1. State responses of $x_{i1}(t)$ $(i = 1,2,\ldots,5)$.

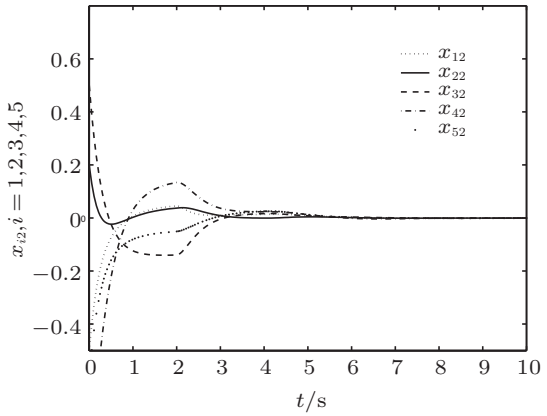


Fig. 2. State responses of $x_{i2}(t)$ $(i = 1,2,\ldots,5)$.

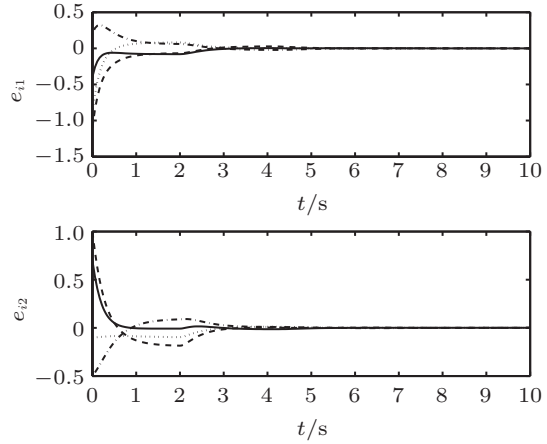


Fig. 3. The errors between different nodes, $e_{ik} = x_{ik} - x_{1k}$, $i = 2,\ldots,5$, $k = 1,2$.

Example 2 Now consider a chaotic Chua's system which consists of three identical coupled nodes with the following parameters

$$A = \begin{pmatrix} -2.57 & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.28 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -0.01 & 0 & 0 \\ -0.01 & 0 & 0 \\ 0.02 & 0 & -0.01 \end{pmatrix},$$

$$L = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix},$$

$$f(x_i(t)) = (1.93\,(|x_{i1}(t)+1| - |x_{i1}(t)-1|), 0, 0)^{\mathrm{T}},$$
$$g(x_i(t-\tau)) = (-0.02\sin(0.1x_{i1}(t-\tau)), 0, 0)^{\mathrm{T}},$$
$$C = D = E = F = G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = 1.$$

Just as in Example 1, we can get

$$A + cl_{ii}G = \begin{pmatrix} -2.57 + cl_{ii} & 9 & 0 \\ 1 & -1 + cl_{ii} & 1 \\ 0 & -14.28 & cl_{ii} \end{pmatrix},$$
$$\mu(A + cl_{ii}G) \approx 7.43 + cl_{ii}, \quad \|B\| = 0.0259,$$
$$\left\| D\frac{\partial f(Cy_i(t))}{\partial y_i(t)} \right\| = \left\| DC\frac{\partial f}{\partial \sigma} \right\| = \left\| \begin{pmatrix} N_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\| = N_1,$$
$$\left\| F\frac{\partial g(Ey_i(t-\tau))}{\partial y_i(t-\tau)} \right\| = \left\| FE\frac{\partial g}{\partial \widetilde{\sigma}(t-\tau)} \right\| = \left\| \begin{pmatrix} -0.02\cos(0.1x_{i1}(t-\tau)) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\| = N_2.$$

From the definitions of the function $f(x_i(t))$ and $g(x_i(t-\tau))$, we have $0 \le N_1 \le 1.93 \times 2$, $0 \le N_2 \le 0.002$, so $\beta = cl_{ii} + 7.43 + 0.0259 + 3.86 + 0.002$. Different from Example 1, now we take $l_{ii} = \max\{l_{ii}\} = -2$, then $c > 5.66$ can ensure $\beta < 0$. Therefore when $c > 5.66$, system (5) will be globally asymptotically synchronized. With $c = 6$, figures 4–6 depict the simulation results of the system (5) in which nodes are coupled in Chua's circuits. Figure 4 describes the chaotic attractor of the Chua's system. Figure 5 shows different state trajectories of $x_1(t)$ in Chua's system. And the synchronization errors between different nodes are presented in Fig. 6.

