Bound entanglement and teleportation for arbitrary bipartite systems*

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We construct bound entangled states that are entangled but from which no entanglement can be distilled if all parties are allowed only by performing local operations and classical communications. Moreover, as applications, a detailed example is presented. This example can illuminate that the fidelity of transmission using a bound entangled state is not bigger than a classical scheme.

Keywords: bound entanglement, teleportation, partial transpose

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1. Introduction

Entanglement is one of the most fascinating features of quantum theory and has numerous applications in quantum information processing. Historically, Einstein, Podolsky and Rosen (EPR)[1] first recognized entanglement. The significant progress towards the resolution of the EPR problem was made by Bell.[2] There have many criterions to determine whether a given state is entangled, such as the PPT (Positive Partial Transpose) criterion,[3] the reduction criterion,[4] the range criterion,[5] the matrix realignment criterion,[6] and so on. More efforts were done to analyze the entanglement of quantum states.[7–12] States which are entangled but not distillable are called bound entanglement. Bound entanglement has played very important roles in quantum information, especially in quantum teleportation.

The existence of bound entanglement has been the most intriguing surprising in quantum information. The first example of bound entanglement was constructed by Horodecki.[5] Bennett *et al.*[13] constructed a class of bound entangled states based on the unextendible product basis (UPB). Breuer *et al.*[14] presented a systematic method for the explicit construction of high-dimensional manifolds of bound entangled states. Derkacz *et al.* gave some conditions to construct bound entangled states by using a set of mutually unbiased basis (MUB).[15,16] The existence of thermal-state bound entanglement for several quantum many-body systems was shown by Cavalcanti *et al.*[17] Augusiak *et al.*[18] investigated the dependence of efficiency of detecting entanglement or bound entanglement. Although some examples of bound entangled states are found, the physical characters and mathematical structures of bound entangled states have not been well understood.

In this paper we mainly construct bound entangled states for arbitrary bipartite systems. This paper is organized as follows. In Section 2, we give a detailed description about the construction of bound entangled states in $2\otimes n$ systems. In Section 3, we generalize these results to $n\otimes n$ systems. In Section 4, as applications, a detailed example is presented to illuminate that the fidelity of bound entanglement is not bigger than a classical scheme. Conclusion and discussion are given in Section 5.

2. Construction of bound entangled states in $2\otimes n$ systems

Firstly, we give a theorem to prove that a matrix is positive.

Theorem 1 A real quadratic form $f(x_1,x_2,\ldots,x_n) = X^{\mathrm{T}}AX$ is positive if and only if the positive index of inertia of the quadratic form equals its rank.

Proof We first prove its sufficiency. Every quadratic form $X^{\mathrm{T}}AX$ of rank n can be reduced by a non-degenerate transformation $X = CY$ to the normal form

$$y_1^2+y_2^2+\cdots+y_n^2, \tag{1}$$

where the matrix C is non-degenerate. For any nonzero real vector $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)^{\mathrm{T}}$ and $\beta = C^{-1}\alpha = (\beta_1,\beta_2,\ldots,\beta_n)^{\mathrm{T}}$, we have $f(\alpha_1,\alpha_2,\ldots,\alpha_n) = \beta_1^2+\beta_2^2+\cdots+\beta_n^2 \geq 0$. Hence the quadratic form is positive.

We then prove its necessity. Let p and r denote the positive index of inertia and the rank of the quadratic form, respectively. Assume that $p < r$, the quadratic form $f(x_1,x_2,\ldots,x_n)$ can be reduced to the normal form

$$y_1^2+y_2^2+\cdots+y_p^2-y_{p+1}^2-\cdots-y_r^2, \tag{2}$$

by a non-degenerate transformation $X = CY$. For any real vector $\beta = (0,\ldots,0,1,0,\ldots,0)^{\mathrm{T}}$, only the $(p+1)$-th component

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is 1, and the remaining components are 0. Let $\alpha = C\beta = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}$, then $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = -1 < 0$, which contradicts the fact that $f(x_1, x_2, \ldots, x_n)$ is positive. Thus $p = r$.

