Relativistic treatment of the spin-zero particles subject to the second Pöschl–Teller-like potential

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The three-dimensional Klein–Gordon equation is solved for the case of equal vector and scalar second Pöschl–Teller potential by proper approximation of the centrifugal term within the framework of the asymptotic iteration method. Energy eigenvalues and the corresponding wave function are obtained analytically. Eigenvalues are computed numerically for some values of n and ℓ. It is found that the results are in good agreement with the findings of other methods for short-range potential.

Keywords: second Pöschl–Teller potential, asymptotic iteration, eigenvalues, eigenfunction

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1. Introduction

The increasing interest in finding exact solutions of the relativistic equations is due to the fact that any strong potential field determines the relativistic effects and it gives correction to the non-relativistic results. A moving particle in a potential field should be described by the Klein–Gordon equation and the Dirac equation because of the relativistic effects. In recent years, numerous studies have been made in finding the exact solutions of the Klein–Gordon equation. In most of these studies, the authors assumed that the scalar potential is equal to the vector potential and obtained the s-wave or the ℓ-wave bound state solutions of the Klein–Gordon equation in some typical potential fields by using different methods. These typical potential fields include the second Pöschl–Teller potential,[1,2] Hulthén potential,[5–7] double ring-shaped oscillator potential,[8,9] Columbic ring-shaped potential,[10–12] central potentials,[13] Kratzer potential,[14] Coulomb potential,[15] and the Deng–Fan oscillator potential.[16] These methods which have been used in solving the differential equation arising from each consideration include the asymptotic iteration method,[17–22] the Nikiforov–Uvarov method,[23–27] the exact quantization rule method,[28–34] and the standard method.[1,35–37]

The asymptotic iteration method (AIM) has been introduced by Ciftci *et al.*[20,21] and has been used in many physical systems to obtain the whole spectra.[17–21,38–49] This method reproduces exact solutions of many exactly solvable potentials and also gives an accurate result for the non-solvable potentials such as sextic oscillator, deformed Coulomb potential, etc. Although the AIM has been applied to solve the Schrödinger equation for many potentials, there have been very few studies on its application to relativistic equations such as the Klein–Gordon equation.

In this paper, we attempt to find the bound state solutions of the Klein–Gordon equation with equal scalar and vector second Pöschl–Teller potential. This is given by[1–4]

$$V_{\rm s}(r) = V_v(r) = \frac{V_1 - V_2\cosh\alpha r}{\sinh^2\alpha r}, \tag{1}$$

where the parameters V_1 and V_2 describe the property of the potential well, while the parameter α is related with the range of the potential. The behavior of this potential with respect to three different values of α are shown in Fig. 1. We aim to extend the study of Qiang and Dong[1] by using an improved approximation scheme to deal with the centrifugal term. Furthermore, we use a more practical method, the AIM, to study the bound state solutions of the Klein–Gordon equation with this potential.

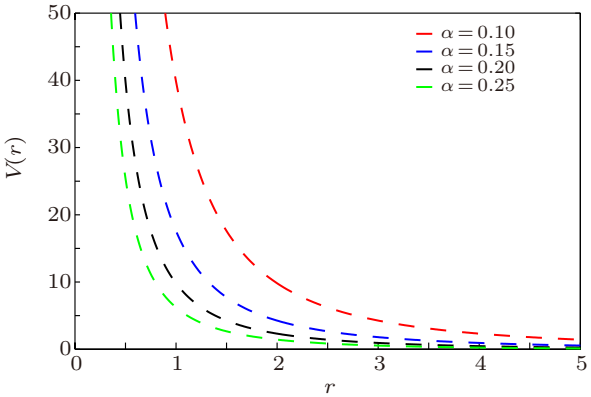


Fig. 1. (color online) The behavior of the second Pöschl–Teller potential with respect to three different values of α, and for $V_1 = 0.5$, $V_2 = 0.1$.

