Eigen-spectra in the Dirac-attractive radial problem plus a tensor interaction under pseudospin and spin symmetry with the SUSY approach

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We approximately solve the Dirac equation for attractive radial potential including a Coulomb-like tensor interaction under pseudospin and the spin symmetry limit for any arbitrary spin–orbit quantum number, by employing the supersymmetric (SUSY) quantum mechanics and supersymmetric shape invariance technique. We obtain the energy eigenvalue equation under the pseudospin and spin conditions. Some numerical results are compared with those obtained by the Nikiforove–Uvarov (NU) method.

Keywords: Dirac equation, attractive radial, SUSY, Coulomb-like tensor potential

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1. Introduction

Relativistic symmetries of the Dirac Hamiltonian were discovered many years ago. However, these symmetries have been recently recognized empirically in nuclear and hadronic spectroscopies.[1] Spin symmetry is relevant to mesons,[2] and the pseudospin symmetry is used to explain features of deformed nuclei[3] and super-deformation[4] and to establish an effective shell-model coupling scheme.[5] The pseudospin symmetry refers to a quasi-degeneracy of single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n,l,j=l+1/2)$ and $(n-1,l+2,j=l+3/2)$, where n, l, and j denote the single-nucleon radial, orbital, and total angular momentum quantum numbers, respectively. This quasi-degenerate doublet structure can be expressed in terms of a pseudo-orbital angular momentum $\tilde{l}=l+1$ and a pseudospin angular momentum $\tilde{s}=1/2$. For example, $(3s_{1/2},2d_{3/2})$ can be dealt with pseudo-spin doublets $(2\tilde{p}_{1/2},2\tilde{p}_{3/2})$, where $j=\tilde{l}\pm\tilde{s}$ for the two states in the doublet. The spin symmetry appears when the magnitudes of scalar and vector potentials are nearly equal, i.e., $V(r)\cong S(r)$, in the nuclei (i.e., when the potential difference $\Delta(r)=V(r)-S(r)=\text{const}$). However the pseudospin symmetry occurs when $V(r)\cong -S(r)$ (i.e., when the sum potential $\Sigma(r)=V(r)+S(r)=\text{const}$). The bound states of nucleons seem to be sensitive to some mixtures of these potentials. The $\Delta(r)=0$ and $\Sigma(r)=0$ correspond to SU(2) symmetries of the Dirac Hamiltonian.[6] In recent years, there has been an increased interest in finding the exact solutions for non-relativistic or relativistic particles with various vector and scalar potentials.[7–30] These equations have been used for solving many problems of nuclear and high-energy physics.

On the one hand, concepts of pseudospin and spin symmetries and tensor potentials have been found to have interesting applications in the field of nuclear physics.[31–33] On the other hand, tensor potentials were introduced into the Dirac equation with the substitution $\boldsymbol{p}\to\boldsymbol{p}-\mathrm{i}m\omega\beta\hat{r}U(r)$.[34,35] In this way, a new spin–orbit coupling term is added to the Dirac Hamiltonian. In recent years, tensor couplings have been used widely in the studies of nuclear properties. In this regard, see Refs. [36]–[41].

The attractive radial potential, introduced by Williams and Poulios,[42] is given by

$$V(r)=\frac{V_1\,\mathrm{e}^{-4\alpha r}+V_2\,\mathrm{e}^{-2\alpha r}+V_3}{(1-\mathrm{e}^{-2\alpha r})^2}, \tag{1}$$

where $V_1=\alpha^2/4$, $V_2=[(A-8)\alpha^2]/4$, $V_3=[(4-A)\alpha^2]/4$, parameters α and A are real and $\alpha>1/2$, $4<A<8$. By using the concept of suppersymmetry quantum mechanics, Zou *et al.*[43] obtained the exact energy equation of the Dirac equation for s-wave bound states of the attractive radial potential. In Fig. 1, we plot the potentials for the values of two sets of parameters α and A. From Fig. 1, one can see that the above potential behaves like a Coulomb attractive potential.

Tensor Coulomb-like potential[44–46] is

$$U(r)=\frac{-H}{r},\quad H=\frac{Z_aZ_b\,\mathrm{e}^2}{4\pi\varepsilon_0},\quad r\geq R_c, \tag{2}$$

where $R_c=7.78$ fm ($1\ \text{fm}=10^{-15}$ m) is the Coulomb radius,

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Z_a and Z_b denote the charges of the projectile a and the target nuclei b, respectively. We should solve the Dirac equation for attractive radial potential including a Coulomb-like tensor coupling under the condition of pseudospin and spin symmetry. We obtain the energy eigenvalue equation and the corresponding spinor wave functions by using the SUSY shape invariance technique and with the help of choosing appropriate parameters in the six-parameter exponential-type potential (SPEP).

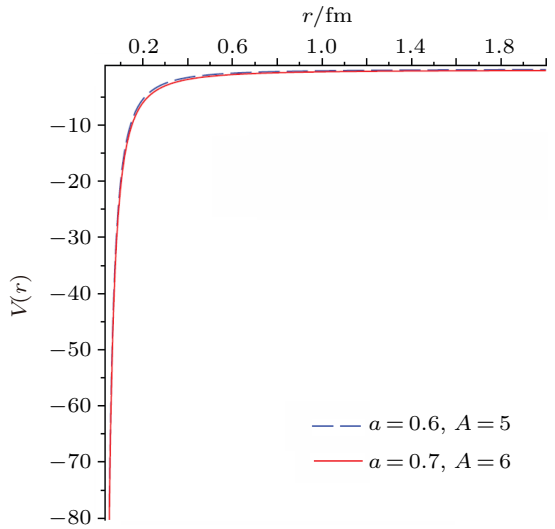


Fig. 1. (color online) Shapes of the attractive radial potentials.

