Reliability of linear coupling synchronization of hyperchaotic systems with unknown parameters*

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Complete synchronization could be reached between some chaotic and/or hyperchaotic systems under linear coupling. More generally, the conditional Lyapunov exponents are often calculated to confirm the stability of synchronization and reliability of linear controllers. In this paper, detailed proof and measurement of the reliability of linear controllers are given by constructing a Lyapunov function in the exponential form. It is confirmed that two hyperchaotic systems can reach complete synchronization when two linear controllers are imposed on the driven system unidirectionally and the unknown parameters in the driving systems are estimated completely. Finally, it gives the general guidance to reach complete synchronization under linear coupling for other chaotic and hyperchaotic systems with unknown parameters.

Keywords: parameter estimation, exponential Lyapunov function, parameter observer, linear coupling

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1. Introduction

Hyperchaos is an important subject in the field of nonlinear dynamics, which is often observed in the physical, chemical, and biological systems.[1–8] For example, Rössler[3] presented a four-variable nonlinear equation and generated hyperchaos in this scheme. Yang *et al.*[7] confirmed the chaos occurrence of the time series of an human electrocardiogram by using nonlinear analysis. Krese *et al.*[8] studied the transition of chaos to chaos in the laser droplet generation. In a general chaotic/ hyperchaotic system there often emerge finite strange attractors, and it is confirmed that the multiscroll hyperchaotic attractors have much better properties and complex dynamics than the general hyperchaotic attractors. For example, Lü and Chen[9] and Yu *et al.*[10] even gave an instructive guidance to generate multiscroll chaotic attractors in a theoretical way and investigated its potential applications. One potential application is that time series generated from chaotic or hyperchaotic systems could be used as a secure key or carrier wave for secure communication.[11–18] For example, Kanso *et al.*[11] presented a keyed hash function from a chaotic map. Li *et al.*[12] proposed another keyed hash function from a modified coupled chaotic map lattice for encryption. More interesting, Huang[13] designed a class of parallel keyed hash functions from the neural network. On the other hand, secure communication based on the synchronization between hyperchaotic and/or spatiotemporal systems has also received much attention.[19–26]

Complex systems often contain a large number of oscillators (or cells), for example, the neuronal system is composed of a large number of oscillators, and these oscillators can communicate with other oscillators in a feasible way such as coupling.[27–31] Phase synchronization is often reached under appropriate coupling intensity, and it is confirmed that phase synchronization is conducible to memory processes.[30] Chaotic systems can also reach synchronization due to coupling, for example, Rosenblum *et al.*[32] investigated the phase synchronization between two chaotic Rössler systems and it was also confirmed that phase synchronization could be approached between two chaotic oscillators when the coupling intensity exceeded a certain threshold even if the mismatch of parameters in the two systems occurred. Furthermore, the transition from phase synchronization to lag synchronization was also investigated in two coupled chaotic oscillators, and the synchronization state was dependent on the coupling intensity.[33] More often, much work has discussed the adaptive synchronization between chaotic or hyperchaotic systems,[34–41] the controllers and parameter observers could be approached analytically by using the Lyapunov function control scheme, particularly, parameters with different orders of magnitudes could be estimated completely.[37]

However, the controllers could be in the complex form by using the original Lyapunov function scheme. Therefore, it is important to find a simpler controller such as a linear controller to realize synchronization and parameter estimation. It is confirmed that some chaotic or hyperchaotic systems[42,43] can reach complete or phase synchronization under the linear coupling scheme,[44,45] and its reliability of linear coupling on realizing synchronization is often checked by calculating the conditional Lyapunov exponents, the threshold for coupling intensity could be approached so that the conditional Lyapunov exponents of the controlled systems should be negative.

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On the other hand, the topic about the complex networks[46–48] has received increasing attention from various fields of science and engineering because various synchronization phenomena are found in the network. For example, in Ref. [46] some synchronization phenomenon were investigated and several network synchronization theorems were proved in a time-varying complex dynamical network model. The complex network shows better application than a single oscillator, for example, Squartini *et al.*[47] introduced a neural paradigm for complex systems and the new algorithms and discussed its applications in detail. As a result, the synchronization and control of complex networks has become more interesting and challenging.[48] It is also important to discuss the effectiveness of this scheme. In the case of a ring network with a nearest-neighbor connection (similar to the regular network type), a similar Lyapunov function could be designed and the error function could be marked with a subscript (i). In the case of a two-dimensional regular network, in which each node (oscillator) is connected with the four nearest-neighbor nodes (oscillators), the error function in the Lyapunov function will be marked with two subscripts (i, j), and this scheme could be extended to realize the synchronization in network under linear coupling. However, it could encounter much difficulty in the case of a network with a small-world-type connection because the long-range connection with a certain probability should be considered.