The paper is organized as follows: In Section 2, we derive arbitrary ℓ-state solutions of the Klein–Gordon equation with equal scalar and vector second Pöschl–Teller potential. We present our numerical results and then compare them with

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numerical simulations of the energy levels for both the approximate and non-approximate problem in Section 3. Finally, we give the concluding remarks in Section 4.

2. Any ℓ-state solutions

We begin with the Klein–Gordon equation with the scalar and vector potential ($\hbar = c = 1$)

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]+(M+V_{\rm s})^2-(E_{\rm R}-V_{\rm v})^2\right\}\psi(r,\theta,\phi)=0, \quad (2)$$

where $E_{\rm R}$ is the relativistic energy of the system. Taking

$$\psi(r,\theta,\phi)=\frac{R(r)\Theta(\theta)\,{\rm e}^{\pm im\phi}}{r}, \quad (3)$$

and substituting it into Eq. (2), we obtain the radial part of the Klein–Gordon equation as:

$$\left\{\frac{{\rm d}^2}{{\rm d}r^2}+(E_{\rm R}-V_{\rm v}(r))^2-(M+V_{\rm s}(r))^2-\frac{\ell(\ell+1)}{r^2}\right\}R(r)=0. \quad (4)$$

This equation cannot be solved analytically for $\ell\neq 0$ due to the centrifugal term. Therefore, we must use an approximation to this centrifugal term. Wei *et al.*[37] found that for short-range potential, the following formula

$$\frac{1}{r^2}=\alpha^2\left[\frac{\varpi\,{\rm e}^{-\alpha r}}{(1-{\rm e}^{-\alpha r})}+\frac{{\rm e}^{-2\alpha r}}{(1-{\rm e}^{-\alpha r})^2}\right] \quad (5)$$

is a proper approximation to $1/r^2$. The parameter $\varpi = 1.05$ is a dimensionless constant. However when $\varpi = 1$, the new improved approximation scheme becomes the convectional approximation scheme.[1] By considering potential equation (1) and using the following definitions of hyperbolic functions:

$$\sinh x=\frac{{\rm e}^x-{\rm e}^{-x}}{2},\quad \cosh x=\frac{{\rm e}^x+{\rm e}^{-x}}{2},\quad \tanh x=\frac{\sinh x}{\cosh x},$$
$$\frac{\rm d}{{\rm d}x}\cosh x=\sinh x,\quad \frac{\rm d}{{\rm d}x}\sinh x=\cosh x,$$
$$\frac{\rm d}{{\rm d}x}\tanh x=\frac{1}{\cosh^2 x}, \quad (6)$$

equation (4) takes the following form:

$$s(1-s)\frac{{\rm d}^2R(s)}{{\rm d}s^2}+\frac{s(1-3s)}{2}\frac{{\rm d}R(s)}{{\rm d}s}+\left[\frac{As^2+Bs+C}{s(1-s)}\right]R(s)=0, \quad (7)$$

where we have introduced a new transformation of the form $s=\tanh^2(\alpha r/2)$ and the following notations for simplicity

$$\begin{aligned}A &= -\frac{(V_1+V_2)(E_{\rm R}+M)}{2\alpha^2},\\ B &= \frac{(E_{\rm R}+M)(E_{\rm R}-M+V_1)}{\alpha^2}+\frac{\varpi\ell(\ell+1)}{4},\\ C &= \frac{(V_2-V_1)(E_{\rm R}+M)}{2\alpha^2}-\frac{\varpi\ell(\ell+1)}{4}.\end{aligned} \quad (8)$$

In order to solve this equation with the AIM, we should transform this equation to the form of Eq. (A1) in Appendix A. Therefore, the reasonable wave function we proposed is as follows:

$$R(s)=s^{\eta}(1-s)^{\gamma}f(s), \quad (9)$$

where

$$\eta=\frac{1}{4}\left(1+\sqrt{1-16C}\right),\quad \gamma=j\sqrt{A+B+C}. \quad (10)$$