2. Dirac equation including tensor interaction

According to Refs. [36]–[41], the Dirac equation with the scalar potential $S(r)$, the vector potential $V(r)$, and a tensor potential $U(r)$ reads ($\hbar = c = 1$),

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta(M + S(r)) - \mathrm{i}\beta\boldsymbol{\alpha}\cdot\hat{r}U(r)]\Psi_{n\kappa}(r) = [E_{n\kappa} - V(r)]\Psi_{n\kappa}(r), \tag{3}$$

where $\boldsymbol{p} = -\mathrm{i}\nabla$ is the three-dimensional momentum operator, $E_{n\kappa}$ is the relativistic energy of the system, M is the mass of the fermionic particle, and $\boldsymbol{\alpha}$ and β are the 4×4 Dirac matrices given as[47]

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{4}$$

where I is a 2×2 unitary matrix and $\boldsymbol{\sigma}$ are three-vector spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

For spherical nuclei, the total angular momentum $\boldsymbol{J}$ and spin matrix operator $\hat{K} = -\beta(\boldsymbol{\sigma}\cdot\boldsymbol{L} + 1)$ commute with the Dirac Hamiltonian. The eigenvalues of spin–orbit coupling operator $\hat{K}$ are $\kappa = (j + 1/2) > 0$ and $\kappa = -(j + 1/2) < 0$ for unaligned spin $j = l - 1/2$ and the aligned spin $j = l + 1/2$, respectively. In the relativistic quantum mechanics, the set (H^2, K, J^2, J_z) can be taken as the complete set of the conservative quantities. The spinor wave functions can be classified according to their angular momentum j, spin–orbit quantum number κ, and the radial quantum number n, and can be written as follows:

$$\Psi_{n\kappa}(\boldsymbol{r}) = \begin{pmatrix} f_{n\kappa}(\boldsymbol{r}) \\ g_{n\kappa}(\boldsymbol{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{n\kappa}(r)\mathrm{Y}^{l}_{jm}(\theta,\phi) \\ \mathrm{i}G_{n\kappa}(r)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi) \end{pmatrix}, \tag{6}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the radial wave functions of the upper and the lower spinor components of the Dirac spinors, respectively. $\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)$ and $\mathrm{Y}^{l}_{jm}(\theta,\phi)$ are the spherical harmonic functions, and m is the projection of the total angular momentum on the z axis. The pseudo-orbital and the orbital angular momentum quantum numbers for pseudospin symmetry $\tilde{l}$ and spin symmetry l, are denoted with the labels of upper and lower components, respectively. Substituting Eq. (6) into Eq. (3) and using the following relations:[48]

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B} + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}), \tag{7}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{P}) = \boldsymbol{\sigma}\cdot\hat{r}\left(\hat{r}\boldsymbol{P} + \mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}\right), \tag{8}$$

with the following properties:

$$\boldsymbol{\sigma}\cdot\boldsymbol{L}\begin{Bmatrix}\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi) \\ \mathrm{Y}^{l}_{jm}(\theta,\phi)\end{Bmatrix} = \begin{cases}(\kappa-1)\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi), \\ -(\kappa-1)\mathrm{Y}^{l}_{jm}(\theta,\phi),\end{cases} \tag{9}$$

$$\boldsymbol{\sigma}\cdot\hat{r}\begin{Bmatrix}\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi) \\ \mathrm{Y}^{l}_{jm}(\theta,\varphi)\end{Bmatrix} = \begin{cases}-\mathrm{Y}^{l}_{jm}(\theta,\varphi), \\ -\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\varphi),\end{cases} \tag{10}$$

one obtains two coupled differential equations for the upper and lower radial wave functions, $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right]F_{n\kappa}(r) = [M + E_{n\kappa} - \Delta(r)]G_{n\kappa}(r), \tag{11a}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\right]G_{n\kappa}(r) = [M - E_{n\kappa} + \Sigma(r)]F_{n\kappa}(r), \tag{11b}$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$.

By eliminating $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ from Eqs. (11a) and (11b), we obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components, respectively,

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{\mathrm{d}U(r)}{\mathrm{d}r} - U^2(r)\right. \\ &\left. + \frac{\mathrm{d}\Delta(r)/\mathrm{d}r}{E_{n\kappa} + M - \Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right)\right]F_{n\kappa}(r) \\ &= [M + E_{n\kappa} - \Delta(r)][M - E_{n\kappa} + \Sigma(r)]F_{n\kappa}(r),\end{aligned} \tag{12a}$$

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{\mathrm{d}U(r)}{\mathrm{d}r} - U^2(r)\right. \\ &\left. - \frac{\mathrm{d}\Sigma(r)/\mathrm{d}r}{M - E_{n\kappa} + \Sigma(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\right)\right]G_{n\kappa}(r) \\ &= [M + E_{n\kappa} - \Delta(r)][M - E_{n\kappa} + \Sigma(r)]G_{n\kappa}(r),\end{aligned} \tag{12b}$$

where $\kappa(\kappa+1) = l(l+1)$ and $\kappa(\kappa-1) = \tilde{l}(\tilde{l}+1)$. The quantum number κ is related to the quantum numbers for pseudospin symmetry $\tilde{l}$ and spin symmetry l as follows:

$$\kappa = \begin{cases} -(l+1) = -\left(j+\frac{1}{2}\right)(\mathrm{s}_{1/2},\mathrm{p}_{3/2},\mathrm{etc.}), & j = l+\frac{1}{2}, \quad \text{aligned spin} \quad (\kappa<0), \\ +l = +\left(j+\frac{1}{2}\right)(\mathrm{p}_{1/2},\mathrm{d}_{3/2},\mathrm{etc.}), & j = l-\frac{1}{2} \quad \text{unaligned spin} \quad (\kappa>0), \end{cases}$$

and the quasi-degenerate doublet structure can be expressed in terms of a pseudospin angular momentum $\tilde{s} = 1/2$ and pseudo-orbital angular momentum $\tilde{l}$, which is defined as

$$\kappa = \begin{cases} -\tilde{l} = -\left(j+\frac{1}{2}\right)(\mathrm{s}_{1/2},\mathrm{p}_{3/2},\mathrm{etc.}), & j = \tilde{l}-\frac{1}{2}, \quad \text{aligned pseudospin} \quad (\kappa<0), \\ +(\tilde{l}+1) = +\left(j+\frac{1}{2}\right)(\mathrm{d}_{3/2},\mathrm{f}_{5/2},\mathrm{etc.}), & j = \tilde{l}+\frac{1}{2}, \quad \text{unaligned pseudospin} \quad (\kappa>0), \end{cases}$$

where $\kappa = \pm1, \pm2, \ldots$. For example, $(1\mathrm{s}_{1/2}, 0\mathrm{d}_{3/2})$ and $(1\mathrm{p}_{3/2}, 0\mathrm{f}_{5/2})$ can be considered as pseudospin doublets.

