Analytic solutions of the double ring-shaped Coulomb potential in quantum mechanics*

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The exact solutions of the Schrödinger equation with the double ring-shaped Coulomb potential are presented, including the bound states, continuous states on the "$\kappa/2\pi$ scale", and the calculation formula of the phase shifts. The polar angular wave functions are expressed by constructing the so-called super-universal associated Legendre polynomials. Some special cases are discussed in detail.

Keywords: double ring-shaped Coulomb potential, super-universal associated-Legendre polynomial, functional analysis method

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1. Introduction

As is well known, an important part of quantum chemistry and also of nuclear physics is to study ro-vibrational energy levels of the molecules, atoms, and distorted nucleus. Up to now, the non-central potentials have been extensively studied in the literature since they provide a useful theoretical foundation for describing the interaction between the ring-shaped molecules and that between distorted nucleus.[1–29] For example, the Hartmann potential[1] is a kind of non-central physical potential, which arises from adding a potential proportional to $(r\sin\theta)^{-2}$ to a Coulomb potential. In the spherical coordinates (r,θ,ϕ), it is defined by

$$V(r,\theta)=\eta\sigma^2\left(\frac{2a_0}{r}-q\eta\frac{a_0^2}{r^2\sin^2\theta}\right)\varepsilon_0, \tag{1}$$

where $a_0=\hbar^2/Me^2$ and $\varepsilon_0=-Me^4/2\hbar^2$ represent the Bohr radius and ground state energy of the hydrogen atom, respectively, and η, σ, q are three dimensionless parameters. When $q=0$ and $\eta\sigma^2=Z$, the Hartmann potential reduces to the Coulomb potential. Due to its possible applications to ring-shaped organic molecules like cyclic polygenes and benzene, a number of contributions have been devoted to this potential since 1972.[1–15]

Double ring-shaped Coulomb potential,[18] the Coulomb potential that surrounds two ring-shaped inversed square potentials, may be used to describe the properties of ring-shaped organic molecules. This potential is given by

$$V(r,\theta)=\eta\sigma^2\left(\frac{2a_0}{r}-q\eta\frac{a_0^2}{r^2\sin^2\theta}\right)\varepsilon_0+\frac{\hbar^2}{2M}\frac{c}{r^2\cos^2\theta},\quad c\geq 0, \tag{2}$$

where c is a dimensionless parameter. When $c=0$ and $q=c=0$, $\eta\sigma^2=Z$, it goes back to the Hartmann potential and the Coulomb potential, respectively. In general, the ranges of a unique set of spherical coordinates for the Hartmann potential and the Coulomb potential[1–15] are chosen as $0<r<\infty$, $0\leq\theta\leq\pi$, and $0\leq\varphi\leq 2\pi$. However, there exists a singularity at $\theta=\pi/2$ for potential (2), but the ranges of other two variables r and φ are kept unchanged. The bound states of the Schrödinger equation with the double ring-shaped Coulomb potential have been investigated by supersymetry quantum mechanics (SUSYQM) and shape invariance.[18] Recently, other more complicated double ring-shaped Coulomb potentials have also been proposed.[19–29] Unfortunately, a fatal mathematical problem has occurred in all double ring-shaped Coulomb potentials, i.e., these results cannot go back to those of the Hartmann potential and the Coulomb potential, respectively when the potential parameters $c=0$ and $q=c=0$, $\eta\sigma^2=Z$. This motivates us to addressing this problem.

The purpose of this paper is two-fold. First, we attempt to study the Schrödinger equation with the double ring-shaped Coulomb potential and present the exact solutions using traditional functional analysis method. Second, it is also the

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most important contribution to this topic that we show how the present results go back to those of the Hartmann potential and the Coulomb potential when these potential parameters $c=0$ and $q=c=0$, $\eta\sigma^2=Z$. This can be realized by introducing a new variable $x=\cos\theta$ rather than $x=\cos^2\theta$ or $\sin^2\theta$,[18–27] and constructing the super-universal associated Legendre polynomial. The difference between $x=\cos\theta\in[-1,1]$ and $x=\cos^2\theta\in[0,1]$ might become the main reason why the previously obtained results cannot reduce to those of the Hartmann potential and the Coulomb potential. On the other hand, the introduced super-universal associated Legendre polynomial has its own interest in mathematical physics.

The rest of this paper is organized as follows. In Section 2 we present the exact solutions of polar angular differential equation. The key point is how to solve it since there exists a singularity at $\theta=\pi/2$. After some complicated algebraic manipulations, we find that the polar angular equation can be transformed into a super-universal associated-Legendre differential equation. In Section 3 we give the normalized eigenfunctions and eigenvalues of bound states. Section 4 is devoted to the exact solutions of the continuous states. The normalized radial wave functions of the continuous states on the "$\kappa/2\pi$ scale" and the calculation formula of the phase shifts are presented explicitly. In Section 5 we discuss some analytical properties of the scattering amplitude. Finally, we summarize this work in Section 6.