Substituting them into Eq. (7) allows us to obtain the second-order homogenous linear differential equation in the following form:

$$\frac{{\rm d}^2f(s)}{{\rm d}s^2}+\left[\frac{(1/2+2\eta)-s(2\eta+2\gamma+3/2)}{s(1-s)}\right]\frac{{\rm d}f(s)}{{\rm d}s}+\left[\frac{B-(2\eta^2+\gamma/2+2\eta\gamma)}{s(1-s)}\right]f(s)=0, \quad (11)$$

which is now amenable to AIM solutions. To apply the AIM, it is necessary to compare Eq. (11) with Eq. (A1). Subsequently, by using Eq. (A7), the values of $\lambda_k(s)$ and $s_k(s)$ are calculated as follows:

$$\begin{aligned}\lambda_o(s) &= \frac{s(2\eta+2\gamma+3/2)-(2\eta+1/2)}{s(1-s)},\quad s_0(s)=\frac{2\eta^2+2\eta\gamma-B+\gamma/2}{s(1-s)},\\ \lambda_1(s) &= \frac{(-24\eta^2 s-4Bs+8\eta^2s^2+4Bs^2+16s^2\gamma^2+16\eta+15s^2-10s)}{4s^2(1-s)^2}\\ &\quad+\frac{32\eta s^2+30s^2\gamma^2-6s\gamma+3-48\eta s+16\eta^2-24\eta\gamma s+24\eta\gamma s^2}{4s^2(1-s)^2},\\ s_1(s) &= \frac{(4\eta^2+\gamma+4\eta\gamma-2B)(-3+7s+4\eta s+4s\gamma-4\eta)}{s^2(1-s^2)}.\end{aligned} \quad (12)$$

Employing the termination condition (A10), we arrive at the following eigenvalues expressions:

$$\begin{aligned}&\text{for } k=1: \ s_0(s)\lambda_1(s)=\lambda_0(s)s_0(s)\Rightarrow B_0=0+2\eta^2+\frac{1}{2}\gamma+2\eta\gamma+0,\\ &\text{for } k=2: \ s_1(s)\lambda_2(s)=\lambda_1(s)s_2(s)\Rightarrow B_1=\frac{3}{2}+2\eta^2+\frac{5}{2}\gamma+2\eta\gamma+2\eta,\\ &\text{for } k=3: \ s_2(s)\lambda_3(s)=\lambda_2(s)s_3(s)\Rightarrow B_2=5+2\eta^2+\frac{9}{2}\gamma+2\eta\gamma+4\eta,\\ &\text{for } k=4: \ s_3(s)\lambda_4(s)=\lambda_3(s)s_4(s)\Rightarrow B_3=\frac{21}{2}+2\eta^2+\frac{13}{2}\gamma+2\eta\gamma+6\eta,\end{aligned}$$

$$\text{for } k=5: \quad s_4(s)\lambda_5(s)=\lambda_4(s)s_5(s)\Rightarrow B_4=18+2\eta^2+\frac{17}{2}\gamma+2\eta\gamma+8\eta,$$
$$\ldots,\text{etc.} \tag{13}$$

When the above expressions are generalized, the eigenvalues turn out as:

$$B_n=\frac{2n^2+n}{2}+2\eta^2+\left(\frac{4n+1}{2}\right)\gamma+2\eta\gamma+(2n)\eta, \quad n=0,1,2,\ldots \tag{14}$$

If one inserts the values of A, B, C, η, and γ into Eq. (14), the energy spectrum of the radial Klein–Gordon equation with the second Pöschl–Teller potential becomes

$$\frac{2\left[\frac{(M+E_{\rm R})(E_{\rm R}-M+V_1)}{\alpha^2}-\left(\frac{4n+1}{2}\right)\sqrt{\frac{M^2-E_{\rm R}^2}{\alpha^2}}+\frac{\varpi\ell(\ell+1)}{4}-\frac{2n^2+n}{2}\right]}{\left[\left(n+\frac{1}{4}\right)+\frac{1}{4\alpha}\sqrt{\varpi\alpha^2\ell(\ell+1)+\alpha^2+8(V_1-V_2)(E_{\rm R}+M)}+\sqrt{\frac{M^2-E_{\rm R}^2}{\alpha^2}}\right]}-1=\frac{1}{\alpha}\sqrt{\varpi\alpha^2\ell(\ell+1)+\alpha^2+8(V_1-V_2)(E+M)}. \tag{15}$$