2.1. Pseudospin symmetry limit

The condition of pseudospin symmetry (ps) is $\mathrm{d}\Sigma/\mathrm{d}r = 0$ or $\Sigma = C_{\mathrm{ps}} = \text{constant}$,[49,50] then equation (12b) with $\Delta(r)$ as an attractive radial potential and a Coulomb-like tensor potential becomes

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\kappa+H)(\kappa+H-1)}{r^2} + (E_{n\kappa} - M - C_{\mathrm{ps}})\frac{V_1\mathrm{e}^{-4\alpha r} + V_2\mathrm{e}^{-2\alpha r} + V_3}{(1-\mathrm{e}^{-2\alpha r})^2}\right] G_{n\kappa}(r) = [E_{n\kappa}^2 - M^2 - C_{\mathrm{ps}}(E_{n\kappa}+M)]G_{n\kappa}(r), \tag{13}$$

where $\kappa = -\tilde{l}$ and $\kappa = \tilde{l}+1$ for $\kappa < 0$ and $\kappa > 0$, respectively.

2.2. Spin symmetry limit

The condition of spin symmetry is $\mathrm{d}\Delta/\mathrm{d}r = 0$ or $\Delta = C_{\mathrm{s}} = \text{constant}$, then Eq. (12a) with $\Sigma(r)$ as an attractive radial potential and a Coulomb-like tensor potential becomes

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(\kappa+H)(\kappa+H+1)}{r^2} + (E_{n\kappa} + M - C_{\mathrm{s}})\frac{V_1\mathrm{e}^{-4\alpha r} + V_2\mathrm{e}^{-2\alpha r} + V_3}{(1-\mathrm{e}^{-2\alpha r})^2}\right] F_{n\kappa}(r) = E_{n\kappa}^2 - M^2 - C_{\mathrm{s}}(E_{n\kappa} - M)]F_{n\kappa}(r), \tag{14}$$

where $\kappa = l$ and $\kappa = -l-1$ for $\kappa > 0$ and $\kappa < 0$ respectively.

Equations (13) and (14) cannot be solved analytically due to the pseudo-centrifugal term $1/r^2$. Thus, we use the approximation scheme suggested by Greene and Aldrich[51] as

$$\frac{1}{r^2} \approx 4\alpha^2\left[C_0 + \frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}\right], \tag{15}$$

where the parameter $C_0 = 1/12$ is dimensionless constant.

2.3. Solution of Dirac equation with the pseudospin symmetry limit

Inserting Eq. (15) into Eq. (13) and defining

$$\tilde{E}_{n\kappa} = E_{n\kappa}^2 - M^2 - C_{\mathrm{ps}}[E_{n\kappa} + M]$$

$$-4\alpha^2 C_0(\kappa+H)(\kappa+H-1), \tag{16a}$$

$$\tilde{V}_1 = [E_{n\kappa} - M - C_{\mathrm{ps}}]V_1, \tag{16b}$$

$$\tilde{V}_2 = [E_{n\kappa} - M - C_{\mathrm{ps}}]V_2 + 4\alpha^2(\kappa+H)(\kappa+H-1), \tag{16c}$$

$$\tilde{V}_3 = [E_{n\kappa} - M - C_{\mathrm{ps}}]V_3, \tag{16d}$$

we obtain the following Schrödinger-like equation for the lower wave function $G_{n\kappa}(r)$ as

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \tilde{V}_1\frac{\mathrm{e}^{-4\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} + \tilde{V}_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} + \tilde{V}_3\frac{1}{(1-\mathrm{e}^{-2\alpha r})^2}\right] G_{n\kappa}(r) = \tilde{E}_{n\kappa}G_{n\kappa}(r). \tag{17}$$

By using the basic concept of supersymmetry shape invariance technique,[52,53] we can solve Eq. (17). The ground state lower component $G_{0,\kappa}(r)$ can be written as

$$G_{0,\kappa}(r) = \exp\left(-\int W(r)\,\mathrm{d}r\right), \tag{18}$$

where $W(r)$ is called superpotential in SUSY quantum mechanics. Inserting Eq. (18) into Eq. (17) leads to the following equation for $W(r)$

$$W^2(r) - \frac{\mathrm{d}W(r)}{\mathrm{d}r} = \tilde{V}_1\frac{\mathrm{e}^{-4\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} + \tilde{V}_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} + \tilde{V}_3\frac{1}{(1-\mathrm{e}^{-2\alpha r})^2} - \tilde{E}_{0,\kappa}, \tag{19}$$

where $\tilde{E}_{0,\kappa}$ is the ground-state energy. Equation (19) is a non-linear Riccati equation. By choosing the appropriate parameters $P_i = (i = 1, \ldots, 4)$, α and q in the six-parameter exponential type potential,[54] we can solve Eq. (19). If we make the replacements $P_1 = \tilde{V}_1$, $P_2 = \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3$, $P_3 = 2\tilde{V}_1 + \tilde{V}_2$, $P_4 = P_5 = 0$, $\alpha = -\alpha$, and $q = -1$, the SPEP will yield the attractive radial potential, introduced by Williams and Poulios,[42] so we write the superpotential in the following form

$$W(r) = -Q_1 + Q_2\frac{1}{(1-\mathrm{e}^{-2\alpha r})}. \tag{20}$$

Substituting Eq. (20) into Eq. (18) leads to the ground-state lower component $G_{0,\kappa}(r)$,

$$G_{0,\kappa}(r) = N\mathrm{e}^{Q_1 r}\left(\frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right)^{+Q_2/2\alpha}. \tag{21}$$