Based on the Lyapunov function scheme and stability theory, the scheme of adaptive synchronization[49–54] between chaotic or hyperchaotic systems is often used to estimate unknown parameters in the driving system and control chaos, and it sounds reasonable in the analytical and numerical way. In fact, this scheme and most of the previous similar schemes of adaptive synchronization deserve further investigation and improvement for some realistic nonlinear systems. In Refs. [55]–[59] the parmaeter identification in chaotic or hyperchaotic systems was also discussed, and these results sound attractive. However, for realistic nonlinear systems, the problem of parameter estimation is often challenging and is worth further investigation by considering the external noise, finite signal series, diversity of order magnitudes of parameters, sudden shifts in parameters, and other uncertain factors. Synchronization between fractional order chaotic systems is also an interesting problem, for example, Huang and Qi[60] suggested that adaptive sliding mode control could be used to realize the synchronization between two different fractional order chaotic systems. It is expected that this continuous linear coupling-induced synchronization and parameter estimation scheme could be improved, and can be used in some realistic nonlinear systems.

In this paper, the four-variable Rössler equations with four unknown parameters will be investigated. The aim of this paper is to prove that linear coupling can realize the complete synchronization between two hyperchaotic Rössler systems under linear coupling, and the unknown parameters in the drive system will be estimated completely. The proof will be given in a theoretical way and then corresponding parameter observers will also be approached analytically. And the design of the exponential Lypunov function could be useful to approach the parameter observer and linear controller for other hyperchaotic or chaotic systems, and thus the control target could be approached completely.

2. Model and scheme

At first, the principle of this scheme is introduced. For any given nonlinear dynamical systems in chaotic or hyperchaotic state, the driving and response systems are defined as follows:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1,x_2,x_3,\ldots x_n,a,\ b,\ c,\ d,\ldots),\\
\dot{x}_2 &= f_2(x_1,x_2,x_3,\ldots x_n,a,\ b,\ c,\ d,\ldots),\\
\dot{x}_3 &= f_3(x_1,x_2,x_3,\ldots x_n,a,\ b,\ c,\ d,\ldots),\\
\dot{x}_4 &= f_4(x_1,x_2,x_3,\ldots x_n,a,\ b,\ c,\ d,\ldots),\\
&\cdots,\\
\dot{x}_n &= f_n(x_1,x_2,x_3,\ldots x_n,a,\ b,\ c,\ d,\ldots);
\end{aligned} \tag{1}$$

$$\begin{aligned}
\dot{y}_1 &= g_1(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots),\\
\dot{y}_2 &= g_2(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots),\\
\dot{y}_3 &= g_3(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots),\\
\dot{y}_4 &= g_4(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots),\\
&\cdots,\\
\dot{y}_n &= g_n(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots),
\end{aligned} \tag{2}$$

where x_i $(i = 1,\ 2,\ 3,\ \ldots, n)$, $a,\ b,\ c,\ d,\ \ldots$, are variables and parameters and $f_1,\ f_2,\ f_3,\ \ldots$ are linear or nonlinear functions in the driving system (1), which determine the dynamics of the systems; y_i $(i = 1,2,3,\ldots,n)$, $\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ \ldots$ are variables and parameters and $g_1,\ g_2,\ g_3,\ \ldots$ are linear or nonlinear functions in the response system (2), more generally, linear controllers could be imposed on the response system (2) as follows:

$$\begin{aligned}
\dot{y}_1 &= g_1(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots)+k_1(x_1-y_1),\\
\dot{y}_2 &= g_2(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots)+k_2(x_2-y_2),\\
\dot{y}_3 &= g_3(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots)+k_3(x_3-y_3),\\
\dot{y}_4 &= g_4(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots)+k_4(x_4-y_4),\\
&\cdots,\\
\dot{y}_n &= g_n(y_1,y_2,y_3\ldots y_n,\hat{a},\ \hat{b},\ \hat{c},\ \hat{d},\ldots)+k_n(x_n-y_n),
\end{aligned} \tag{3}$$

where $k_1,\ k_2,\ k_3,\ k_4,\ \ldots,$ are the coupling intensities, and for simplicity, these coupling intensities could be selected to be the same value, i.e., $k_1 = k_2 = k_3 = k_4 = \cdots = k$. The output error series for the locked variables in the drive and driven systems are marked as follows:

$$e_i = x_i - y_i,\ \ i = 1,2,3,\ldots,n, \tag{4}$$

$$e_a = a - \hat{a},\ e_b = b - \hat{b}, \quad e_c = c - \hat{c}, \quad e_d = d - \hat{d}, \ldots. \tag{5}$$