As indicated in Section 2, we can construct the corresponding eigenfunctions by using Eq. (A13). Equation (11) provides the Eq. (A13) for $m=\eta-3/4$, $N=-1$, $a=\gamma+1/2$, $b=1$, $\rho=2(\gamma+\eta+1/4)$ and so, the solution of Eq. (11) can easily be found

$$R_{n\ell}(r)=C_{n\ell}(-1)^n\tanh^{2\eta}(\alpha r)\,\mathrm{csch}^{2\gamma}(\alpha r)\frac{\Gamma(2\eta+n+1/2)}{\Gamma(2\eta+1/2)}$$
$$\times{}_2\mathrm{F}_1\left[-n,2\left(\eta+\gamma+\frac{n}{2}+1/4\right);2\eta+\frac{1}{2};\tanh^2\alpha r\right], \tag{16}$$

which is found to be identical to Ref. [1]. The normalization constant $C_{n\ell}$, obtained from the normalization condition $\int_0^\infty|R_{n\ell}(r)|^2\,\mathrm{d}r=1$, can be expressed as follows:

$$C_{n\ell}=\frac{1}{\Gamma(2\eta+\frac{1}{2})}\sqrt{\frac{2\alpha\Gamma(n+2\eta+\frac{1}{2})\Gamma(2\eta+2\gamma+n+\frac{1}{2})}{n!\Gamma(1+n+2\gamma)}}. \tag{17}$$

3. Numerical results

To show the accuracy of our results, we calculate energy eigenvalues for various n and ℓ quantum numbers. Also we show the absolute difference between the exact values. As can be seen from the results presented in Tables 1 and 2, the AIM results are in good agreement with findings of the other methods for short potential ranges. However, the differences between them will appear for a long potential range (large α), this implies that Eq. (5) is not a good approximation for a centrifugal term when the potential parameter becomes large.

Table 1. Comparison of the energy eigenvalues (in eV) obtained by using the AIM with other methods for 2p, 3p, 3d, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f, and 6g states in atomic units ($\hbar=M=c=1$) for $V_1=2V_2=4$.

States	α	NINT[1]	AIM	Standard method[1]	$\lvert E_{\rm AIM}-E_{\rm NINT}\rvert$
	0.05	0.494253	0.49428242141	0.494480	0.000029421
	0.10	0.524141	0.52425170317	0.525014	0.000110703
2p	0.15	0.553673	0.55390694829	0.555562	0.028857052
	0.20	0.582764	0.58315463980	0.585997	0.000390640
	0.25	0.611338	0.61190949392	0.616202	0.000571494
	0.30	0.639324	0.64009362935	0.646071	0.028755629
	0.05	0.549262	0.54928818608	0.549473	0.000026186
	0.10	0.624543	0.62463252966	0.625312	0.000089529
3p	0.15	0.691365	0.69153874410	0.692955	0.000173744
	0.20	0.750677	0.75094215002	0.753288	0.000265150
	0.25	0.803123	0.80347767522	0.806902	0.000354675
	0.30	0.849139	0.84957196568	0.854180	0.000432966
	0.05	0.495428	0.49551619042	0.496107	0.000088190
	0.10	0.528541	0.52886842690	0.531139	0.000327427
3d	0.15	0.562902	0.56358387412	0.868479	0.000681874
	0.20	0.598008	0.59912239112	0.607456	0.001114391
	0.25	0.633399	0.63499136572	0.647456	0.001592366
	0.30	0.668663	0.67075065548	0.687934	0.002087655
	0.05	0.599558	0.59956822767	0.599757	0.000010227
4p	0.10	0.709171	0.70910942347	0.709862	0.000061576
	0.15	0.798289	0.79875448897	0.799663	0.000465480
	0.20	0.869834	0.86877836745	0.871998	0.001055633
	0.05	0.550315	0.55043416634	0.550949	0.000119166

