The spin-one Duffin–Kemmer–Petiau equation in the presence of pseudo-harmonic oscillatory ring-shaped potential

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The Duffin–Kemmer–Petiau equation (DKP) is studied in the presence of a pseudo-harmonic oscillatory ring-shaped potential in $(1+3)$-dimensional space–time for spin-one particles. The exact energy eigenvalues and the eigenfunctions are obtained using the Nikiforov–Uvarov method.

Keywords: DKP equation, pseudo-harmonic oscillatory ring-shaped potential, Nikiforov–Uvarov method, energy eigenvalues, eigenfunctions

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1. Introduction

The solutions of the relativistic wave equations give us an invaluable source of knowledge in particle and nuclear physics. Among these equations, the Klein–Gordon and Dirac equations have been considerably investigated in the literature. On the other hand, there are only a few studies on the spin-one particles. The latter class can be investigated via the Duffin–Kemmer–Petiau (DKP) and Proca equations. The DKP equation has two versions and is capable of investigating both spin-zero and spin-one bosons. This equation has already been used to study of deuteron–nucleus scattering,[1] meson–nuclear interaction,[2] and α–nucleus elastic scattering.[3] Interesting work on the interfaces with quantum chromodynamics (QCD),[4] the covariant Hamiltonian,[5] and the casual approach[6,7] have been released. A survey on the analytical solutions of the equation can be found in Refs. [8]–[23]. Here, we are going to solve the equation for a pseudo-harmonic oscillatory ring-shaped potential. The motivation behind the present study is threefold. The first is the high number of spin-one mesons in particles physics. The second is the complexity of the counterpart of the spin-one DKP equation, i.e. the Proca equation. The third is our choice of the potential. The radial interactions, despite all their notable advantages, do have limitations in some cases including the study of deformed nuclei. Our considered potential includes a physical angle-dependent term besides the well-established pseudo-harmonic interaction. We will soon show that we will recover a Schrödinger-like equation with a different effective potential. As a result, depending on the choice of the potential, we might use various analytical ideas of quantum mechanics including the supersymmetry quantum mechanics, point canonical transformation, Lie groups, factorization technique, Wentzel–Kramers–Brillouin (WKB) approximation, path integral formalism, series expansion, asymptotic iteration method, Green functions, master function approach, super-integrability, etc. Here, to provide the solution as simply as possible, we will use the Nikiforov–Uvarov (NU) technique. The remainder of this paper is organized as follows. In Section 2, we review the basic concepts of the DKP equation for spin-one particles in three-dimensional space. We will introduce the NU method in Section 3 and derive the energy eigenvalues and eigenfunctions in Section 4. Finally some conclusions are drawn in Section 5.

2. DKP equation

The DKP equation for free scalar and vector bosons is $(\hbar = c = 1)$,[24–26]

$$(\mathrm{i}\beta^{\mu}\partial_{\mu} - m)\psi = 0, \tag{1}$$

where β^{μ} are DKP matrices and for a spin-one field

$$\beta^{\mu}\beta^{\nu}\beta^{\lambda} + \beta^{\lambda}\beta^{\nu}\beta^{\mu} = g^{\mu\nu}\beta^{\lambda} + g^{\lambda\nu}\beta^{\mu} \tag{2}$$

with

$$g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \quad \left(g^{\mu\nu}\right)^2 = 1. \tag{3}$$

In the case of vector bosons, the β^{μ} matrices are[27]

$$\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^{\mathrm{T}} & 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\ \bar{0}^{\mathrm{T}} & I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ \bar{0}^{\mathrm{T}} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \end{pmatrix},$$

$$\beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ \bar{0}^{\mathrm{T}} & 0_{3\times3} & 0_{3\times3} & -\mathrm{i}S_i \\ -e_i^{\mathrm{T}} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ \bar{0}^{\mathrm{T}} & -iS_i & 0_{3\times3} & 0_{3\times3} \end{pmatrix}, \tag{4}$$

where $(S_i)_{jk} = -\mathrm{i}\varepsilon_{ijk}$ are 3×3 matrices, with ε_{ijk} being 1, -1, 0 for an even permutation, an odd permutation, and repeated

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indices, respectively; $(e_i)_{1j} = \delta_{ij}$ matrices are 1 matrices, i.e.,

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1);$$

matrices, $I_{3\times3}$ and $0_{3\times3}$, represent unit and null 3×3 matrices, respectively.