By inserting $W(r)$ into Eq. (19) and comparing the equal powers on two sides, we obtain the following relations:

$$Q_1^2 = \tilde{V}_1 - \tilde{E}_{0,\kappa}, \tag{22a}$$

$$Q_2^2 + 2\alpha Q_2 = \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3, \tag{22b}$$

$$2Q_1Q_2 + 2\alpha Q_2 = 2\tilde{V}_1 + \tilde{V}_2. \tag{22c}$$

For a central potential, the radial part of the wave function $\Psi_{n\kappa}(r)$ must satisfy the regularity conditions that $G_{n,\kappa}(r)/r$ becomes zero when $r \to \infty$ and $G_{n,\kappa}(r)/r$ is finite if $r = 0$. It is clear that when $r \to \infty$, $G_{0,\kappa}(r)$ is finite and $G_{0,\kappa}(r) = 0$ at the origin point $r = 0$. From Eq. (21) we know that the ground state lower component $G_{0,\kappa}(r)$ can satisfy the boundary conditions in the case of $Q_2 < 0$. By using these restriction conditions and solving Eqs. (22b) and (22c), we obtain Q_1 and Q_2 as follows:

$$Q_1 = \frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2} + \frac{Q_2}{2}, \tag{23}$$

$$Q_2 = -\alpha\left[1 + \sqrt{1 + \frac{\tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3}{\alpha^2}}\right]. \tag{24}$$

Putting Eq. (23) into Eq. (20) and Eq. (22a) leads to $W(r)$ and $\tilde{E}_{0,\kappa}$,

$$W(r) = \frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2} + \frac{Q_2}{2} + Q_2\frac{1}{1-\mathrm{e}^{-2\alpha r}}, \tag{25}$$

$$\tilde{E}_{0,\kappa} = \tilde{V}_1 - \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2} + \frac{Q_2}{2}\right]^2. \tag{26}$$

Now, we construct these two supersymmetric partner potentials $V_{\pm}(r)$ by using the expression given in Eq. (25),

$$\begin{aligned} V_+(r) = W^2(r) + \frac{\mathrm{d}W(r)}{\mathrm{d}r} &= \frac{Q_2^2}{4} + \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2}\right]^2 \\ &+ Q_2^2\frac{1}{(1-\mathrm{e}^{-2\alpha r})^2} - 2\alpha Q_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} \\ &- \left[Q_2^2 + \tilde{V}_1 - \tilde{V}_3\right]\frac{1}{1-\mathrm{e}^{-2\alpha r}} + \frac{\tilde{V}_1 - \tilde{V}_3}{2}, \end{aligned} \tag{27}$$

$$\begin{aligned} V_-(r) = W^2(r) - \frac{\mathrm{d}W(r)}{\mathrm{d}r} &= \frac{Q_2^2}{4} + \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2}\right]^2 \\ &+ Q_2^2\frac{1}{(1-\mathrm{e}^{-2\alpha r})^2} + 2\alpha Q_2\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} \\ &- \left[Q_2^2 + \tilde{V}_1 - \tilde{V}_3\right]\frac{1}{1-\mathrm{e}^{-2\alpha r}} + \frac{\tilde{V}_1 - \tilde{V}_3}{2}. \end{aligned} \tag{28}$$

Considering $a_0 = Q_2$ and $a_1 = Q_2 - 2\alpha$, we can easily show that the two partner potentials $V_+(r,a_0)$ and $V_-(r,a_1)$ satisfy the following relationship

$$V_+(r,a_0) = V_-(r,a_1) + R(a_1), \tag{29}$$

and it shows that $V_+(r,a_0)$ and $V_-(r,a_1)$ can be determined by using the shape invariance approach,

$$\begin{aligned} R(a_1) &= \frac{Q_2^2}{4} - \frac{(Q_2-2\alpha)^2}{4} + \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2}\right]^2 - \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2\alpha)}\right]^2 \\ &= \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2} + \frac{Q_2}{2}\right]^2 - \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2\alpha)} + \frac{Q_2-2\alpha}{2}\right]^2. \end{aligned} \tag{30}$$

The remainder $R(a_1)$ is independent of r. The energy eigenvalues of the shape invariance potential $V_-(r)$ are given by

$$\tilde{E}_{0,\kappa}^{(-)} = 0, \tag{31}$$

$$\begin{aligned} \tilde{E}_{n,\kappa}^{(-)} &= \sum_{\kappa=1}^{n} R(a_\kappa) = R(a_1) + R(a_2) + \cdots + R(a_n) \\ &= \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2} + \frac{Q_2}{2}\right]^2 - \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2\alpha)} + \frac{Q_2-2\alpha}{2}\right]^2 \\ &+ \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2\alpha)} + \frac{Q_2-2\alpha}{2}\right]^2 \\ &- \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2\times 2\alpha)} + \frac{Q_2-2\times 2\alpha}{2}\right]^2 + \cdots \\ &+ \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2(n-1)\alpha)} + \frac{Q_2-2(n-1)\alpha}{2}\right]^2 \\ &- \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2n\alpha)} + \frac{Q_2-2n\alpha}{2}\right]^2 \\ &= \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2} + \frac{Q_2}{2}\right]^2 - \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2n\alpha)} + \frac{Q_2-2n\alpha}{2}\right]^2, \end{aligned} \tag{32}$$

where quantum number $n = 1,2,3,\ldots$.