By calculating the Lyapunov exponent spectra for the driving system (1) and driven system (3), the threshold for coupling intensity $k = k_c$ could be approached when the largest conditional Lyapunov exponent keeps negative by increasing the value of k. According to the Lyapunov stability theory, a general positive Lyapunov function in the exponential form could be constructed as follows:

$$\begin{aligned} V \ =\ & \exp\Bigg[A_1(e_1^2+e_2^2+e_3^2+\cdots+e_n^2)+\sum_{i=1}^{n}A_{ij}e_ie_j \\ & + B(e_a^2+e_b^2+e_c^2+\cdots) \\ & - \int h(e_i,e_a,e_b,\ldots,\hat{a},\hat{b},\hat{c},\ldots,A_i,B,k_c)\,\mathrm{d}t\Bigg] \\ & > 0, \end{aligned} \tag{6}$$

where A_{ij} $(i,j = 1,2,3,\ldots,n)$ is certain coefficient, A_1, B are gain coefficients, and k_c is the critical coupling intensity for realizing the complete synchronization. The nonlinear function $h(e_i,e_a,e_b,\ldots,\hat{a},\hat{b},\hat{c},\ldots,A_i,B,k_c)$ could be approached by calculating the differential coefficient of the Lyapunov function in Eq. (6) on the condition that the differential coefficient of the Lyapunov function versus time should be negative

$$\begin{aligned} \dot{V} \ =\ & V\{[e_a(\dot{e}_a/B - A_1 g_a(x_i,y_i,e_i))] \\ & + [e_b(\dot{e}_b/B - A_1 g_b(x_i,y_i,e_i))] \\ & + [e_c(\dot{e}_c/B - A_1 g_c(x_i,y_i,e_i))] + \cdots \\ & - \sum_{i=1}^{n}(k_i - k_c)e_i^2 + f(**) \\ & - h(e_i,e_a,e_b\ldots,\hat{a},\hat{b},\hat{c},\ldots,A_1,B,k_c)\}, \end{aligned} \tag{7}$$

where $f(**)$ is a superfluous term, which is often used to offset the function $h(e_i,e_a,e_b,\ldots,\hat{a},\hat{b},\hat{c},\ldots,A_i,B,k_c)$ in Eqs. (6) and (7). As a result, equation (7) is redefined as

$$\begin{aligned} \dot{V} \ =\ & V\Bigg\{[e_a(\dot{e}_a/B - A_1 g_a(x_i,y_i,e_i))] \\ & + [e_b(\dot{e}_b/B - A_1 g_b(x_i,y_i,e_i))] \\ & + [e_c(\dot{e}_c/B - A_1 g_c(x_i,y_i,e_i))] + \cdots \\ & - \sum_{i=1}^{n}(k_i - k_c)e_i^2\Bigg\}. \end{aligned} \tag{8}$$

The parameter observers are approached from Eq. (8) as follows:

$$\begin{aligned} & e_a(\dot{e}_a/B - A_1 g_a(x_i,y_i,e_i)) = 0, \\ & e_b(\dot{e}_b/B - A_1 g_b(x_i,y_i,e_i)) = 0, \\ & e_c(\dot{e}_c/B - A_1 g_c(x_i,y_i,e_i)) = 0, \\ & \cdots, \\ & \dot{e}_a/B - A_1 g_a(x_i,y_i,e_i) = 0, \\ & \dot{e}_b/B - A_1 g_b(x_i,y_i,e_i) = 0, \end{aligned} \tag{9a}$$

$$\begin{aligned} & \dot{e}_c/B - A_1 g_c(x_i,y_i,e_i) = 0, \\ & \cdots. \end{aligned} \tag{9b}$$

The parameters in the driving system are often constants, and thus the parameter observers are found as follows:

$$\begin{aligned} & \dot{\hat{a}} = -\dot{e}_a = -A_1 B g_a(x_i,y_i,e_i), \\ & \dot{\hat{b}} = -\dot{e}_b = -A_1 B g_b(x_i,y_i,e_i), \\ & \dot{\hat{c}} = -\dot{e}_c = -A_1 B g_c(x_i,y_i,e_i), \\ & \cdots. \end{aligned} \tag{9c}$$

As a result, equation (8) is redefined as follows:

$$\dot{V} = -V\sum_{i=1}^{n}(k_i - k_c)e_i^2. \tag{10}$$

Equation (10) becomes negative completely under $k_i > k_c$, so the Lyapunov stability criterion is satisfied, and the controlled (driven) system will keep pace with the driving system and the unknown parameters in driving system are estimated according to Eq. (9c). Clearly, the Lyapunov function in exponential type shown in Eq. (6) keeps positive completely, and the differential coefficient of the Lyapunov function versus time becomes negative when linear controllers are imposed on the response system. The parameters are estimated completely when the two hyperchaotic systems reach the complete syschronization. For further understanding, this scheme is used to realize the parameter estimation based on an improved linear coupling-induced synchronization between two hyperchaotic systems.