The more general form of the interaction is[1]

$$U = s_1(r) + Ps_2(r) + \beta^0 v_1(r) + \beta^0 P v_2(r), \tag{5}$$

and the equation is written as

$$(\mathrm{i}\beta^\mu \partial_\mu - m - U)\psi = 0. \tag{6}$$

Therefore,

$$(\mathrm{i}\beta^\mu \partial_\mu - m - S_1(r) - PS_2(r) - \beta^0 V_1(r) - \beta^0 P V_2(r))\psi = 0, \tag{7}$$

where the projection operator is

$$P = \mathrm{diag}(1,1,1,1,0,0,0,0,0,0), \tag{8}$$

and the wave function can be written as

$$\psi = \left(\mathrm{i}\phi, F^1, F^2, F^3, W^1, W^2, W^3, X^1, X^2, X^3\right)^{\mathrm{T}}. \tag{9}$$

Therefore,

$$\left[\begin{pmatrix}
-m & 0 & 0 & 0 & \mathrm{i}\partial 1 & \mathrm{i}\partial_2 & \mathrm{i}\partial_3 & 0 & 0 & 0 \\
0 & -m & 0 & 0 & \mathrm{i}\partial_0 & 0 & 0 & 0 & -\mathrm{i}\partial_3 & \mathrm{i}\partial_2 \\
0 & 0 & -m & 0 & 0 & \mathrm{i}\partial_0 & 0 & \mathrm{i}\partial_3 & 0 & -\mathrm{i}\partial_1 \\
0 & 0 & 0 & -m & 0 & 0 & \mathrm{i}\partial_0 & -\mathrm{i}\partial_2 & \mathrm{i}\partial_1 & 0 \\
-\mathrm{i}\partial 1 & \mathrm{i}\partial_0 - V_2 & 0 & 0 & -m & 0 & 0 & 0 & 0 & 0 \\
-\mathrm{i}\partial_2 & 0 & \mathrm{i}\partial_0 - V_2 & 0 & 0 & -m & 0 & 0 & 0 & 0 \\
-\mathrm{i}\partial_3 & 0 & 0 & \mathrm{i}\partial_0 - V_2 & 0 & 0 & -m & 0 & 0 & 0 \\
0 & 0 & -\mathrm{i}\partial_3 & \mathrm{i}\partial_2 & 0 & 0 & 0 & -m & 0 & 0 \\
0 & \mathrm{i}\partial_3 & 0 & -\mathrm{i}\partial_1 & 0 & 0 & 0 & 0 & -m & 0 \\
0 & -\mathrm{i}\partial_2 & \mathrm{i}\partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & -m
\end{pmatrix}
-
\begin{pmatrix}
S_1+S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & S1+S_2 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & S_1+S_2 & 0 & 0 & V_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & S_1+S_2 & 0 & 0 & V_1 & 0 & 0 & 0 \\
0 & V_1 & 0 & 0 & S_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & V_1 & 0 & 0 & S_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & V_1 & 0 & 0 & S_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & S_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_1
\end{pmatrix}\right]
\begin{pmatrix}
\mathrm{i}\phi \\ F^1 \\ F^2 \\ F^3 \\ W^1 \\ W^2 \\ W^3 \\ X^1 \\ X^2 8 \\ X^3
\end{pmatrix} = 0, \tag{10}$$

From Eq. (10), we can obtain ten coupled equations which, after a rather cumbersome algebra, give

$$\nabla \cdot \boldsymbol{W} = (m + S_1 + S_2)\varphi, \tag{11a}$$

$$(E - V_1)\boldsymbol{W} + \mathrm{i}(\nabla \times \boldsymbol{X}) = (m + S_1 + S_2)\boldsymbol{F}, \tag{11b}$$

$$\nabla\varphi + (E - V_1 - V_2)\boldsymbol{F} = (m + S_1)\boldsymbol{W}, \tag{11c}$$

$$\mathrm{i}(\nabla \times \boldsymbol{F}) = (m + S_1)\boldsymbol{X}. \tag{11d}$$