Consider the Hilbert space $\mathscr{H} = \mathscr{C}^2 \otimes \mathscr{C}^n$. Let $\{e_i\}_{i=1}^n$ denote the standard bases of $\mathscr{C}^n$. We construct an inseparable state

$$\rho_{\mathrm{insep}} = \frac{2}{2n-1}\sum_{i=1}^{n-1} P_{\varphi_i} + \frac{1}{2n-1} P_{e_1 \otimes e_n}, \tag{3}$$

where we define vectors $\varphi_i \equiv (e_1 \otimes e_i + e_2 \otimes e_{i+1})/\sqrt{2}$ $(i = 1, 2, \ldots, n-1)$, and the projection operator $P_\phi \equiv |\phi\rangle\langle\phi|$. The inseparability of state ρ_{insep} can be easily verified by using the PPT criterion. Next we can get the state

$$\rho_a = \frac{(2n-1)a}{(2n-1)a+1}\rho_{\mathrm{insep}} + \frac{1}{(2n-1)a+1} P_{\psi_a}, \tag{4}$$

where the vectors

$$\psi_a \equiv e_2 \otimes \left(\sqrt{\frac{1+a}{2}}e_1 + \sqrt{\frac{1-a}{2}}e_n\right), \quad 0 \le a \le 1.$$

Theorem 2 The state ρ_a defined as Eq. (4) in $2 \otimes n$ $(n \ge 4)$ systems is a bound entangled state.

Proof Part 1: First we prove that ρ_a is a PPT state, i.e, the partial transposition $\rho_a^{\mathrm{T}_2}$ is positive. The corresponding matrices are

$$\rho_a = \frac{1}{(2n-1)a+1} \times \begin{bmatrix} a & 0 & \cdots & 0 & 0 & a & \cdots & 0 \\ 0 & a & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & (1+a)/2 & 0 & \cdots & \sqrt{1-a^2}/2 \\ a & 0 & \cdots & 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{1-a^2}/2 & 0 & \cdots & (1+a)/2 \end{bmatrix},$$

$$\rho_a^{\mathrm{T}_2} = \frac{1}{(2n-1)a+1} \times \begin{bmatrix} a & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 & a & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a & 0 & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 & (1+a)/2 & 0 & \cdots & \sqrt{1-a^2}/2 \\ 0 & 0 & \cdots & 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{1-a^2}/2 & 0 & \cdots & (1+a)/2 \end{bmatrix}. \tag{5}$$

For any nonzero real vector $X = (x_1, x_2, \ldots, x_{2n})^{\mathrm{T}}$, we have

$$\begin{aligned} X^{\mathrm{T}}\rho_a^{\mathrm{T}_2}X &= f(x_1, x_2, \ldots, x_{2n}) = ax_1^2 + a(x_2 + x_{n+1})^2 \\ &\quad + a(x_3 + x_{n+2})^2 + \cdots + a(x_n + x_{2n-1})^2 \\ &\quad + \left(\sqrt{\frac{1-a}{2}}x_{n+1} + \sqrt{\frac{1+a}{2}}x_{2n}\right)^2. \end{aligned} \tag{6}$$

Obviously, the positive index of inertia is $p = n+1$, and the rank of $\rho_a^{\mathrm{T}_2}$ is $r = n+1$. So we straightforwardly verify that the state $\rho_a^{\mathrm{T}_2}$ is positive by Theorem 1.

Part 2: We now prove that ρ_a is an entangled state by using the range criterion. Note that a state acting on $2 \otimes 2$ or $2 \otimes 3$ systems is separable if and only if its partial transposition is a positive operator.[19] Hence we only consider the case of the state acting on $2 \otimes n$ $(n \ge 4)$ quantum systems. Assume that $a \ne 0, 1$, on the one hand, any vector belonging to the range of ρ_a can be presented as

$$\begin{aligned} \mu &= (A_1, A_2, \ldots, A_{n-1}, A_n; A_{n+1}, A_1, \ldots, A_{n-2}, A_{n-1} + xA_{n+1}), \\ &\quad A_i \in \mathscr{C}, i = 1, 2, \ldots, n. \end{aligned} \tag{7}$$