Table 1. (*continued*)

States	α	NINT[1]	AIM	Standard method[1]	$\lvert E_{\rm AIM} - E_{\rm NINT}\rvert$
4d	0.10	0.628122	0.62824535614	0.630417	0.000123356
	0.15	0.698231	0.69790688873	0.702945	0.000324111
	0.20	0.761069	0.76008976965	0.768756	0.000979230
	0.05	0.497183	0.49714788069	0.498537	0.000035119
4f	0.10	0.535032	0.53616134525	0.540168	0.001129345
	0.15	0.576268	0.57768034355	0.587169	0.001412344
	0.20	0.619571	0.61708829305	0.637798	0.002482707
5p	0.10	0.780705	0.78064316613	0.781332	0.000061833
	0.20	0.949245	0.95140065712	0.950986	0.002155657
5d	0.10	0.712122	0.71212213422	0.714186	0.002063866
	0.20	0.876833	0.87767454143	0.883217	0.000841541
5f	0.10	0.633413	0.63364132340	0.637962	0.000228323
	0.20	0.775872	0.77584535712	0.790842	0.000026642
5g	0.10	0.543490	0.54522001195	0.551926	0.001730012
	0.20	0.646182	0.64675534732	0.675279	0.000573347
6p	0.10	0.840336	0.84056462412	0.841408	0.000228624
6d	0.10	0.783146	0.78221413234	0.785019	0.000931868
6f	0.10	0.716490	0.71648453146	0.720585	0.000005468
6g	0.10	0.640326	0.64021235157	0.647818	0.000113648

Table 2. Comparison of the energy eigenvalues (in eV) obtained by using the AIM with other methods for 2p, 3p, 3d, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f, and 6g states in atomic units ($\hbar = M = c = 1$) for $V_1 = 5/3V_2 = 10$.

States	α	NINT[1]	AIM	Standard method[1]	$\lvert E_{\rm AIM} - E_{\rm NINT}\rvert$
	0.05	−0.826518	−0.826290834168	−0.825172	0.000227166
	0.10	−0.722685	−0.722136127400	−0.719364	0.000548873
	0.15	−0.634790	−0.633880879372	−0.629178	0.000909121
	0.20	−0.555888	−0.554596745072	−0.547762	0.001291255
	0.25	−0.483124	−0.481438916748	−0.472309	0.001685083
	0.30	−0.414980	−0.412893880749	−0.401329	0.002086119
	0.05	−0.652741	−0.652630435572	−0.652062	0.000110564
	0.10	−0.457226	−0.298132523016	−0.455509	0.000267306
	0.15	−0.298574	−0.298132523016	−0.295615	0.000441477
	0.20	−0.161405	−0.160780515605	−0.157048	0.000624484
	0.25	−0.039298	−0.384876554770	−0.033415	0.000810345
	0.30	0.071200	0.072195858950	0.078720	0.000995859
	0.05	−0.817762	−0.817131661405	−0.814030	0.000630339
	0.10	−0.701944	−0.700478213459	−0.693056	0.001465787
	0.15	−0.600968	−0.598614287312	−0.586302	0.002353713
	0.20	−0.508497	−0.505240126875	−0.487637	0.003256873
	0.25	−0.421986	−0.417831781145	−0.394610	0.004154219
	0.30	−0.340109	−0.335072580752	−0.305948	0.005036419
	0.05	−0.519884	−0.519806782659	−0.519395	0.000077217
	0.10	−0.257770	−0.257585518760	−0.256522	0.000184481
	0.15	−0.050483	−0.050183054722	−0.048320	0.000299945
	0.20	0.124021	0.12443835804	0.127220	0.000417358
	0.05	−0.648374	−0.648354634896	−0.626382	0.000019365
	0.10	−0.446732	−0.445046737831	−0.441747	0.001685262
	0.15	−0.281330	−0.282343278514	−0.272813	0.164388721
	0.20	−0.137162	−0.135452315063	−0.124714	0.001709685
	0.05	−0.805722	−0.805646744525	−0.798969	0.000075255
	0.10	−0.674512	−0.674543625567	−0.658909	0.000031625
	0.15	−0.557510	−0.674543625567	−0.532235	0.117033626
	0.20	−0.449032	−0.448574834567	−0.413521	0.000457165
	0.10	−0.091788	−0.091745356783	−0.090777	0.000042643
	0.20	0.350035	0.34967863689	0.352635	0.000356363
	0.10	−0.250470	−0.249453675661	−0.246791	0.001016324
	0.20	0.140402	0.14046553710	0.149750	0.000063537
	0.10	−0.431650	−0.429516451957	−0.422115	0.002133548
	0.20	−0.103321	−0.100465724561	−0.079957	0.002855275