Incorporating Eqs. (19), (28), and (32), we can find the solution for $\tilde{E}_{n,\kappa}$ in Eq. (17) as follows:

$$\begin{aligned} \tilde{E}_{n,\kappa} &= \tilde{E}_{n,\kappa}^{(-)} + \tilde{E}_{0,\kappa} \\ &= \tilde{V}_1 - \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2(Q_2-2n\alpha)} + \frac{Q_2-2n\alpha}{2}\right]^2, \quad n = 0,1,2,\ldots, \end{aligned} \tag{33}$$

where we have employed

$$\tilde{E}_{0,\kappa} = \tilde{V}_1 - \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2} + \frac{Q_2}{2}\right]^2.$$

The effective potential in the Schrödinger-like equation (17) can be written as

$$\begin{aligned} V(r) &= \frac{\tilde{V}_1\mathrm{e}^{-4\alpha r} + \tilde{V}_2\mathrm{e}^{-2\alpha r} + \tilde{V}_3}{(1-\mathrm{e}^{-2\alpha r})^2} \\ &= V_-(r,a_0) + \tilde{V}_1 - \left[\frac{\tilde{V}_1 - \tilde{V}_3}{2Q_2} + \frac{Q_2}{2}\right]^2. \end{aligned} \tag{34}$$

By substituting Eq. (24) into Eq. (23) and using Eq. (16a), the energy eigenvalue equation of the attractive radial potential introduced by Williams and Poulios[42] are obtained under the condition of pseudospin symmetry,

$$E_{n,\kappa}^2 - M^2 - C_{\rm ps}[E_{n,\kappa}+M] - 4\alpha^2 C_0(\kappa+H)(\kappa+H-1)$$
$$= -\left[\frac{(E_{n,\kappa}-M-C_{\rm ps})^2(V_1-V_3)^2}{4\left[-\alpha\left(1+\sqrt{1+4(\kappa+H)(\kappa+H-1)+(E_{n,\kappa}-M-C_{\rm ps})(V_1+V_2+V_3)^2/\alpha^2}\right)-2n\alpha\right]^2}\right]$$
$$-\frac{1}{4}\left[-\alpha\left(1+\sqrt{1+4(\kappa+H)(\kappa+H-1)+\frac{(E_{n,\kappa}-M-C_{\rm ps})(V_1+V_2+V_3)^2}{\alpha^2}}\right)-2n\alpha\right]^2$$
$$+(E_{n,\kappa}-M-C_{\rm ps})V_1-\frac{1}{2}(E_{n,\kappa}-M-C_{\rm ps})(V_1-V_3), \tag{35}$$

where the quantum number $n = 0, 1, 2, \ldots$.

For obtaining the energy eigenfunctions, we define a new variable of the form $x = {\rm e}^{-2\alpha r}$, then the lower spinor radial wave functions $G_{n\kappa}(x)$ can be written as[55]

$$G_{n\kappa}(x) = B_n x^{-A}(1-x)^{(1+D)/2} \times P_n^{(-2A,D)}(1-2x), \tag{36}$$

where B_n is the normalized constant and A and D are

$$A = \left[(\kappa+H)(\kappa+H-1)C_0 + \frac{(E_{n,\kappa}-M-C_{\rm ps})V_3}{4\alpha^2} + \frac{(M+E_{n,\kappa})(M-E_{n,\kappa}+C_{\rm ps})}{4\alpha^2}\right]^{1/2},$$
$$D = \left[(2\kappa+2H-1)^2 + \frac{(E_{n,\kappa}-M-C_{\rm ps})}{\alpha^2}(V_1+V_2+V_3)\right]^{1/2}. \tag{37}$$

The upper component of the Dirac equation can be calculated by applying Eq. (11b) as

$$F_{n,\kappa}(r) = \frac{1}{[M-E_{n,\kappa}+C_{\rm ps}]}\left(\frac{{\rm d}}{{\rm d}r} - \frac{\kappa+H}{r}\right)G_{n\kappa}(r), \tag{38}$$

where $E_{n,\kappa} \neq M + C_{\rm ps}$.

2.4. Solution of the Dirac equation with the spin symmetry limit

To avoid repetition in the solution of Eq. (14), we follow the same procedures explained in Subsection 2.3. and hence obtain the following energy eigenvalue equation corresponding to the spin symmetry condition:

$$E_{n,\kappa}^2 - M^2 - C_{\rm s}[E_{n,\kappa}-M] - 4\alpha^2 C_0(\kappa+H)(\kappa+H+1)$$
$$= -\left[\frac{(E_{n,\kappa}+M-C_{\rm s})^2(V_1-V_3)^2}{4\left[-\alpha\left(1+\sqrt{1+4(\kappa+H)(\kappa+H+1)+(E_{n,\kappa}+M-C_{\rm s})(V_1+V_2+V_3)^2/\alpha^2}\right)-2n\alpha\right]^2}\right]$$
$$-\frac{1}{4}\left[-\alpha\left(1+\sqrt{1+4(\kappa+H)(\kappa+H+1)+(E_{n,\kappa}+M-C_{\rm s})(V_1+V_2+V_3)^2/\alpha^2}\right)-2n\alpha\right]^2$$
$$+(E_{n,\kappa}+M-C_{\rm s})V_1-\frac{1}{2}(E_{n,\kappa}+M-C_{\rm s})(V_1-V_3). \tag{39}$$

Further, we can easily obtain the upper spinor radial wave functions $F_{n\kappa}(x)$ directly via the pseudo-spin symmetry solution through the parametric mappings

$$G_{n\kappa} \leftrightarrow F_{n\kappa}, \quad E_{n,\kappa} \leftrightarrow -E_{n,\kappa},$$
$$C_{\rm ps} \leftrightarrow -C_{\rm s}, \quad V(r) \to -V(r), \quad \kappa \to \kappa+1, \tag{40}$$

and the corresponding wave functions in spin symmetry limit are obtained as

$$F_{n\kappa}(x) = N_n x^{A'}(1-x)^{1(1+D')/2} \times P_n^{(2A',D')}(1-2x), \tag{41}$$

where N_n is the normalized constant and A' and D' are

$$A' = \left[(\kappa+H+1)(\kappa+H)C_0 + \frac{(M+E_{n,\kappa}-C_{\rm s})V_3}{4\alpha^2} + \frac{(M-E_{n,\kappa})(M+E_{n,\kappa}-C_{\rm s})}{4\alpha^2}\right]^{1/2},$$
$$D' = \left[(2\kappa+2H+1)^2 + \frac{M+E_{n,\kappa}-C_{\rm s}}{\alpha^2}(V_1+V_2+V_3)\right]^{1/2}. \tag{42}$$