The dynamical equations for the four-variable Rössler model[3] is often described by

$$\begin{aligned} & \dot{x} = -y - z, \\ & \dot{y} = x + ay + w, \\ & \dot{z} = b + xz, \\ & \dot{w} = -cz + dw, \end{aligned} \tag{11}$$

where x, y, z, w are output variables; a, b, c, d are control parameters in system (11). Hyperchaotic state will be observed in this system when parameters are selected as $a = 0.25$, $b = 3.0$, $c = 0.5$, $d = 0.05$. More generally, controllers are often imposed on the second and fourth formula to change the dynamics of the nonlinear system as shown in Eq. (11). The driven system with nonidentical parameters is often given as

$$\begin{aligned} & \dot{\hat{x}} = -\hat{y} - \hat{z}, \\ & \dot{\hat{y}} = \hat{x} + \hat{a}\hat{y} + \hat{w} + u_3, \\ & \dot{\hat{z}} = \hat{b} + \hat{x}\hat{z}, \\ & \dot{\hat{w}} = -\hat{c}\hat{z} + \hat{d}\hat{w} + u_4, \end{aligned} \tag{12}$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ are output variables and $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ are parameters in Eq. (12). The linear controllers are defined as follows:

$$u_3 = k_3(y - \hat{y}) \ \text{ and } \ u_4 = k_4(w - \hat{w}), \tag{13}$$

where k_3 and k_4 are coupling intensities. The two nonlinear systems will become identical when the parameters in the two systems are the same and then the two identical systems could reach the complete synchronization only when the intensities of coupling exceed certain thresholds. It is important to exactly prove that the controllers, as shown in Eq. (13), are effective to realize the complete synchronization and the parameter observers for the parameters in Eq. (12) will also be approached analytically.

According to Eq. (6), an appropriate positive Lyapunov function is determined as follows:

$$\begin{aligned} V \;=\; & \exp[g_1(e_x^2+e_y^2+e_z^2+e_w^2-2e_xe_y-2e_xe_z-2e_xe_w)/2 \\ & +(e_a^2+e_b^2+e_c^2+e_d^2)/2g_2 \\ & -\int g_1(e_x-e_y-e_z-e_w)(-e_y-e_z)\,\mathrm{d}t \\ & -\int g_1(e_y-e_x)(\hat{a}e_y+e_w)\,\mathrm{d}t-\int g_1(k_3+1)e_ye_x\,\mathrm{d}t \\ & -\int g_1(e_z-e_x)(xz-\hat{x}\hat{z})\,\mathrm{d}t \\ & -\int g_1(e_w-e_x)(-\hat{c}e_z+\hat{d}e_w)\,\mathrm{d}t \\ & -\int g_1k_4e_xe_w\,\mathrm{d}t-\int k_3e_ye_b/g_2\,\mathrm{d}t]>0, \end{aligned} \tag{14}$$

where g_1 and g_2 are positive gain coefficients to be selected, and the errors for corresponding variables, parameters in the two systems are determined as follows:

$$e_x=x-\hat{x},\;\; e_y=y-\hat{y},\;\; e_z=z-\hat{z},\;\; e_w=w-\hat{w}, \tag{15a}$$

$$\theta(e_x,\,e_y,\,e_z,\,e_w)=\sqrt{(e_x^2+e_y^2+e_z^2+e_w^2)}. \tag{15b}$$

Then the differential coefficient of the Lyapunov function as shown in Eq. (14) versus time is approached by

$$\begin{aligned} \dot{V} \;=\; & V[(e_x\dot{e}_x+e_y\dot{e}_y+e_z\dot{e}_z+e_w\dot{e}_w-\dot{e}_xe_y-e_x\dot{e}_y \\ & -\dot{e}_xe_z-e_x\dot{e}_z-\dot{e}_xe_w-e_x\dot{e}_w)g_1 \\ & +(e_a\dot{e}_a+e_b\dot{e}_b+e_c\dot{e}_c+e_d\dot{e}_d)/g_2 \\ & -(e_x-e_y-e_z-e_w)(-e_y-e_z)g_1 \\ & -(e_y-e_x)(\hat{a}e_y+e_w)g_1-(k_3+1)e_ye_xg_1 \\ & -(e_z-e_x)(xz-\hat{x}\hat{z})g_1 \\ & -(e_w-e_x)(-\hat{c}e_z+\hat{d}e_w)g_1-g_1k_4e_xe_w \\ & -k_3e_ye_b/g_2] \\ =\; & \{e_a[\dot{e}_a/g_2+g_1y(e_y-e_x)]+e_b[\dot{e}_b/g_2+g_1(e_z-e_x) \\ & -k_3e_y/g_2]+e_c[\dot{e}_c/g_2-g_1z(e_w-e_x)] \\ & -e_d[\dot{e}_d/g_2+g_1w(e_w-e_x)] \\ & -g_1e_x^2-k_3g_1e_y^2-k_4g_1e_w^2\}V. \end{aligned} \tag{16}$$