For $x = \sqrt{(1-a)/(1+a)} \ne 0, 1$, if μ is positive, it must be of the form

$$\begin{aligned} \mu_{\mathrm{prod}} &\equiv (r, s) \otimes (\widetilde{A_1}, \widetilde{A_2}, \ldots, \widetilde{A_n}) \\ &= (r\widetilde{A_1}, r\widetilde{A_2}, \ldots, r\widetilde{A_n}; s\widetilde{A_1}, s\widetilde{A_2}, \ldots, s\widetilde{A_n}), \\ &\quad r, s, \widetilde{A_i} \in \mathscr{C}, i = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

Comparing the two forms of vector μ, we can discuss the following cases.

(i) If $r \ne 0, s = 0$, then we have $\mu_{\mathrm{prod}} = (r\widetilde{A_1}, r\widetilde{A_2}, \ldots, r\widetilde{A_n}; 0, 0, \ldots, 0)$. So $A_{n+1} = A_1 = A_2 = \cdots = A_{n-2} = A_{n-1} + xA_{n+1} = 0$, $r\widetilde{A_1} = r\widetilde{A_2} = \cdots = r\widetilde{A_{n-1}} = 0$, and $r\widetilde{A_n} = A_n$. Thus we obtain the vectors

$$\begin{aligned} \mu_1 &= (0, 0, \ldots, 0, A_n; 0, 0, \ldots, 0) \\ &= \widetilde{A_n}(r, 0) \otimes (0, 0, \ldots, 1), \quad r, A_n, \widetilde{A_n} \in \mathscr{C}. \end{aligned} \tag{9}$$

(ii) If $r \ne 0, s \ne 0$, because of $A_1 = r\widetilde{A_1}$ and $A_{n+1} = s\widetilde{A_1}$, we obtain that $A_{n+1} = (s/r)A_1$. Analogously, $A_2 = (r/s)A_1, \ldots, A_{n-1} = (r/s)^{n-2}A_1$ and $A_n = (r/s)^{n-1}A_1 + xA_1$. So we get the vectors

$$\begin{aligned} \mu_2 &= \left(A_1, \frac{r}{s}A_1, \ldots, \left(\frac{r}{s}\right)^{n-2}A_1, \left(\frac{r}{s}\right)^{n-1}A_1 + xA_1; \frac{s}{r}A_1, A_1, \right. \\ &\quad \left. \ldots, \left(\frac{r}{s}\right)^{n-3}A_1, \left(\frac{r}{s}\right)^{n-2}A_1 + x\frac{s}{r}A_1\right) \\ &= A_1(r, s) \otimes \left(\frac{1}{r}, \frac{1}{s}, \ldots, \frac{r^{n-3}}{s^{n-2}}, \frac{r^{n-2}}{s^{n-1}} + \frac{x}{r}\right), \\ &\quad r, s, A_1 \in \mathscr{C}. \end{aligned} \tag{10}$$

(iii) If $r = 0$, so $A_1 = r\widetilde{A_1} = 0$, $A_2 = r\widetilde{A_2} = 0, \ldots, A_n = r\widetilde{A_n} = 0$. If $s = 0$, then $A_{n+1} = 0$, and μ is a zero vector. Obviously, a zero vector does not belong to the range of ρ_a. If $s \ne 0$, then $A_{n+1} = s\widetilde{A_1}$, $A_{n-1} + xA_{n+1} = s\widetilde{A_n} = xs\widetilde{A_1}$. So we can obtain the vectors

$$\begin{aligned} \mu_3 &= (0, 0, \ldots, 0; A_{n+1}, 0, \ldots, xA_{n+1}) \\ &= \widetilde{A_1}(0, s) \otimes (1, 0, \ldots, x), \quad s, A_{n+1}, \widetilde{A_1} \in \mathscr{C}. \end{aligned} \tag{11}$$