In all simulation experiments, the initial conditions are selected randomly as $x_1 = (0.1, 0.2, 0.1)^{\mathrm{T}}$, $x_2 = (-0.1, 0.15, 0.2)^{\mathrm{T}}$, $x_3 = (-0.1, -0.4, 0.2)^{\mathrm{T}}$.

Remark 4 From the simulations, we can find that the system in Example 1 is stable, since the final states of the coupled Lurie network converge to stable state. And the system in Example 2 is oscillatory, but both of them are synchronized, since the errors between any two nodes converge to 0.

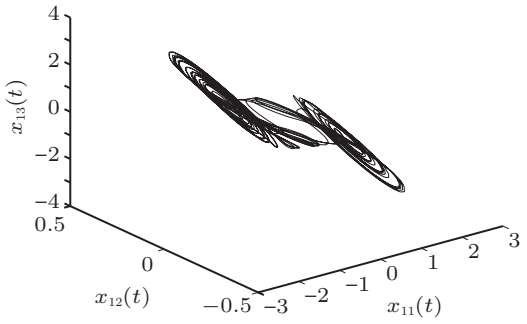


Fig. 4. The chaotic attractor of the Chua's system.

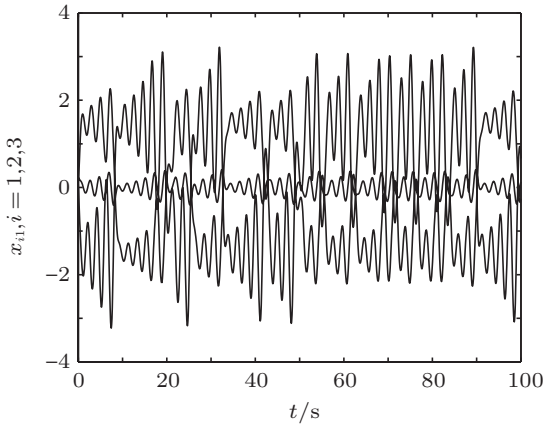


Fig. 5. State trajectories of $x_1(t)$ in Chua's circuit.

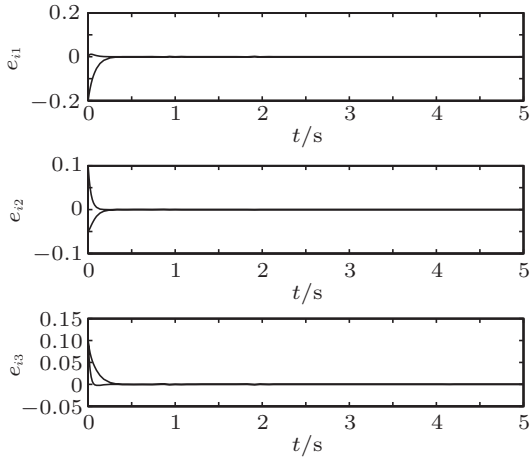


Fig. 6. Synchronization errors between different nodes, $e_{ik} = x_{ik} - x_{1k}$, $i = 2, 3$, $k = 1, 2, 3$.

Remark 5 In Examples 1 and 2, the selection of l_{ii} is different. In Example 1, we get $c > 0.24/l_{ii}$, while in Example 2, we get $c > -11.3179/l_{ii}$. Hence, the value of l_{ii} depends on the symbol of the right inequality.