Table 2. (*continued*)

States	α	NINT[1]	AIM	Standard method[1]	$\lvert E_{\rm AIM}-E_{\rm NINT}\rvert$
	0.10	−0.642637	−0.642158890786	−0.619793	0.000478109
	0.20	−0.382972	−0.379453784123	−0.331980	0.003518216
	0.10	0.051946	0.05195643678	0.052810	0.000010436
	0.10	−0.086137	−0.085956462667	−0.083139	0.000180537
	0.10	−0.239788	−0.239745332568	−0.232606	0.000042667
	0.10	−0.412613	−0.413454333124	−0.397543	0.000841333

4. Concluding remarks

A different approach, the asymptotic iteration method, to solve the ℓ-wave radial Klein–Gordon equation with equal scalar and vector second Pöschl–Teller potential is presented. The energy eigenvalues are calculated and the normalized wave functions are presented by generalized hypergeometric functions. To show the accuracy of our results, we have calculated the eigenvalues numerically for arbitrary n and ℓ. We found that the results obtained by Eq. (15) are in excellent agreement with those obtained by standard method and numerical integration method for short-range potential (small α).

The method presented in this paper is an elegant and powerful technique. The advantage of the method is that it gives the eigenvalues directly by transforming the second-order differential equation into a form of $y''(r)=\lambda_0(r)y'(r)+s_0(r)y(r)$. The method yields exact analytical solutions and provides the closed forms for the energy eigenvalues as well as the corresponding eigenfunctions. However, if there is no such solution, the eigenvalues are obtained by using an iterative approach.[38–40] As we have demonstrated, this method puts no constraint on the potential parameter values involved and it is easy to be implemented.

Appendix A: The asymptotic iteration method

We briefly outline the AIM here; the details can be found in Refs. [20] and [21].

A1. Energy eigenvalues

The AIM is proposed to solve the homogenous linear second-order differential equation of the form

$$y_n''(x)=\lambda_0(x)y_n'(x)+s_0(x)y_n(x), \tag{A1}$$

where $\lambda_0(x)\neq 0$ and the prime denotes the derivative with respect to x, the extral parameter n is thought as a radial quantum number. The variables, $s_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. To find a general solution to this equation, we differentiate Eq. (A1) with respect to x, we find

$$y_n'''(x)=\lambda_1(x)y_n'(x)+s_1(x)y_n(x), \tag{A2}$$

where

$$\lambda_1(x)=\lambda_0'(x)+s_0(x)+\lambda_0^2(x),$$
$$s_1(x)=s_0'(x)+s_0(x)\lambda_0(x), \tag{A3}$$

the second derivative of Eq. (A1) is obtained as

$$y_n''''(x)=\lambda_2(x)y_n'(x)+s_2(x)y_n(x), \tag{A4}$$

with

$$\lambda_2(x)=\lambda_1'(x)+s_1(x)+\lambda_0(x)\lambda_1(x),$$
$$s_2(x)=s_1'(x)+s_0(x)\lambda_1(x). \tag{A5}$$