All partial complex conjugations of vectors μ_1, μ_2, μ_3 are

$$\begin{aligned}
\mu_1^{\star 2} &= A_n(0,0,\ldots,0,1;0,0,\ldots,0),\\
\mu_2^{\star 2} &= A_1\Bigg(\frac{r}{r^\star},\frac{r}{s^\star},\ldots,\frac{r(r^\star)^{n-2}}{(s^\star)^{n-1}}+\frac{r}{r^\star}x;\frac{s}{r^\star},\frac{s}{s^\star},\ldots,\\
&\quad \frac{s(r^\star)^{n-2}}{(s^\star)^{n-1}}+\frac{s}{r^\star}x\Bigg),\\
\mu_3^{\star 2} &= A_{n+1}(0,0,\ldots,0;1,0,\ldots,x). \qquad (12)
\end{aligned}$$

On the other hand, any vector that belongs to the range of $\rho_a^{\mathrm{T}_2}$ can be written as

$$\begin{aligned}
\mu' &= (A_1',A_2',\ldots,A_n';A_2'+xA_{2n}',A_3',\ldots,A_n',A_{2n}'),\\
&\quad A_i'\in\mathscr{C}, i=1,2,\ldots,n,2n. \qquad (13)
\end{aligned}$$

Let us check now whether the vectors $\mu_1^{\star 2},\mu_2^{\star 2},\mu_3^{\star 2}$ can be written in the above form.

i) For the vector $\mu_1^{\star 2}$, we get immediately that it must hold $A_n=A_n'=0$, then $\mu_1^{\star 2}$ is insignificant.

ii) For the $\mu_2^{\star 2}$, assuming that it is of the form μ'. Taking into account the coefficients A_3', A_4', …, A_{n-1}' respectively, we obtain $rr^\star=ss^\star$, and taking into account A_n', we get $x=0$. Which contradicts the fact that $x=\sqrt{(1-a)/(1+a)}\neq 0,1$.

iii) For the $\mu_3^{\star 2}$, considering $A_1'=A_2'=\cdots=A_n'=0, A_2'+xA_{2n}'=xA_{2n}'=s\widetilde{A_1}$, and $A_{2n}'=xs\widetilde{A_1}$, so we obtain $x^2=1$. Which also contradicts the fact that $x=\sqrt{(1-a)/(1+a)}\neq 0,1$.

In summary, it leads to the conclusion that none of vectors $\mu_i^{\star 2}$ $(i=1,2,3)$ belong to the range of $\rho_a^{\mathrm{T}_2}$. For any $a\neq 0,1$, the state ρ_a is entangled. Thus it is a bound entangled state.

3. Construction of bound entangled states in $n\otimes n$ systems

Consider the Hilbert space $\mathscr{H}=\mathscr{C}^n\otimes\mathscr{C}^n$, we define the vectors

$$\begin{aligned}
\Phi &\equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_i\otimes e_i,\\
\Psi_\alpha &\equiv e_n\otimes\left(\sqrt{\frac{1+\alpha}{2}}e_1+\sqrt{\frac{1-\alpha}{2}}e_n\right),\ 0\le\alpha\le 1, \qquad (14)
\end{aligned}$$

and consider the following state

$$\sigma_{\mathrm{insep}}\equiv\frac{n}{n^2-1}P_\Phi+\frac{1}{n^2-1}Q, \qquad (15)$$

where a projection operator $Q\equiv I\otimes I-(\sum_{i=1}^n P_{e_i}\otimes P_{e_i}+P_{e_n}\otimes P_{e_1})$. As $\det(\lambda I-\sigma_{\mathrm{insep}}^{\mathrm{T}_2})=\lambda^{n-1}(\lambda-1)^{n^2-2n}(\lambda-2)^{n-1}(\lambda^2-\lambda-1)$, $\sigma_{\mathrm{insep}}^{\mathrm{T}_2}$ possesses a negative eigenvalue $\lambda=(1-\sqrt{5})/2$. Hence the state σ_{insep} is inseparable.