Example 3 Consider the system (12) with the following parameters

$$A = \begin{pmatrix} -1.5 & 0.1 \\ 0.1 & -1.5 \end{pmatrix}, \quad B = \begin{pmatrix} 0.1 & 0.08 \\ 0.09 & 0.05 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & -0.1 \\ -5 & 3 \end{pmatrix}, \quad F = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix},$$

$$C = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix},$$

$$\tau(t) = 0.65 + 0.45\sin(10t) + 0.05\cos^2(100t),$$

$$0.2 \le \tau(t) \le 1.15, \dot{\tau}(t) \le 9.5 < +\infty,$$

$$f_i(x_i) = \tanh(x_i), g_i(x_i) = |x_i + 1| + |x_i - 1|.$$

Based on Theorem 2, the corresponding coefficients of the virtual system (15) are shown as follows:

$$\widetilde{A} = I_6 \otimes A, \widetilde{B} = I_6 \otimes B, \tilde{D} = (I_6 \otimes D)\tilde{N}_1, \widetilde{F} = (I_6 \otimes F)\tilde{N}_2,$$

$$\widetilde{G}_1 = \begin{pmatrix} -12 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & 0 & -12 \end{pmatrix},$$

$$\widetilde{G}_2 = \begin{pmatrix} -0.75 & 0 & 0.25 & 0 & 0.5 & 0 \\ 0 & -0.75 & 0 & 0.25 & 0 & 0.5 \\ 0.25 & 0 & -0.5 & 0 & 0.25 & 0 \\ 0 & 0.25 & 0 & -0.5 & 0 & 0.25 \\ 0.5 & 0 & 0.25 & 0 & -0.75 & 0 \\ 0 & 0.5 & 0 & 0.25 & 0 & -0.75 \end{pmatrix},$$

where $\tilde{N}_1 = \mathrm{sech}^2(x_i(t))$, $\tilde{N}_2 = \partial g/\partial x_i(t - \tau(t))$. Then

$$M = \widetilde{A} + c\widetilde{G}_1 = \begin{pmatrix} -1.5 - 12c & 0.1 & 0 & 0 & 0 & 0 \\ 0.1 & -1.5 - 12c & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 - 8c & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & -1.5 - 8c & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.5 - 12c & 0.1 \\ 0 & 0 & 0 & 0 & 0.1 & -1.5 - 12c \end{pmatrix}.$$

It is easy to calculate that $\lambda_{\max}(M + M^{\mathrm{T}}) = -2.8 - 16c$, so

$$\mu(\widetilde{A} + c\widetilde{G}_1) = -1.4 - 8c.$$

At the same time, after calculation, we can get $\|c\widetilde{G}_2\| = 1.25c$, $\|\widetilde{B}\| = 0.1638$, $\|\tilde{D}\| = 6.0989$, $\|\widetilde{F}\| = 5.0453$. To make sure

$$\widetilde{\beta} = \mu(\widetilde{A} + c\widetilde{G}_1) + \|c\widetilde{G}_2\| + \|\widetilde{B}\| + \|\tilde{D}\| + \|\widetilde{F}\| < 0,$$

we have $c > 1.47$. So when $c > 1.47$, system (12) will achieve global asymptotic synchronization. Take $c = 3$, figure 7 shows behaviors about the dynamical system (12), figure 8 shows the time responses of synchronization error. In this experiment, initial conditions are taken randomly as $x_1 = (0.4, -0.2)^{\mathrm{T}}$, $x_2 = (-0.2, 5)^{\mathrm{T}}$, $x_3 = (1, -4)^{\mathrm{T}}$. Thus the simulation results in Figs. 7 and 8 can illustrate the efficiency of our theorem.

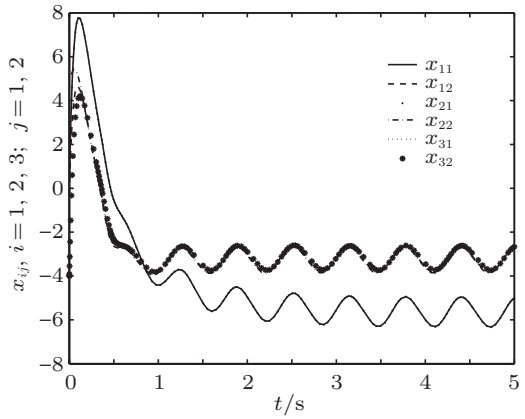


Fig. 7. Behaviors of system (12) in Example 3.