The lower component of the Dirac equation can be calculated by using Eq. (11a) as

$$G_{n\kappa}(r) = \frac{1}{M+E_{n,\kappa}-C_{\rm s}}\left(\frac{{\rm d}}{{\rm d}r} + \frac{\kappa-H}{r}\right)F_{n\kappa}(r), \tag{43}$$

where $E_{n,\kappa} \neq -M + C_{\rm s}$.

3. Numerical results

In Figs. 2 and 3 we plot the relativistic energy eigenvalues under the pseudospin and spin symmetry limitations versus $\alpha = [0.51\text{–}0.63]$ for several states. In these figures the lines with dots denote the states of $\kappa < 0$ and the lines with circles represent the states of $\kappa > 0$. It is seen that when α increases, the negative energies increase. Also, in these figures we investigate the effects of the α parameter on the bound state under the conditions of the pseudospin and spin symmetries limits for $H = 0$. It is readily seen that if α parameter increases, the values of the bound state negative energy eigenvalues of this potential increase for several states and the values of the bound state positive energy eigenvalues of this potential decrease for several states.

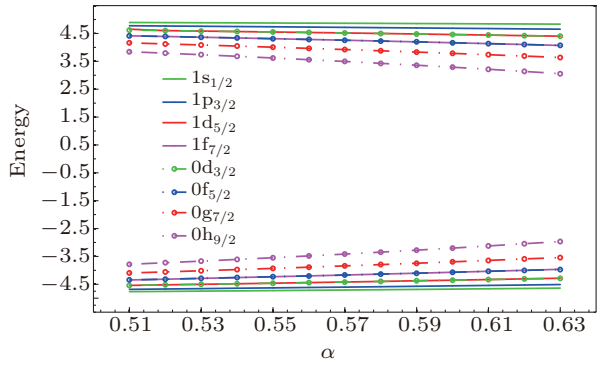


Fig. 2. (color online) Energy eigenvalues for pseudospin symmetry limit for several values of n and κ with $A = 5$, $M = 5$ fm^{-1}, $\alpha = [0.51-0.63]$, and $C_{\rm ps} = 0$.

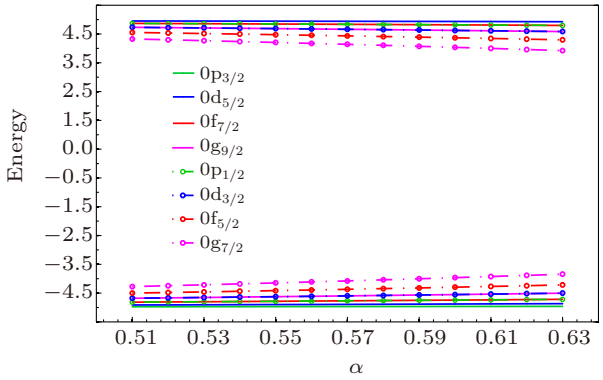


Fig. 3. (color online) Energy eigenvalues of spin symmetry limit for several values of n and κ with $A = 5$, $M = 5$ fm^{-1}, $\alpha = [0.51$–$0.63]$, and $C_{\rm s} = 0$.

In addition, in Tables 1 and 2 the results are obtained by using the following parameters: $\alpha = 0.6$, $A = 5$, $M = 5$ fm^{-1}, and $C_{\rm ps}$, $C_{\rm s} = 0$. We consider the same set of pseudospin doublets as $(np_{1/2}, np_{3/2})$, $(nd_{3/2}, nd_{5/2})$, $(nf_{5/2}, nf_{7/2})$, $(ng_{7/2}, ng_{9/2}), \ldots$ (where each pair is considered as a pseudospin doublet) and the same set of spin doublets as $(ns_{1/2}, (n-1)d_{3/2})$, $(np_{3/2}, (n-1)f_{5/2})$, $(nd_{5/2}, (n-1)g_{7/2}) \ldots$, (where each pair is considered as a spin doublet). We see that the energies of the bound states in the absence of the tensor interaction, i.e., when $H = 0$, are degenerate but in the presence of the tensor interaction, i.e., when $H = 1$, the degeneracies are removed. For example, in pseudospin doublet $(1s_{1/2}, 0d_{3/2})$ when $H = 0$, $E_{1,-1} = E_{1,2} = -4.556531$ but when $H = 1$, $E_{1,-1} = -4.672750$ with $\kappa < 0$ and $E_{1,2} = -4.352819$ with $\kappa > 0$. In conclusion, the term $2\kappa H$ is responsible for this pseudospin doublet splitting.

In Tables 1 and 2 it is shown that the positive energy eigenvalues increase by increasing the tensor strength H for $\kappa < 0$ and the negative energy eigenvalues decrease by increasing the tensor strength H for $\kappa < 0$. Also, the positive energy eigenvalues decrease by increasing the tensor strength H for $\kappa > 0$ and the negative energy eigenvalues increase by increasing the tensor strength H for $\kappa > 0$. Furthermore, the obtained results for the Williams and Poulios potential are compared with the results obtained through the Nikiforove–Uvarov (NU) method.

Table 1. Bound state energy eigenvalues of the attractive radial potential under the pseudospin symmetry limit in comparison with the corresponding results of Ref. [36] for several values of n and κ with $A = 5$, $M = 5$ fm^{-1}, $\alpha = 0.6$, and $C_{\rm ps} = 0$.