Mixing the states σ_{insep} and P_{Ψ_α} in the following way, we will have a state

$$\sigma_\alpha=\frac{(n^2-1)\alpha}{(n^2-1)\alpha+1}\sigma_{\mathrm{insep}}+\frac{1}{(n^2-1)\alpha+1}P_{\Psi_\alpha}. \qquad (16)$$

Theorem 3 The state σ_α defined in Eq. (16) in $n\otimes n$ $(n\ge 3)$ systems is a bound entangled state.

Proof Firstly, the corresponding matrices of σ_α and $\sigma_\alpha^{\mathrm{T}_2}$ are

$$\begin{aligned}
&\sigma_\alpha\\
&=\frac{1}{(n^2-1)\alpha+1}\\
&\times\begin{bmatrix}
\alpha & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & \alpha\\
0 & \alpha & \cdots & 0 & \cdots & 0 & 0 & \ldots & 0\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & \alpha & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & 0 & \cdots & (1+\alpha)/2 & 0 & \cdots & \sqrt{1-\alpha^2}/2\\
0 & 0 & \cdots & 0 & \cdots & 0 & \alpha & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
\alpha & 0 & \cdots & 0 & \cdots & (\sqrt{1-\alpha^2})/2 & 0 & \cdots & (1+\alpha)/2
\end{bmatrix},\\
&\sigma_\alpha^{\mathrm{T}_2}\\
&=\frac{1}{(n^2-1)\alpha+1}\\
&\times\begin{bmatrix}
\alpha & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0\\
0 & \alpha & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & \alpha & \cdots & \alpha & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & \alpha & \cdots & (1+\alpha)/2 & 0 & \cdots & \sqrt{1-\alpha^2}/2\\
0 & 0 & \cdots & 0 & \cdots & 0 & \alpha & \cdots & 0\\
A: & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & 0 & \cdots & \sqrt{1-\alpha^2}/2 & 0 & \cdots & (1+\alpha)/2
\end{bmatrix}. \qquad (17)
\end{aligned}$$

The real symmetric matrix $\sigma_\alpha^{\mathrm{T}_2}$ corresponding to the quadric form is

$$\begin{aligned}
&f(x_1,x_2,\ldots,x_{n^2})\\
&=\alpha x_1^2+\alpha(x_2+x_{n+1})^2+\cdots+\alpha(x_n+x_{n^2-n+1})^2\\
&\quad+\alpha x_{n+2}^2+\alpha(x_{n+3}+x_{2n+2})^2+\cdots\\
&\quad+\alpha(x_{2n}+x_{n^2-n+2})^2+\cdots\\
&\quad+\left(\sqrt{\frac{1-\alpha}{2}}x_{n^2-n+1}+\sqrt{\frac{1+\alpha}{2}}x_{n^2}\right)^2. \qquad (18)
\end{aligned}$$

Obviously, the positive index of inertia $p=n(n+1)/2$, and the rank of $\sigma_\alpha^{\mathrm{T}_2}$ is $r=n(n+1)/2$. According to the Theorem 1, we drive that the state $\sigma_\alpha^{\mathrm{T}_2}$ is positive. So we have proved that the state σ_α is a PPT state.

Next, we will show that the state σ_α is entangled. Note that a state acting on $2\otimes 2$ systems is separable if and only if its partial transposition is a positive operator.[19] Hence we only consider the case of the state acting on $n\otimes n$ $(n\ge 3)$ quantum systems. For any vector belonging to the range of $\sigma_\alpha^{\mathrm{T}_2}$ can be

presented as

$$\widetilde{\omega} = (A_1^{(1)},A_2^{(1)},\ldots,A_n^{(1)};A_2^{(1)},A_2^{(2)},\ldots,A_n^{(2)};\ldots;A_n^{(1)}+A,A_n^{(2)},\ldots,A_n^{(n-1)},yA),\ A_i^{(j)},A\in\mathscr{C},i,j=1,2,\ldots,n. \quad (19)$$

Where the superscript (j) denotes the j-th subsystem. For $y=\sqrt{(1+\alpha)/(1-\alpha)}\neq 0,1$, if $\widetilde{\omega}$ is positive, it must be of the form

$$\widetilde{\omega}_{\text{prod}} = (r_1,r_2,\ldots,r_n)\otimes(\widetilde{A_1},\widetilde{A_2},\ldots,\widetilde{A_n}),\quad r_i,\widetilde{A_i}\in\mathscr{C},\quad i=1,2,\ldots,n. \quad (20)$$

Let us now consider the following cases.