By combining Eqs. (11a) and (11d), we have

$$\nabla\left(\frac{1}{(m+S_1+S_2)}\right)\left[\begin{array}{l}
\nabla\left(\dfrac{(m+S_1+S_2)}{(E-V_1)}\right)\cdot\boldsymbol{F} + \dfrac{(m+S_1+S_2)}{(E-V_1)}\nabla\cdot\boldsymbol{F} - \\
\nabla\left(\dfrac{\mathrm{i}}{(E-V_1)}\right)\cdot\left\{\nabla\left(\nabla\left(\dfrac{\mathrm{i}}{m+S_1}\right)\cdot\boldsymbol{F}\right) - \left(\nabla\left(\dfrac{\mathrm{i}}{m+S_1}\right)\cdot\nabla\right)\boldsymbol{F} + \dfrac{\mathrm{i}}{m+S_1}\left(\nabla(\nabla\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}\right)\right\}
\end{array}\right]$$

$$+\frac{1}{(m+S_1+S_2)}\nabla\left[\begin{array}{l}
\nabla\left(\dfrac{(m+S_1+S_2)}{(E-V_1)}\right)\cdot\boldsymbol{F} + \dfrac{(m+S_1+S_2)}{(E-V_1)}\nabla\cdot\boldsymbol{F} - \\
\nabla\left(\dfrac{\mathrm{i}}{(E-V_1)}\right)\left\{\nabla\left(\nabla\left(\dfrac{\mathrm{i}}{m+S_1}\right)\cdot\boldsymbol{F}\right) - \left(\nabla\left(\dfrac{\mathrm{i}}{m+S_1}\right)\cdot\nabla\right)\boldsymbol{F} + \dfrac{\mathrm{i}}{m+S_1}\left(\nabla(\nabla\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}\right)\right\}
\end{array}\right] + (E - V_1 - V_2)\boldsymbol{F}$$

$$= (m+S_1)\left[\begin{array}{l}
\dfrac{(m+S_1+S_2)}{(E-V_1)}\boldsymbol{F} - \dfrac{\mathrm{i}}{(E-V_1)}\nabla\left(\nabla\left(\dfrac{\mathrm{i}}{m+S_1}\right)\cdot\boldsymbol{F}\right) \\
+\dfrac{\mathrm{i}}{(E-V_1)}\left(\nabla\left(\dfrac{\mathrm{i}}{m+S_1}\right)\cdot\nabla\right)\boldsymbol{F} + \dfrac{1}{(m+S_1)(E-V_1)}\left(\nabla(\nabla\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F}\right)
\end{array}\right]. \tag{12}$$

In the special case of $S_1 = S_2 = V_1 = V_2 = 0$, we have

$$(E^2_{n,l,\lambda} - m^2)\boldsymbol{F} + \nabla^2 \boldsymbol{F} = 0, \tag{13a}$$

$$\nabla^2 = -p^2, \tag{13b}$$

$$(E^2_{n,l,\lambda} - m^2 - p^2)\boldsymbol{F} = 0. \tag{13c}$$

The other special case is $S_1 = S_2 = V_1 = 0$,

$$(E_{n,l,\lambda}(E_{n,l,\lambda} - V_2) - m^2)\boldsymbol{F} + \nabla^2 \boldsymbol{F} = 0. \tag{14}$$

Now, we introduce the pseudo-harmonic oscillatory ring-shaped potential as follows:

$$V_2 = \frac{A}{2}r^2 + \frac{B}{r^2} + \frac{\eta + \beta\cos^2\theta + \gamma\cos^4\theta}{r^2\sin^2\theta\cos^2\theta}, \tag{15}$$

as well as an ansatz in the form of

$$F = F(r,\theta,\phi) = R(r)Q(\theta)\mathrm{e}^{\mathrm{i}\lambda\phi}. \tag{16}$$