Equation (A1) can be iterated up to $(k+1)$-th and $(k+2)$-th derivatives, $k=1,2,3,\ldots$ Therefore we have

$$y_n^{(k+1)}(x)=\lambda_{k-1}(x)y_n'(x)+s_{k-1}(x)y_n(x),$$
$$y_n^{(k+2)}(x)=\lambda_k(x)y_n'(x)+s_k(x)y_n(x), \tag{A6}$$

where

$$\lambda_k(x)=\lambda_{k-1}'(x)+s_{k-1}(x)+\lambda_0(x)\lambda_{k-1}(x),$$
$$s_k(x)=s_{k-1}'(x)+s_0(x)\lambda_{k-1}(x). \tag{A7}$$

From the ratio of the $(k+2)$-th and $(k+1)$-th derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[y_n^{k+1}(x)\right]=\frac{y_n^{(k+2)}(x)}{y_n^{(k+1)}(x)}=\frac{\lambda_k(x)\left[y_n'(x)+\dfrac{s_k(x)}{\lambda_k(x)}y_n(x)\right]}{\lambda_{k-1}(x)\left[y_n'(x)+\dfrac{s_{k-1}(x)}{\lambda_{k-1}(x)}y_n(x)\right]}, \tag{A8}$$

if $k>0$, for sufficiently large k, we obtain α values from

$$\frac{s_k(x)}{\lambda_k(x)}=\frac{s_{k-1}(x)}{\lambda_{k-1}(x)}=\alpha(x), \tag{A9}$$

with quantization condition

$$\delta_k(x)=\begin{vmatrix}\lambda_k(x) & s_k(x)\\ \lambda_{k-1}(x) & s_{k-1}(x)\end{vmatrix}=0\ ,\quad k=1,2,3,\ldots. \tag{A10}$$

Then equation (A8) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[y_n^{(k+1)}(x)\right]=\frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{A11}$$

which yields the general solution of Eq. (A1)

$$y_n(x)=\exp\left(-\int^x\alpha(x')\,\mathrm{d}x'\right)\times\left[C_2+C_1\int^x\exp\left(\int^{x'}\left[\lambda_0(x'')2\alpha(x'')\right]\mathrm{d}x''\right)\mathrm{d}x'\right]. \tag{A12}$$

For a given potential, the idea is to convert the radial Klein–Gordon equation to the form of Eq. (A1). Then $\lambda_0(x)$ and $s_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are cal-

culated by the recurrence relations given by Eq. (A7). The energy eigenvalues are then obtained by the condition given by Eq. (A10) if the problem is exactly solvable. For nontrivial potentials that have no exact solutions, for a specific principal quantum number n, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential[20,21,38–40] and the approximate energy eigenvalues are obtained from the roots of Eq. (A10) for sufficiently large values of k with iteration for which k is always larger than n in the numerical solutions.

A2. Energy eigenfunction

Suppose we wish to solve the radial Klein–Gordon equation for which the homogenous linear second-order differential equation takes the following general form

$$y''(x) = 2\left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{m+1}{x}\right)y'(x) - \frac{Wx^N y(x)}{1-bx^{N+2}}. \quad \text{(A13)}$$

The exact solution $y_n(x)$ can be expressed as[21]

$$y_n(x) = (-1)^n C_2 (N+2)^n (\sigma)_{n2}\mathrm{F}_1(-n, \rho+n; \sigma; bx^{N+2}), \quad \text{(A14)}$$

where the following notations has been used

$$(\sigma)_n = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}, \quad \sigma = \frac{2m+N+3}{N+2},$$
$$\rho = \frac{(2m+1)b+2a}{(N+2)b}. \quad \text{(A15)}$$

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