(I) While $r_1\neq 0$, without loss of generality we put $r_1=1$, then we can get the vectors

$$\begin{aligned}
&\widetilde{\omega}_1 = A_1^{(1)}(1,r_2,\ldots,r_{n-2},r_{n-1},0)\otimes(1,r_2,\ldots,r_{n-2},r_{n-1},0),\\
&\widetilde{\omega}_2 = A_1^{(1)}(1,r_2,\ldots,r_{n-2},0,0)\otimes(1,r_2,\ldots,r_{n-2},0,0),\\
&\quad\cdots,\\
&\widetilde{\omega}_{2^{n-2}} = A_1^{(1)}(1,0,\ldots,0,0,r_n)\otimes\left(1,0,\ldots,0,0,\frac{yr_n}{y+r_n}\right),\\
&\widetilde{\omega}_{2^{n-2}+1} = A_1^{(1)}(1,0,\ldots,0,0,0)\otimes(1,0,\ldots,0,0,0).
\end{aligned} \quad (21)$$

(II) While $r_1=0$, we can get the vectors

$$\begin{aligned}
\widetilde{\omega}_{2^{n-2}+2} &= A_2^{(2)}(0,1,r_3,\ldots,r_{n-2},r_{n-1},0)\\
&\quad\otimes(0,1,r_3,\ldots,r_{n-2},r_{n-1},0),\\
\widetilde{\omega}_{2^{n-2}+3} &= A_2^{(2)}(0,1,r_3,\ldots,r_{n-2},0,0)\\
&\quad\otimes(0,1,r_3,\ldots,r_{n-2},0,0),\\
&\quad\cdots,\\
\widetilde{\omega}_{2^{n-1}+1} &= \widetilde{A_1}(0,0,\ldots,0,0,r_n)\otimes(1,0,\ldots,0,0,y).
\end{aligned} \quad (22)$$

The partial complex conjugations of all $(2^{n-1}+1)$ vectors are

$$\begin{aligned}
&\widetilde{\omega}_1^{\star 2} = A_1^{(1)}(1,r_2,\ldots,r_{n-2},r_{n-1},0)\otimes(1,r_2^\star,\ldots,r_{n-2}^\star,r_{n-1}^\star,0),\\
&\widetilde{\omega}_2^{\star 2} = A_1^{(1)}(1,r_2,\ldots,r_{n-2},0,0)\otimes(1,r_2^\star,\ldots,r_{n-2}^\star,0,0),\\
&\quad\cdots,\\
&\widetilde{\omega}_{2^{n-1}+1}^{\star 2} = \widetilde{A_1}(0,0,\ldots,0,0,r_n)\otimes(1,0,\ldots,0,0,y).
\end{aligned} \quad (23)$$

The above vectors cannot span the range of σ_α as they are orthogonal to the vector

$$\omega=(0,0,\ldots,0,0,1)\otimes(0,1,\ldots,1,1,0), \quad (24)$$

which belongs just to the range of σ_α. Hence for any $\alpha\neq 0,1$, the state σ_α is entangled as it violates the range criterion.

4. Applications of bound entanglement

To analyze bound entanglement, one needs as many examples of bound entangled states as possible. Hence there is a very exciting physical motivation for the search for bound entangled states. By the definition of bound entangled states, it cannot be distilled, and hence it is impossible to obtain faithful teleportation. Moreover, it was shown that no bound entangled states can offer better fidelity than classical communication.[20–22] It seems that bound entangled states are useless in quantum communication. But the existence of bound entanglement constituted a fundamental restriction on entanglement processing. Hence it seems important to develop an approach of combining bound entanglement and the entanglement measure. On the other hand, bound entanglement can produce a nonclassical effect called activation of bound entanglement.[23] The four-part unlockable bound entanglement were used for remote concentration of quantum information.[24] Yeo *et al.*[25] presented the teleportation fidelity of some states was not more than the classical scheme which fidelity is no more than 2/3. In Ref. [22], at least for some of the Horodecki density matrices, it is not possible to teleport spin-1 states with better fidelity than classical ones. In the following, we will prove that for some bound entanglement in $2\otimes 4$ systems, teleportation cannot be achieved with better fidelity than classical ones.