According to the Lyapunov stability theory, the Lyapunov function in Eq. (14) will decrease to zero with a long transient period and thus the two systems become synchronized only when the result in Eq. (16) is negative completely. As a result, it is required that the condition as shown in Eq. (17) should be satisfied, i.e.,

$$\begin{aligned} 0 \;>\; & e_a[\dot{e}_a/g_2+g_1y(e_y-e_x)] \\ & +e_b[\dot{e}_b/g_2+g_1(e_z-e_x)-k_3e_y/g_2] \\ & +e_c[\dot{e}_c/g_2-g_1z(e_w-e_x)]+e_d[\dot{e}_d/g_2+g_1w(e_w-e_x)] \\ & -g_1e_x^2-k_3g_1e_y^2-k_4g_1e_w^2. \end{aligned} \tag{17}$$

A general but simple solution for inequality (17) could be given as

$$\begin{aligned} & e_a[\dot{e}_a/g_2+g_1y(e_y-e_x)]=0, \\ & e_b[\dot{e}_b/g_2+g_1(e_z-e_x)-k_3e_y/g_2]=0, \\ & 0=e_c[\dot{e}_c/g_2-g_1z(e_w-e_x)], \\ & 0=e_d[\dot{e}_d/g_2+g_1w(e_w-e_x)]. \end{aligned} \tag{18}$$

And the condition as shown in Eq. (17) is replaced as follows:

$$\dot{V}=V(-g_1e_x^2-k_3g_1e_y^2-k_4g_1e_w^2)<0. \tag{19}$$

When the parameters in the driving system (11) are not time-varying, and the parameter observers are approached as follows:

$$\begin{aligned} & \dot{\hat{a}}=-\dot{e}_a=g_1g_2y(e_y-e_x), \\ & \dot{\hat{b}}=-\dot{e}_b=g_1g_2(e_z-e_x)-k_3e_y, \\ & \dot{\hat{c}}=-\dot{e}_c=-g_1g_2z(e_w-e_x), \\ & \dot{\hat{d}}=-\dot{e}_d=g_1g_2w(e_w-e_x). \end{aligned} \tag{20}$$

Inequality (19) is satisfied when appropriate gain coefficients and coupling intensity are used, and thus the parameter observers in Eq. (20) will become stable to estimate all the unknown parameters in the driving system as shown in Eq. (11). According to Eqs. (14) and (19), the Lyapunov function decreases with time gradually only when the gain coefficients and coupling intensity are selected to be positive values because $(-g_1e_x^2-k_3g_1e_y^2-k_4g_1e_w^2)<0$, and the convergence rate of error function is dependent on the selection of gain coefficients and coupling intensity.

3. Numerical results and discussion

In the numerical studies, the fourth Runge–Kutta algorithm is used to calculate the nonlinear dynamical equations, in time steps of $h=0.001$. The initial values for the drive system and driven system are used as (−20, 0, 0, 15), (−25, 0, 0, 20), respectively. The transient period is about 500 time units.

The results in Figs. 1 and 2 confirm that the two hyperchaotic Rössler systems could reach the complete synchronization and all the unknown parameters are identified completely under appropriate gain coefficients and coupling intensity. Extensive numerical results confirm that the transient period decreases when bigger gain coefficients and coupling intensity are used. Furthermore, it is also investigated that

some parameters in the drive system jump suddenly, for example, parameter shift could occur when some electric elements (resistance, capacitor, inductance) in the circuit are damaged. The numerical results confirm that the unknown parameters in the drive system are estimated completely. In Fig. 3, the error function series are plotted, and the estimated results for the unknown parameters are plotted in Fig. 4 when parameter b is jumped from 3 to 4 at $t = 250$ time units.

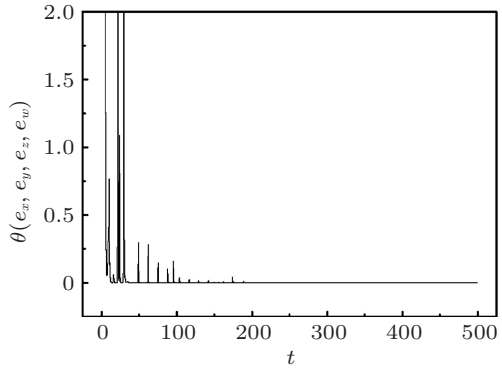


Fig. 1. Time evolution of the error function as defined in Eq. (15b), with the two gain coefficients being $g_1 = 5.0$ and $g_2 = 0.05$, and the coupling intensity for controllers being $k_3 = k_4 = 5.0$.