By substituting Eq. (16) into Eq. (14), one arrives at

$$\nabla^2 \boldsymbol{F}(\boldsymbol{r},\theta,\varphi) - E_{n,l,\lambda}V_2\boldsymbol{F}(\boldsymbol{r},\theta,\varphi) + (E^2_{n,l,\lambda} - m^2)\boldsymbol{F}(\boldsymbol{r},\theta,\varphi) = 0, \tag{17}$$

where

$$\nabla^2 = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2\frac{\mathrm{d}}{\mathrm{d}r}\Big) + \frac{1}{r^2}\Big[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\Big)\Big] + \frac{1}{r^2\sin^2\theta}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}. \tag{18}$$

3. NU method

The NU method provides us a simple solution of some second-order differential equations. For simplicity, we will use the parametric form of the idea, which solves[28]

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\psi_n(s) + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{\mathrm{d}}{\mathrm{d}s}\psi_n(s) + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{[s(1-\alpha_3 s)]^2}\psi_n(s) = 0. \tag{19a}$$

Here, we give only the basic ingredients of the generalized NU method. The NU method rewrites a second-order differential equation as

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0. \tag{19b}$$

A comparison between these two equations gives

$$\tilde{\tau}(s) = \alpha_1 - \alpha_2 s, \tag{20}$$

$$\sigma(s) = s(1-\alpha_3 s), \tag{21}$$

$$\tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \tag{22}$$

To obtain the solution, we ought to determine

$$\pi(s) = \frac{\sigma'(s) - \tilde{\tau}(s)}{2} \pm \sqrt{(\frac{\sigma'(s) - \tilde{\tau}(s)}{2})^2 - \tilde{\sigma}(s) + k\sigma(s)} = \alpha_4 + \alpha_5 s \pm \sqrt{(\alpha_6 - k\alpha_3)s^2 + (\alpha_7 + k)s + \alpha_8}, \tag{23}$$

where

$$\alpha_4 = \frac{1}{2}(1-\alpha_1), \tag{24}$$

$$\alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \tag{25}$$

$$\alpha_6 = \alpha_5^2 + \xi_1, \tag{26}$$

$$\alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \tag{27}$$

$$\alpha_8 = \alpha_4^2 + \xi_3, \tag{28}$$

$$k_{1,2} = -(\alpha_7 + 2\alpha_3\alpha_8) \pm 2\sqrt{\alpha_8\alpha_9}, \tag{29}$$

with

$$\alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6. \tag{30}$$

Different k values lead to the different $\pi(s)$ values. If we take

$$k = -(\alpha_7 + 2\alpha_3\alpha_8) - 2\sqrt{\alpha_8\alpha_9}, \tag{31}$$

$\pi(s)$ becomes

$$\pi(s) = \alpha_4 + \alpha_5 s - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}\,], \tag{32}$$

and we find

$$\tau(s) = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_5)s - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}\,]. \tag{33}$$

As a result, the energy eigenvalue equation can be readily obtained as

$$\alpha_2 n - (2n+1)\alpha_5 + (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + n(n-1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0. \tag{34}$$

In order to obtain the wave functions, one can use the following relations:

$$\rho(s) = s^{\alpha_{10}-1}(1-\alpha_3 s)^{(\alpha_{11}/\alpha_3)-\alpha_{10}-1}, \tag{35}$$

$$\Phi_n(s) = \mathrm{P}_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{36}$$

$$W(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}, \tag{37}$$

$$\Psi_n(s) = s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-(\alpha_{13}/\alpha_3)} \times \mathrm{P}_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s), \tag{38}$$

where $\mathrm{P}_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s)$ denotes the Jacobi polynomials. In the special case of

$$\lim_{\alpha_3\to 0}\mathrm{P}_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3 s) = \mathrm{L}_n^{\alpha_{10}-1}(\alpha_{11}s).$$