$\tilde{l}$	$n, \kappa < 0$	(l, j)	$E_{n,\kappa<0}$				$n-1, \kappa > 0$	$(l+2, j+1)$	$E_{n,\kappa>0}$			
			$H = 0$	$H = 1$	$H = 0$ Ref. [36]	$H = 1$ Ref. [36]			$H = 0$	$H = 1$	$H = 0$ Ref. [36]	$H = 1$ Ref. [36]
1	1, −1	$1s_{1/2}$	−4.556531	−4.672750	−4.556531	−4.672750	0, 2	$0d_{3/2}$	−4.556531	−4.352819	−4.556531	−4.352818
			4.685901	4.849765	4.685901	4.849764			4.685901	4.461142	4.685901	4.461142
2	1, −2	$1p_{3/2}$	−4.352819	−4.556531	−4.352818	−4.556531	0, 3	$0f_{5/2}$	−4.352819	−4.069045	−4.352818	−4.069044
			4.461142	4.685901	4.461142	4.685901			4.461142	4.166479	4.461142	4.166478
3	1, −3	$1d_{5/2}$	−4.069045	−4.352819	−4.069044	−4.352818	0, 4	$0g_{7/2}$	−4.069045	−3.694608	−4.069044	−3.694608
			4.166479	4.461142	4.166478	4.461142			4.166479	3.785571	4.166478	3.785571
4	1, −4	$1f_{7/2}$	−3.694608	−4.069045	−3.694608	−4.069044	0, 5	$0h_{9/2}$	−3.694608	−3.201540	−3.694608	−3.201540
			3.785571	4.166479	3.785571	4.166478			3.785571	3.288266	3.785571	3.288266
1	2, −1	$2s_{1/2}$	−4.235693	−4.408844	−4.235693	−4.408843	1, 2	$1d_{3/2}$	−4.235693	−3.941172	−4.235693	−3.941172
			4.401472	4.650286	4.401472	4.650285			4.401472	4.073719	4.401472	4.073719
2	2, −2	$2p_{3/2}$	−3.941172	−4.235693	−3.941172	−4.235693	1, 3	$1f_{5/2}$	−3.941172	−3.534749	−3.941172	−3.534748
			4.073719	4.401472	4.073719	4.401472			4.073719	3.650186	4.073719	3.650186
3	2, −3	$2d_{5/2}$	−3.534749	−3.941172	−3.534748	−3.941172	1, 4	$1g_{7/2}$	−3.534749	−2.986596	−3.534748	−2.986596
			3.650186	4.073719	3.650186	4.073719			3.650186	3.091851	3.650186	3.091851
4	2, −4	$2f_{7/2}$	−2.986596	−3.534749	−2.986596	−3.534748	1, 5	$1h_{9/2}$	−2.986596	−2.202462	−2.986596	−2.202461
			3.091851	3.650186	3.091851	3.650186			3.091851	2.301026	3.091851	2.301025

Table 2. Bound state energy eigenvalues of the attractive radial potential under the spin symmetry limit in comparison with the corresponding results of Ref. [36] for several values of n and κ with $A = 5$, $M = 5$ fm^{-1}, $\alpha = 0.6$ and $C_s = 0$.

l	$n,\kappa<0$	$(l, j = l+1/2)$	$E_{n,\kappa<0}$				$n,\kappa>0$	$(l, j = l-1/2)$	$E_{n,\kappa>0}$			
			$H=0$	$H=1$	$H=0$ Ref. [36]	$H=1$ Ref. [36]			$H=0$	$H=1$	$H=0$ Ref. [36]	$H=1$ Ref. [36]
1	0, −2	$0p_{3/2}$	−4.879633	−4.964355	−4.880113	−4.964565	0, 1	$0p_{1/2}$	−4.876933	−4.743495	−4.880113	−4.744442
			4.934150	—	—	—			4.934150	4.817315	—	—
2	0, −3	$0d_{5/2}$	−4.743495	−4.879633	−4.744442	−4.880113	0, 2	$0d_{3/2}$	−4.743495	−4.551324	−4.744442	−4.552969
			4.817315	4.934150	4.843292	—			4.817315	4.624799	4.843292	4.655853
3	0, −4	$0f_{7/2}$	−4.551324	−4.743495	−4.552969	−4.744442	0, 3	$0f_{5/2}$	−4.551324	−4.295663	−4.552969	−4.298321
			4.624799	4.817315	4.655853	4.843292			4.624799	4.368386	4.655853	4.401391
4	0, −5	$0g_{9/2}$	−4.295663	−4.551324	−4.298321	−4.552969	0, 4	$0g_{7/2}$	−4.295663	−3.964354	−4.298321	−3.968507
			4.368386	4.624799	4.401391	4.655853			4.368386	4.036429	4.401391	4.071212
1	1, −2	$1p_{3/2}$	−4.694813	−4.857895	−4.696712	−4.859649	1, 1	$1p_{1/2}$	-4.694813	-4.474013	−4.696712	−4.476756
			4.845470	—	—	—			4.845470	4.581261	—	—
2	1, −3	$1d_{5/2}$	−4.474013	−4.694813	−4.476756	−4.696712	1, 2	$1d_{3/2}$	−4.474013	−4.185506	−4.476756	−4.189555
			4.581261	4.845470	4.630310	—			4.581261	4.280613	4.630310	4.325470
3	1, −4	$1f_{7/2}$	−4.185506	−4.474013	−4.189555	−4.476756	1, 3	$1f_{5/2}$	−4.185506	−3.813999	−4.189555	−3.820028
			4.280613	4.581261	4.325470	4.630310			4.280613	3.902866	4.325470	3.947182
4	1, −5	$1g_{9/2}$	−3.813999	−4.185506	−3.820028	−4.189555	1, 4	$1g_{7/2}$	−3.813999	−3.332123	−3.820028	−3.341360
			3.902866	4.280613	3.947182	4.325470			3.902866	3.417119	3.947182	3.463125

4. Conclusion

In this paper we have obtained the exact analytical solutions of the Dirac equation for the attractive radial potential introduced by Williams and Poulios, including the tensor interaction. We investigate this problem in the framework of pseudospin and spin symmetries by using the SUSY. The obtained results for this potential are compared with the results obtained through the NU method. Obviously, the degeneracy between the member of doublet states in pseudospin symmetry and that in spin symmetry is removed by using the tensor interaction. Some numerical results are obtained and the effects of the tensor interaction and α are presented, too.