The results in Fig. 3 confirm that the error function decreases to a stable value quickly, and a small fluctuation occurs when parameter b changes from 3 to 4 at $t = 250$ time units, then the error function still decreases to zero and a complete synchronization state is reached.

The results in Fig. 4 confirm that the unknown parameters still could be identified exactly even though one parameter is jumped from 3 to 4 at t=250 time units. It indicates that the scheme could be effective to estimate the unknown parameters when some parameters in the drive system jump suddenly. We also check the case where the gain coefficient and coupling intensity are selected to be other values. It is also confirmed that the short transient period costs a lot when a group of bigger values are given to the gain coefficients and coupling intensity. For example in which another group of gain coefficients and coupling intensity are selected, the results are shown in Figs. 5 and 6.

The results in Figs. 5 and 6 confirm that the two nonlinear hyperchaotic systems can still reach synchronization completely, and the unknown parameters could be estimated exactly even if one of the unknown parameters in the drive system decreases from $b = 3$ to $b = 2.0$ at $t = 250$ time units. Furthermore, this exponential Lyapunov function scheme is also used to prove the effectiveness of linear coupling on synchronization between the chaotic Chua circuit and hyperchaotic Chen system, and the linear controllers based on linear coupling and the parameter observers could be approached and proved to be stable and effective.

In summary, this scheme is helpful to confirm the reliability of linear coupling-induced synchronization of other chaotic or hyperchaotic systems. In some hyperchaotic systems there could emerge multiscroll attractors, and the appearance of multiscroll attractors often depends on the dynamics and parameter region of the system. This scheme is also useful when the dynamics of the system are described by deterministic

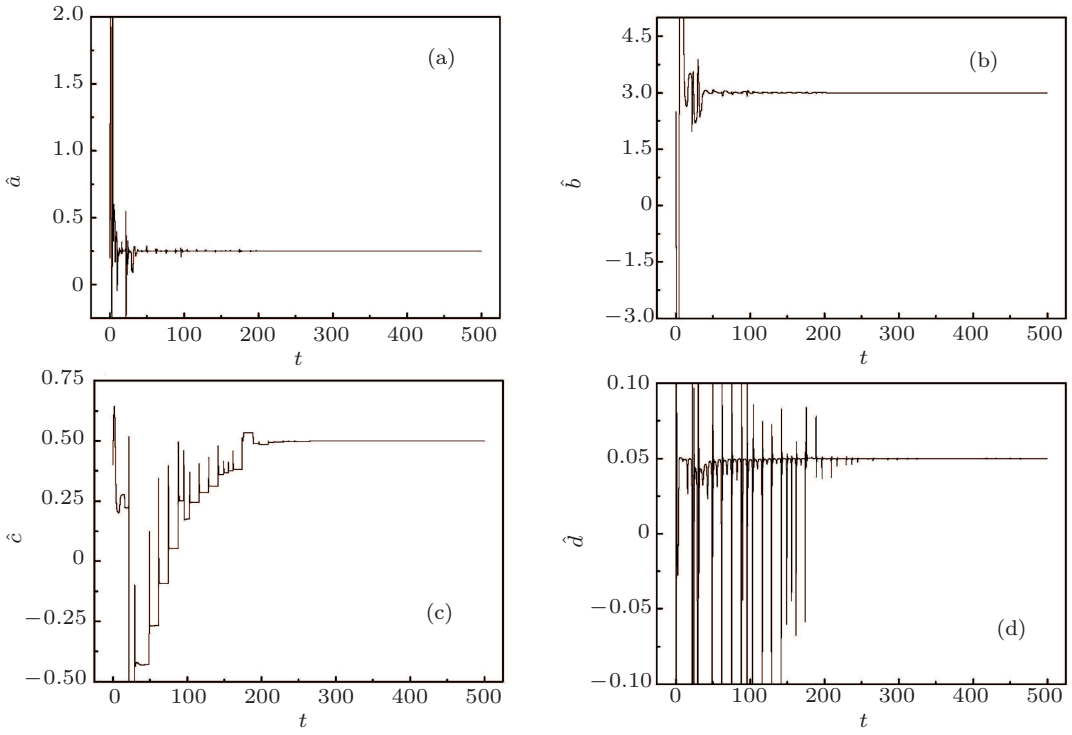


Fig. 2. Estimated results for the four unknown parameters in the driven system, with the two gain coefficients being $g_1 = 5.0$ and $g_2 = 0.05$, and the coupling intensity for controllers being $k_3 = k_4 = 5.0$. The real values for these unknown parameters in the driving system are $a = 0.25$, $b = 3.0$, $c = 0.5$, $d = 0.05$.