Therefore, the wavefunction in this case becomes

$$\psi_n(s) = s^{\alpha_{12}}e^{\alpha_{13}s}\mathrm{L}_n^{\alpha_{10}-1}(\alpha_{11}r) \tag{39}$$

and

$$\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \tag{40}$$

$$\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}), \tag{41}$$

$$\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \tag{42}$$

$$\alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}). \tag{43}$$

4. Energy eigenvalues and eigenfunctions

By substituting Eqs. (18) and (15), into Eq. (17), we find

$$\Big[\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2\frac{\mathrm{d}}{\mathrm{d}r}\Big) + \frac{1}{r^2}\Big(\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\Big)\Big) + \frac{1}{r^2\sin^2\theta}\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\Big]\boldsymbol{F} - E_{n,l,\lambda}\Big(\frac{A}{2}r^2 + \frac{B}{r^2} + \frac{\eta + \beta\cos^2\theta + \gamma\cos^4\theta}{r^2\sin^2\theta\cos^2\theta}\Big)\boldsymbol{F} + (E^2_{n,l,\lambda} - m^2)\boldsymbol{F} = 0, \tag{44}$$

which, after separation of the variables, yields the radial function

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}R(r)}{\mathrm{d}r}-\frac{1}{2}E_{n,l,\lambda}Ar^2R(r)+(E_{n,l,\lambda}^2-m^2)R(r)-\frac{(E_{n,l,\lambda}B+l(l+1))}{r^2}R(r)=0, \tag{45}$$

$$\frac{\mathrm{d}^2Q_l}{\mathrm{d}\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}Q_l}{\mathrm{d}\theta}+\Big\{-\frac{\lambda^2}{\sin^2\theta}-\frac{E_{n,l,\lambda}\eta+E_{n,l,\lambda}\beta\cos^2\theta+\gamma E_{n,l,\lambda}\cos^4\theta}{\sin^2\theta\cos^2\theta}+l(l+1)\Big\}Q_l(\theta)=0. \tag{46}$$

By choosing the change of variable $s=r^2$, the radial equation appears as

$$\frac{\mathrm{d}^2R(s)}{\mathrm{d}s^2}+\frac{3/2}{s}\frac{\mathrm{d}R(s)}{\mathrm{d}s}+\left[\frac{-\frac{1}{8}E_{n,l,\lambda}As^2+\frac{1}{4}(E_{n,l,\lambda}^2-m^2)s-\frac{1}{4}(E_{n,l,\lambda}B+l(l+1))}{s^2}\right]R(s)=0, \tag{47}$$

which corresponds to the NU parameters

$$\alpha_1=\frac{3}{2},\ \alpha_2=0,\ \alpha_3=0,\ \xi_1=\frac{1}{8}E_{n,l,\lambda}A,$$
$$\xi_2=\frac{1}{4}(E_{n,l,\lambda}^2-m^2),\ \xi_3=\frac{1}{4}(E_{n,l,\lambda}B+l(l+1)),$$
$$\alpha_4=-\frac{1}{4},\ \alpha_5=0,\ \alpha_6=\frac{1}{8}E_{n,l,\lambda}A,\ \alpha_7=-\frac{1}{4}(E_{n,l,\lambda}^2-m^2),$$
$$\alpha_8=\frac{1}{16}+\frac{1}{4}(E_{n,l,\lambda}B+l(l+1)),\ \alpha_9=\frac{1}{8}E_{n,l,\lambda}A,$$
$$\alpha_{10}=1+\sqrt{\frac{1}{4}+(E_{n,l,\lambda}B+l(l+1))},\ \alpha_{11}=\sqrt{\frac{1}{2}E_{n,l,\lambda}A},$$
$$\alpha_{12}=-\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{1}{4}(E_{n,l,\lambda}B+l(l+1))},$$
$$\alpha_{13}=-\sqrt{\frac{1}{8}E_{n,l,\lambda}A}. \tag{48}$$

Therefore, the radial energy is

$$E_{n,l,\lambda}^2-(2n+1)\sqrt{2E_{n,l,\lambda}A}-\sqrt{2E_{n,l,\lambda}A\left[\left(l+\frac{1}{2}\right)^2+E_{n,l,\lambda}B\right]}-m^2=0. \tag{49}$$