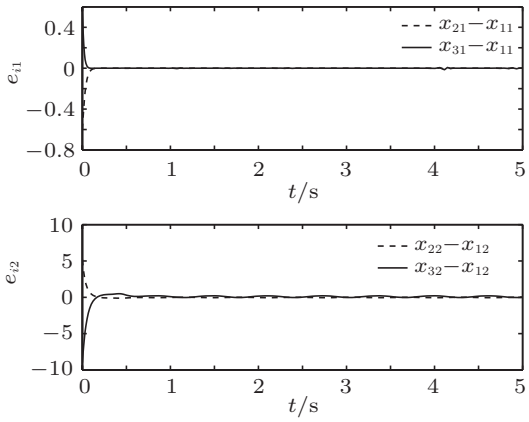


Fig. 8. Time responses of synchronization error.

5. Conclusion

In this paper, the synchronization problem of complex dynamical Lurie networks has been addressed. Some new conditions based on contraction analysis have been proposed. No matter the constant time delay case or the time-varying delay case, these delay-independent conditions can ensure the global synchronization of the system. Finally three examples are given to validate the effectiveness of the proposed results.

References

[1] Lü L, Zhang Q L and Guo Z A 2008 *Chin. Phys. B* **17** 498
[2] Chai Y, Lü L and Chen L Q 2012 *Chin. Phys. B* **21** 030506
[3] Li Z and Han C Z 2002 *Chin. Phys.* **11** 9
[4] Guan X P and Hua C C 2002 *Chin. Phys. Lett.* **19** 1031
[5] Chen S H, Liu J, Feng J W and Lü J H 2002 *Chin. Phys. Lett.* **19** 1257
[6] Li L X, Peng H P, Lu H B and Guan X P 2001 *Chin. Phys.* **10** 796
[7] Zhang J S, Wan J H and Xiao X C 2001 *Chin. Phys* **10** 97
[8] DeLellis P, Bernardo M D, Gorochowski T E and Russo G 2010 *IEEE Circ. Syst. Mag.* **10** 64
[9] Arenas A, Guilera A D, Kurths J, Moreno Y and Zhou C S 2008 *Phys. Rep.* **469** 93
[10] Cao J D, Chen G R and Li P 2008 *IEEE Trans. Syst. Man Cybern. Part B* **38** 488
[11] Slotine J J E and Wang W 2005 *Cooperative Control* (Berlin/Herdelberg: Springer-Verlag), pp. 207–213
[12] Wang W and Slotine J J E 2005 *Biol. Cybern* **92** 38
[13] Wang X F 2002 *Int. J. Bifur. Chaos* **12** 885
[14] Xu S Y and Yang Y 2009 *Commun. Nonlinear Sci. Numer. Simul.* **14** 3230
[15] Suykens J A K, Curran P F and Chua L O 1999 *IEEE Trans. Circ. Syst. I: Fundamental Theory and Applications* **46** 841
[16] He H L, Tu J J and Xiong P 2011 *J. Franklin Inst.* **348** 2257
[17] Liu X, Gao Q and Niu L Y 2010 *Nonlinear Dyn.* **59** 297
[18] Li T, Song A, Fei S and Wang T 2011 *Commun. Nonlinear Sci. Numer. Simul.* **16** 10
[19] Wang T, Li T, Yang X and Fei S M 2012 *Neurocomputing* **83** 72
[20] Li Z K, Duan Z S and Huang L 2007 *Proceedings of the 26th Chinese Control Conference*, July 26–31, 2007, Zhangjiajie, China, p. 304
[21] Li Z K, Duan Z S and Chen G R 2011 *Int. J. Control* **84** 216
[22] Lohmiller W and Slotine J J E 1998 *Automatica* **34** 683
[23] Sharma B B and Kar I N 2011 *Nonlinear Dyn.* **63** 429
[24] Zhang Y P and Sun J T 2004 *Phys. Lett. A* **330** 442
[25] Li C B, He Y and W A G 2012 *Advances in Automation and Robotics* **1** (Berlin/Herdelberg: Springer-verlag), pp. 207–213
[26] Hmamed A 1991 *Int. J. Syst. Sci.* **22** 605
[27] Coppel W A 1965 *Stability and Asymptotic Behavior of Differential Equations* (Boston: D. C. Heath and Company), p. 59
[28] Desoer C A and Vidyasagar M 1975 *Feedbacksystems: Input–Output Properties* (New York: Academic Press), p. 31