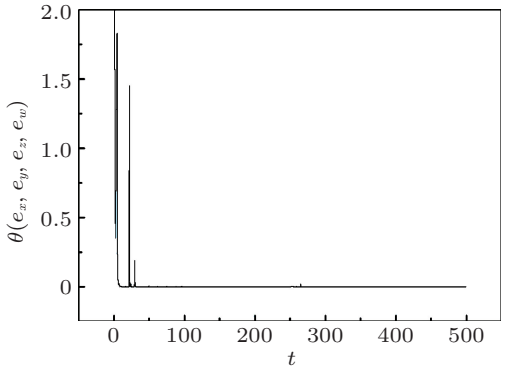


Fig. 3. Time evolution of the error function as defined in Eq. (15b), with the two gain coefficients being $g_1 = 4.0$ and $g_2 = 0.04$, and the coupling intensity for controllers being $k_3 = k_4 = 5.0$.

dynamic equations. It is also important to discuss the potential application of this scheme, and it is also attractive to explain whether this scheme is useful in the networks. In our opinion, this scheme confirms the reliability of linear coupling-induced synchronization between hyperchaotic oscillators and that of the parameter estimation. A liner controller could be the simplest controller and it is practical to be realized experimentally. In most of the previous work about adaptive synchronization, the appropriate Lyapunov function is often constructed, and the controllers are often in complex form. It is critical to find the appropriate exponential Lyapunov function when liner coupling is used to realize the synchronization and control of hyperchaotic systems. Furhthermore, this scheme will be investigated in the nonlinear systems with fractional order and time-delay, and it could provide a simpler and practical way to realize the synchronization between more complex hyperchaotic systems.

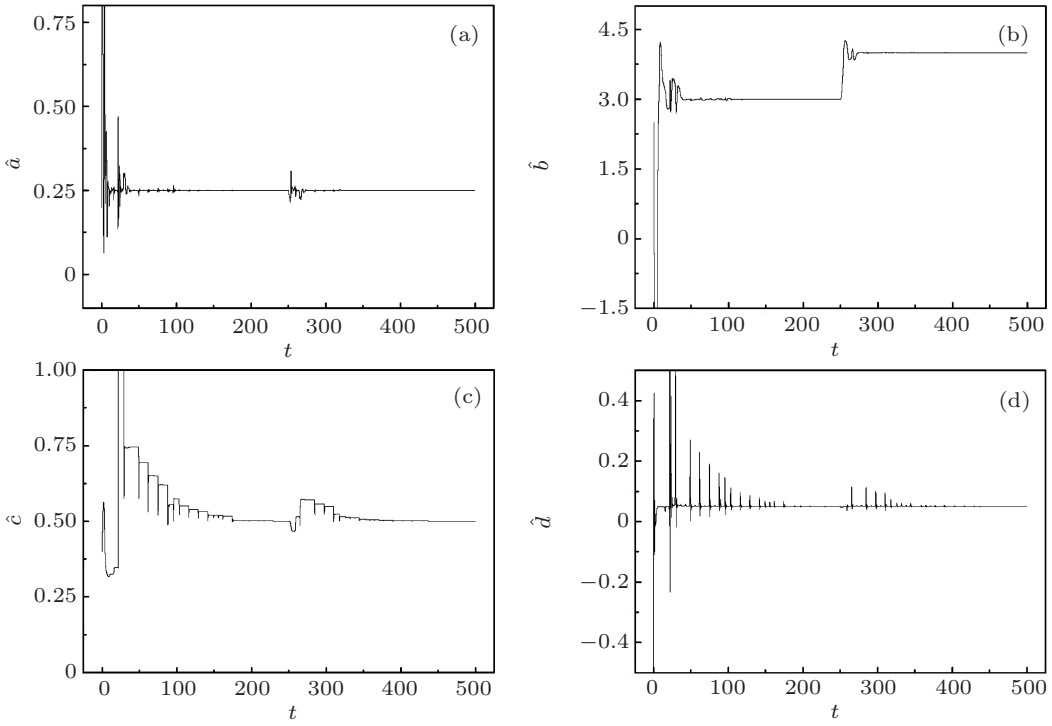


Fig. 4. Estimated results for the four unknown parameters in the drive system, with the two gain coefficients being $g_1 = 4.0$ and $g_2 = 0.04$, and the coupling intensity for controllers being $k_3 = k_4 = 5.0$. The real values for these unknown parameters in the drive system are $a = 0.25$, $b = 3.0$ ($t < 250$ time units), $b = 4.0$ ($t > 250$ time units), $c = 0.5$, $d = 0.05$.