For the angular section, we have to solve

$$\frac{\mathrm{d}^2Q}{\mathrm{d}\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{\mathrm{d}Q}{\mathrm{d}\theta}+\frac{(-\gamma E_{n,l,\lambda}-l(l+1))\cos^4\theta+(l(l+1)-\lambda^2-\beta E_{n,l,\lambda})\cos^2\theta-E_{n,l,\lambda}\eta}{\cos^2\theta(1-\cos^2\theta)}Q(\theta)=0. \tag{50}$$

Using the change of variable $y=\cos\theta$, Eq. (5) comes into the form

$$\frac{\mathrm{d}^2Q(y)}{\mathrm{d}y^2}+\frac{-2y}{1-y^2}\frac{\mathrm{d}Q(y)}{\mathrm{d}y}+\frac{(-\gamma E_{n,l,\lambda}-l(l+1))y^4+(l(l+1)-\lambda^2-\beta E_{n,l,\lambda})y^2-E_{n,l,\lambda}\eta}{y^2(1-y^2)^2}Q(y)=0. \tag{51}$$

Performing the second transformation $z=y^2$, we have

$$\frac{\mathrm{d}^2Q_l(z)}{\mathrm{d}z^2}+\frac{1/2-3z/2}{z(1-z)}\frac{\mathrm{d}Q_l(z)}{\mathrm{d}z}+\frac{1}{[z(1-z)]^2}\left\{-\frac{1}{4}\left(l(l+1)+\gamma E_{n,l,\lambda}\right)z^2+\frac{1}{4}\left(l(l+1)-\beta E_{n,l,\lambda}-\lambda^2\right)z-\frac{1}{4}E_{n,l,\lambda}\eta\right\}Q_l(z)=0, \tag{52}$$

which possesses the NU parameters

$$\alpha_1=\frac{1}{2},\ \alpha_2=\frac{3}{2},\ \alpha_3=1,\ \ \xi_1=\frac{1}{4}\left(l(l+1)+\gamma E_{n,l,\lambda}\right),$$
$$\xi_2=\frac{1}{4}\left(l(l+1)-\beta E_{n,l,\lambda}-\lambda^2\right),\ \xi_3=\frac{1}{4}E_{n,l,\lambda}\eta,$$
$$\alpha_4=\frac{1}{4},\ \alpha_5=-\frac{1}{2},\ \alpha_6=\frac{1}{16}+\frac{1}{4}\left(l(l+1)+\gamma E_{n,l,\lambda}\right),$$
$$\alpha_7=-\frac{1}{8}-\frac{1}{4}\left(l(l+1)-\beta E_{n,l,\lambda}-\lambda^2\right),$$
$$\alpha_8=\frac{1}{16}+\frac{1}{4}E_{n,l,\lambda}\eta,$$
$$\alpha_9=\frac{1}{4}\left(\gamma E_{n,l,\lambda}+\beta E_{n,l,\lambda}+\lambda^2+E_{n,l,\lambda}\eta\right),$$
$$\alpha_{10}=1+\sqrt{\frac{1}{4}+E_{n,l,\lambda}\eta},$$
$$\alpha_{11}=2+\sqrt{\gamma E_{n,l,\lambda}+\beta E_{n,l,\lambda}+\lambda^2+E_{n,l,\lambda}\eta}+\sqrt{\frac{1}{4}+E_{n,l,\lambda}\eta},$$
$$\alpha_{12}=\frac{1}{4}+\sqrt{\frac{1}{16}+\frac{1}{4}E_{n,l,\lambda}\eta},$$
$$\alpha_{13}=-\frac{1}{4}-\sqrt{\frac{1}{4}\left(\gamma E_{n,l,\lambda}+\beta E_{n,l,\lambda}+\lambda^2+E_{n,l,\lambda}\eta\right)}-\sqrt{\frac{1}{16}+\frac{1}{4}E_{n,l,\lambda}\eta}. \tag{53}$$