References

[1] Ginocchio J N 1997 *Phys. Rev. Lett.* **78** 436
[2] Page P R, Goldman T and Ginocchio J N 2001 *Phys. Rev. Lett.* **86** 204
[3] Bohr A, Hamamoto I and Mottelson B R 1982 *Phys. Scr.* **26** 267
[4] Dudek J, Nazarewicz W, Szymanski Z and Leander G A 1987 *Phys. Rev. Lett.* **59** 1405
[5] Troltenier D, Bahri C and Draayer G A 1995 *Nucl. Phys. A* **586** 53
[6] Ginocchio J N 2005 *Phys. Rep.* **414** 165
[7] Zarrinkamar S, Rajabi A A and Hassanabadi H 2010 *Ann. Phys.* **325** 2522
[8] Wei G F and Dong S H 2009 *Europhys. Lett.* **87** 40004
[9] Ikhdair S M and Sever R 2010 *Appl. Math. Comput.* **216** 545
[10] Maghsoodi E, Hassanabadi H and Zarrinkamar S 2012 *Few-Body Syst.* **53** 525
[11] Zhang L H, Li X P and Jia C S 2008 *Phys. Lett. A* **372** 2201
[12] Zhang M C, Sun G H and Dong S H 2010 *Phys. Lett. A* **374** 704
[13] Arda A and Sever R 2012 *J. Math. Chem.* **50** 1484
[14] Dong S H 2009 *Int. J. Quantum Chem.* **109** 701
[15] Sun G H and Dong S H 2012 *Commun. Theor. Phys.* **58** 195
[16] Dong S H, Chen C Y and Lozada-Cassou M 2005 *Int. J. Quantum Chem.* **105** 453
[17] Eshghi M 2012 *Chin. Phys. Lett.* **29** 110304
[18] Eshghi M 2013 *Canad. J. Phys.* **91** 71
[19] Eshghi M, Hamzavi M and Ikhdair S M 2012 *Adv. High Ener. Phys.* **87** 3619
[20] Qiang W C, Sun G H and Dong S H 2012 *Ann. Phys.* (Berlin) **524** 360
[21] Jia C S, Chen T and Gui L G 2009 *Phys. Lett. A* **373** 1621
[22] Zhang F, Wang Y and Guo J Y 2009 *Commun. Theor. Phys.* **52** 813
[23] Setare M R and Haidari S 2009 *Int. J. Theor. Phys.* **48** 3249
[24] Guo J Y, Meng J and Xu F X 2003 *Chin. Phys. Lett.* **20** 602
[25] Jia C S, Liu J Y, Wang P Q and Lin X 2009 *Int. J. Theor. Phys.* **48** 2633
[26] Wei G F and Dong S H 2010 *Phys. Lett. B* **686** 288
[27] Guo J Y and Sheng Z Q 2005 *Phys. Lett. A* **338** 90
[28] Samsonov B F and Pecheritsyn A A 2002 *Russ. Phys. J.* **45** 13
[29] Zhang X C, Liu Q W, Jia C S and Wang L Z 2005 *Phys. Lett. A* **340** 59
[30] Alhaidari A D 2001 *Phys. Rev. Lett.* **87** 210405
[31] Gincchio J N and Leviatan A 1998 *Phys. Lett. B* **425** 1
[32] Alberto P, Lisboa R, Malheiro M and de Castro A S 2005 *Phys. Rev. C* **71** 034313
[33] Furnstahl R J, Rusnak J J and Serot B D 1998 *Nucl. Phys. A* **632** 607
[34] Moshinsky M and Szczepaniak A 1989 *J. Phys. A: Math. Gen.* **22** L817
[35] Mao G 2003 *Phys. Rev. C* **67** 044318
[36] Hamzavi M, Eshghi M and Ikhdair S M 2012 *J. Math. Phys.* **53** 082101
[37] Eshghi M and Hamzavi M 2012 *Commun. Theor. Phys.* **57** 355
[38] Eshghi M and Mehraban H 2012 *Few-Body Syst.* **52** 41
[39] Akçay H 2009 *Phys. Lett. A* **373** 616
[40] Aydoğdu O and Sever R 2010 *Eur. Phys. J. A* **43** 73
[41] Aydoğdu O and Sever R 2011 *Phys. Lett. B* **703** 379
[42] Williams B W and Poulios D P 1993 *Eur. J. Phys.* **14** 222
[43] Zou X, Yi L Z and Jia C S 2005 *Phys. Lett. A* **346** 54
[44] Hamzavi M, Rajabi A A and Hassanabadi H 2010 *Phys. Lett. A* **374** 4303
[45] Akçay H and Tezcan C 2009 *Int. J. Mod. Phys. C* **20** 931
[46] Ikhdair S M and Sever R 2010 *Appl. Math. Comput.* **216** 911
[47] Greiner W 2000 *Relativistic Quantum Mechanics-wave Equation*, 3rd edn. (Berlin: Springer-Verlag)
[48] Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)
[49] Ginocchio J N 1999 *Nucl. Phys. A* **654** 663
[50] Ginocchio J N 1999 *Phys. Rep.* **315** 231
[51] Greene R L and Aldrich C 1976 *Phys. Rev. A* **14** 2363
[52] Gendenshtein L E 1983 *Sov. Phys.: JETP Lett.* **38** 356
[53] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[54] Jia C S, Zeng X L, Li S C, Sun L T and Yang Q B 2002 *Commun. Theor. Phys.* **37** 523
[55] Nikiforov A F and Uvarov V B 1988 *Special Functions of Mathematical Physics* (Basel: Birkhauser, Verlag)