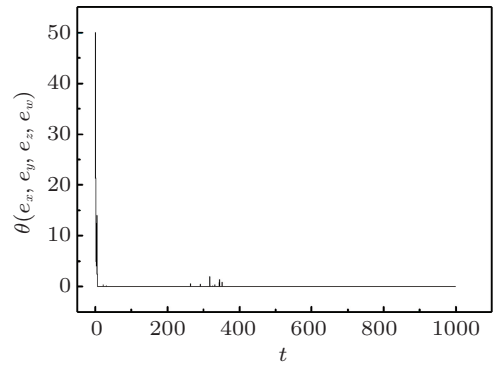


Fig. 5. Time evolution of the error function as defined in Eq. (15b), with the two gain coefficients being $g_1 = 3.0$ and $g_2 = 0.03$, and the coupling intensity for controllers being $k_3 = 4.5$ and $k_4 = 4.0$.

4. Conclusions

In this paper, a kind of exponential Lyapunov function is constructed to prove that linear controllers are very effective to realize the complete synchronization between two hyperchaotic systems, and all the unknown parameters in the driving system could be identified exactly. Our results could give an exact proof about the reliability of linear coupling on realizing the synchronization between chaotic or hyperchaotic systems because most of the previous work focused on checking the possibility of linear coupling by calculating the conditional Lyapunov exponents, while exact analysis was left out. The general Lyapunov function scheme is an

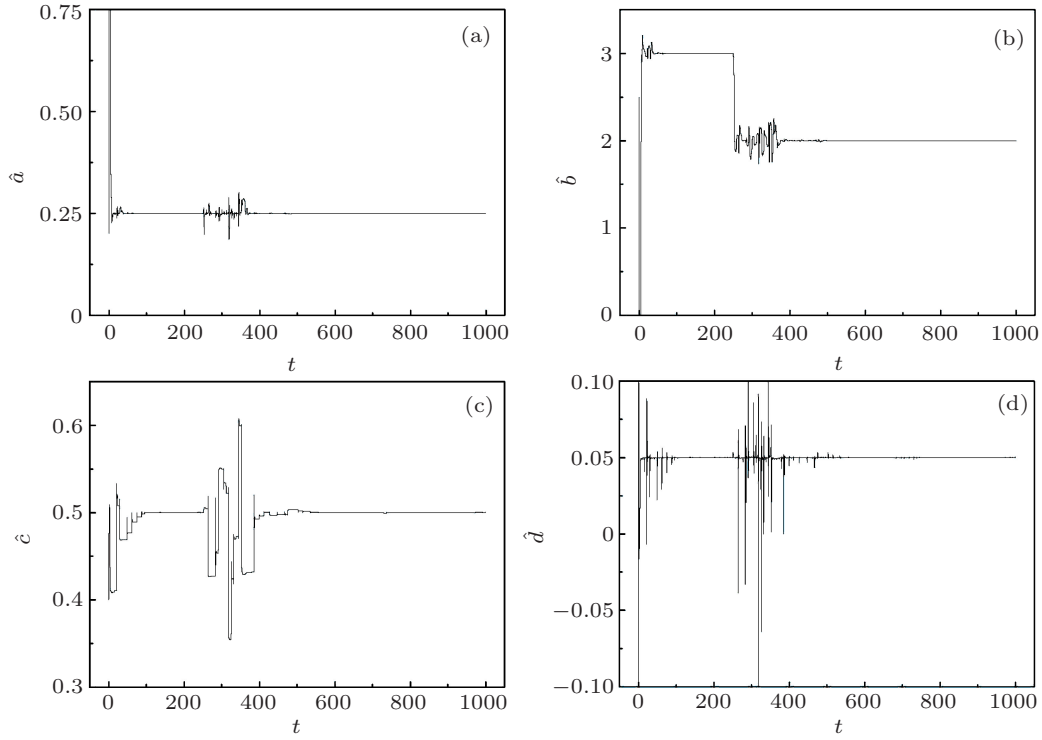


Fig. 6. Estimated results for the four unknown parameters in the drive system, with the two gain coefficients being $g_1 = 3.0$ and $g_2 = 0.03$, and the coupling intensity for controllers being $k_3 = 4.5$ and $k_4 = 4.0$. The real values for these unknown parameters in the drive system are $a = 0.25$, $b = 3.0$ ($t < 250$ time units), $b = 2.0$ ($t > 250$ time units), $c = 0.5$, $d = 0.05$.

effective scheme to construct a controller and parameter observer to realize the synchronization, but sometimes the form of controllers is much too complex and these controllers are difficult to realize experimentally. However, the scheme in this paper could give a feasible way to find appropriate linear controllers and parameter observers by constructing an exponential Lyapunov function, and it is also independent of the model selection and dynamics of the system.

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