Therefore, the wave function is

$$Q(\theta)=(\cos^2\theta)^{\alpha_{12}}(1-\alpha_3(\cos^2\theta))^{-\alpha_{12}-(\alpha_{13}/\alpha_3)}\times \mathrm{P}_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3(\cos^2\theta)), \tag{54}$$

where $\mathrm{P}_n^{(\alpha_{10}-1,(\alpha_{11}/\alpha_3)-\alpha_{10}-1)}(1-2\alpha_3(\cos^2\theta))$ is the Jacobi polynomial. Therefore, the angular part yields the energy relation

$$\begin{aligned}l(l+1)&=4n(n+1)+2+4(2n+1)\left(\sqrt{\frac{1}{16}+\frac{1}{4}E_{n,l,\lambda}\eta}+\sqrt{\frac{1}{4}\left(\gamma E_{n,l,\lambda}+\beta E_{n,l,\lambda}+\lambda^2+E_{n,l,\lambda}\eta\right)}\right)\\&\quad+\beta E_{n,l,\lambda}+\lambda^2+2E_{n,l,\lambda}\eta+2\sqrt{\left(\frac{1}{4}+E_{n,l,\lambda}\eta\right)\left(\gamma E_{n,l,\lambda}+\beta E_{n,l,\lambda}+\lambda^2+E_{n,l,\lambda}\eta\right)}.\end{aligned} \tag{55}$$

Finally, we can derive the one component of the spinor as fol-

lows:

$$F_{n,l,\lambda}(r,\theta,\varphi) = r^{-1/2+\sqrt{1/4+BE_{n,l,\lambda}+l(l+1)}}\exp\left(-\sqrt{\tfrac{1}{8}E_{n,l,\lambda}A}\,r^2\right) \times \mathrm{L}_n^{1/4+BE_{n,l,\lambda}+l(l+1)}\left(\sqrt{\frac{1}{2}E_{n,l,\lambda}A}\,r^2\right) \times \left(\cos^2\theta\right)^{1/4+\sqrt{1/16+\frac{1}{4}E_{n,l,\lambda}\eta}} \times \left(1-\cos^2\theta\right)^{\sqrt{\frac{1}{4}\left(\gamma E_{n,l,\lambda}+\beta E_{n,l,\lambda}+\lambda^2+E_{n,l,\lambda}\eta\right)}} \times \mathrm{P}_n^{\left(\sqrt{1/4+E_{n,l,\lambda}\eta},\sqrt{\gamma E_{n,l,\lambda}+\beta E_{n,l,\lambda}+\lambda^2+E_{n,l,\lambda}\eta}\right)} \times (1-2\cos^2\theta)\,\mathrm{e}^{\mathrm{i}\lambda\varphi}. \quad (56)$$

Our results, for the special case of $B=\eta=\beta=\gamma=0$ reduce to the results of the quadratic term widely used in many branches of physics. The results can be used to study the sub-energy transitions in meson spectroscopy, cross sections, interference patterns, charge transfer, excitation effects, and static multipole polarizabilities of the interacting particles and various static properties of mesons.

5. Conclusions

In the present work, we consider the DKP equation in the presence of pseudo-harmonic oscillatory ring-shaped potential in the three-dimensional space for spin-one particles. The motivation behind the study is twofold. The first is the significant role of vector bosons in particle physics and the second is the successful phenomenological predictions of pseudoharmonic potential which contains an extra inverse square term besides the well-known harmonic term. To obtain a more applied result, we consider a ring-shaped form of the potential which has already been used to describe non-spherical configurations in physics including the ring-shaped organic molecules like cyclic polygenes and benzene as well as non-spherical nuclei. After a rather lengthy calculation, we show that the equation appears as a Schrödinger-like equation. We next use the powerful analytical NU method in our calculations to avoid time-consuming and rather vague numerical techniques. We report on the eigenfunctions in terms of the Jacobi polynomials and on the energy spectrum. The results, for special cases of the potential parameters, yield the pseudoharmonic and harmonic interactions which are frequently present in various physical studies. We can directly use the results to study the mesons spectra, nucleon–meson scattering, binding energies, transition properties, phase shifts, charge radius, electromagnetic form factors, static multipole polarizabilities, charge transfer, etc.

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