The most general scheme of teleportation would be of the following form. Initially Alice and Bob shared an entangled pair of particles, which is described by an arbitrary density matrix ρ in $2\otimes 4$ systems. The two particles, Alice's particle and Bob's particle, have obtained from some sources of entangled particles. A general expression of the density matrix ρ is

$$\rho = \frac{1}{8}I\otimes I+\frac{1}{6}r_i\sigma_i\otimes I+\frac{1}{6}I\otimes s_i\lambda_i+\frac{1}{4}t_{ij}\sigma_i\otimes\lambda_j, \quad (25)$$

where σ_i are the Pauli matrices and λ_i are the traceless Hermitian matrices generators of $SU(4)$, which satisfy $\mathrm{Tr}(\sigma_i)=\mathrm{Tr}(\lambda_i)=0$ and $\mathrm{Tr}(\sigma_i\sigma_j)=\mathrm{Tr}(\lambda_i\lambda_j)=2\delta_{ij}$, the coefficients are $r_i=\mathrm{Tr}(\rho\sigma_i\otimes I)$, $s_i=\mathrm{Tr}(\rho I\otimes\lambda_i)$, and $t_{ij}=\mathrm{Tr}(\rho\sigma_i\otimes\lambda_j)$.

Next, an unknown state $|\phi\rangle$ is given to Alice. We can write the state $|\phi\rangle$ as

$$|\phi\rangle = z_1|1\rangle+z_2|2\rangle+z_3|3\rangle+z_4|4\rangle,\quad z_i\in\mathscr{C},\quad i=1,2,3,4 \quad (26)$$

with respect to some bases $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ and the norm is $\||\phi\rangle\|=1$. A projection operator of the input state is $P_\phi=I/4+\alpha_i\lambda_i/2$. Then Alice performs a measurement P_k as follows:

$$P_k = \frac{1}{8}I\otimes I+\frac{1}{6}R_i^{(k)}\lambda_i\otimes I+\frac{1}{6}I\otimes S_i^{(k)}\sigma_i+\frac{1}{4}T_{ij}^{(k)}\lambda_i\otimes\sigma_j. \quad (27)$$

Via a classical communication channel, Alice sends the measurement result to Bob. Bob can restore the state $|\phi\rangle$ on his particle by performing the correct unitary transformation U_k.

The fidelity of transmission is

$$F = \int \mathrm{d}\mu(\phi)\sum_{k=1}^{8}\mathrm{Tr}_3(p_k\rho_k P_\phi)$$

$$= \int \mathrm{d}\mu(\phi) \sum_{k=1}^{8} \left(\frac{1}{32} + \frac{1}{18} r_q S_q^{(k)} + \frac{1}{24} \alpha_q R_q^{(k)} + \frac{1}{12} \alpha_p T_{pq}^{(k)} r_q + \frac{1}{24} s_q O_{qj}^{(k)} \alpha_j + \frac{1}{18} \alpha_i R_i^{(k)} s_q O_{qj}^{(k)} \alpha_j + \frac{1}{12} S_i^{(k)} t_{iq} O_{qj}^{(k)} \alpha_j + \frac{1}{8} \alpha_p T_{pi}^{(k)} t_{iq} O_{qj}^{(k)} \alpha_j \right). \quad (28)$$

Where the output state ρ_k is

$$\rho_k = \frac{1}{p_k} \mathrm{Tr}_{1,2}(P_k \otimes U_k)(P_\phi \otimes \rho)(P_k \otimes U_k^\dagger), \quad (29)$$

with the probability is

$$p_k = \mathrm{Tr}_{1,2,3}(P_k \otimes U_k)(P_\phi \otimes \rho)(P_k \otimes U_k^\dagger). \quad (30)$$

The orthogonal matrix $O^{(k)}$ is induced by conjugation using the unitary matrix U_k, that is $U^{(k)} x_j \lambda_j U^{(k)\dagger} = x_i O_{ij}^{(k)} \lambda_j$.

We may do integrals over α in the expression for the fidelity using

$$\int \mathrm{d}\alpha \ \alpha_i M_{ij} \alpha_j = \int \mathrm{d}\mu(\phi) \langle\phi|\lambda_i|\phi\rangle M_{ij} \langle\phi|\lambda_j|\phi\rangle = \mathrm{Tr}(M)/2$$

and $\int \mathrm{d}\alpha \ \alpha_i = \int d\mu(\phi) \langle\phi|\lambda_i|\phi\rangle = 0$ (M is an arbitrary density matrix). Then we derive

$$F = \sum_{k=1}^{8} \left(\frac{1}{32} + \frac{1}{18} r_q S_q^{(k)} + \frac{1}{36} R_p^{(k)} s_q O_{qp}^{(k)} + \frac{1}{16} T_{pi}^{(k)} t_{iq} O_{qp}^{(k)} \right), \quad (31)$$

where the projection operators $P^{\max}$, $R^{\max}$, $S^{\max}$, and $T^{\max}$ maximize the summand in Eq. (31). So we get

$$F \leq \frac{1}{4} + \frac{4}{9} r_q S_q^{\max} + \frac{2}{9} s_q R_q^{\max} + \frac{1}{2} t_{ip} T_{pi}^{\max} = \frac{1}{9} r_q S_q^{\max} - \frac{1}{9} s_q R_q^{\max} + 2\mathrm{Tr}(\rho \hat{P}^{\max}). \quad (32)$$

Actually,

$$\mathrm{Tr}(\rho \hat{P}^{\max}) = \frac{1}{8} + \frac{1}{6} r_i S_i^{\max} + \frac{1}{6} s_i R_i^{\max} + \frac{1}{4} t_{ij} T_{ji}^{\max},$$

where the projection operator $\hat{P}^{\max}$ satisfies that $\hat{P}^{\max} = N P^{\max} N$, and the operator N actions on basis vectors, $N e_i \otimes e_j = e_j \otimes e_i$.

Now, we consider the density matrix ρ in $2 \otimes 4$ systems which was introduced in Section 2

$$\rho = \frac{1}{7a+1} \begin{bmatrix} a & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1+a)/2 & 0 & 0 & \sqrt{1-a^2}/2 \\ a & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & a & 0 & \sqrt{1-a^2}/2 & 0 & 0 & (1+a)/2 \end{bmatrix}, \quad (33)$$

all the coefficients r_q for this matrix are zero except $r_3 = (a-1)/(7a+1)$, thus we can get

$$F \leq 2\mathrm{Tr}(\rho \hat{P}^{\max}), \quad (34)$$

choosing the projector $P^{\max}$ such that $\hat{P}^{\max}$ projects to the maximum eigenvalue of ρ.

Now, we derive the eigenvalues of ρ are 0 (it occurs with multiplicity three), $a/(7a+1)$, $2a/(7a+1)$ (it occurs with multiplicity two) and $1/(7a+1)(a+1/2 \pm \sqrt{2a^2-2a+1}/2)$. Thus, in the region $(3\sqrt{41}-1)/46 \ (\approx 0.3959) < a < 1$, the fidelity of this teleportation is less than $2/3$. Therefore, the bound entanglement ρ cannot be achieved with a better fidelity than classical ones.

5. Conclusion and discussion

In conclusion, we have constructed bound entangled states in $2 \otimes n$ systems and $n \otimes n$ systems. And we have also given a relevant example of bound entangled state in $2 \otimes 4$ systems to illuminate that the fidelity of the transmission of bound entanglement is not larger than a classical scheme. These results may be used in experimental entanglement detection and quantum information teleportation.

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