2. Exact solutions of the polar angular equation

In the spherical coordinates the Schrödinger equation with the double ring-shaped Coulomb potential is written as ($\hbar=M=e=1$)

$$\left[-\frac{1}{2}\nabla^2-\frac{\eta\sigma^2}{r}+\frac{q\eta^2\sigma^2}{2r^2\sin^2\theta}+\frac{c}{2r^2\cos^2\theta}\right]\Psi(r,\theta,\varphi)=E\Psi(r,\theta,\varphi). \tag{3}$$

We take the wave functions of the form $\Psi(r,\theta,\varphi)=r^{-1}u(r)H(\theta)\mathrm{e}^{\mathrm{i}m\varphi}$, where $m=0,\pm1,\pm2,\ldots$. Substitution of this into Eq. (3) allows us to obtain a set of second-order differential equations

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2}+\left(2E+\frac{2\eta\sigma^2}{r}-\frac{\lambda}{r^2}\right)u(r)=0, \tag{4}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}H(\theta)}{\mathrm{d}\theta}\right)+\left(\lambda-\frac{q\eta^2\sigma^2+m^2}{\sin^2\theta}-\frac{c}{\cos^2\theta}\right)H(\theta)=0, \tag{5}$$

where λ is a separation constant. The boundary conditions of Eq. (4) are $u(0)=0$ and $u(\infty)=0$ for the bound states, while $u(0)=0$ for the continuous states. For Eq. (5), however, the $H(0)$, $H(\pi/2)$, and $H(\pi)$ are taken as some finite values.

Define

$$m'=\sqrt{q\eta^2\sigma^2+m^2},\quad \lambda=L(L+1),\quad x=\cos\theta, \tag{6}$$

then equation (5) can be written as a super-universal associated-Legendre differential equation (The reason why we call it a super-universal associated Legendre differential equation is that it will reduce to universal and associated Legendre differential equations, respectively, when the potential parameters $c=0$ and $q=c=0$.)

$$(1-x^2)\frac{\mathrm{d}^2H(x)}{\mathrm{d}x^2}-2x\frac{\mathrm{d}H(x)}{\mathrm{d}x}+\left[L(L+1)-\frac{(m')^2}{1-x^2}-\frac{c}{x^2}\right]H(x)=0. \tag{7}$$

When $c=0$, $L=l'=k+\sqrt{q\eta^2\sigma^2+m^2}$, $|m|$, $k=0,1,2,\ldots$, equation (7) reduces to a universal associated-Legendre differential equation, whose solutions are given by universal associated-Legendre polynomials[13,30]

$$\begin{aligned}H_{l'm'}(x)&=N_{l'm'}\mathrm{P}_{l'm'}(x)\\&=\sqrt{\frac{(2l'+1)(l'-m')!}{2\Gamma(l'+m'+1)}}\left(1-x^2\right)^{m'/2}\\&\quad\times\sum_{\nu=0}^{[(l'-m')/2]}\frac{(-1)^\nu\Gamma(2l'-2\nu+1)}{2^{l'}\nu!(l'-m'-2\nu)!\Gamma(l'-\nu+1)}\\&\quad\times x^{l'-m'-2\nu}.\end{aligned} \tag{8}$$

Since $k=l'-m'=0,1,2,\ldots$, the summation part becomes even or odd polynomials for even or odd k.

Since $x=0,\pm1$ are three regular singular points, based on the above super-universal associated-Legendre differential equation, we take the form

$$H(x)=(1-x^2)^{m'/2}x^\gamma y(x), \tag{9}$$

where γ is taken as $\gamma(\gamma-1)-c=0$, i.e.,

$$\gamma=\begin{cases}0\ \text{or}\ 1, & c=0,\\ [1+\sqrt{1+4c}]/2, & c>0.\end{cases} \tag{10}$$

Substitution of Eq. (9) into Eq. (7) leads to the following differential equation

$$(1-x^2)\frac{\mathrm{d}^2y(x)}{\mathrm{d}x^2}+\left[\frac{2\gamma}{x}-2(m'+\gamma+1)x\right]\frac{\mathrm{d}y(x)}{\mathrm{d}x}+\left[L(L+1)-(m'+\gamma)(m'+\gamma+1)\right]y(x)=0. \tag{11}$$

Now, we come to solve this equation by series method. Choosing the following form

$$y(x)=\sum_{\nu=0}^{\infty}a_\nu x^\nu \tag{12}$$

and substituting this into Eq. (11) yield

$$\sum_{\nu=0}^{\infty}[(\nu+2)(\nu+1+2\gamma)a_{\nu+2} - [(\gamma+m'+\nu)(\gamma+m'+\nu+1) - L(L+1)]a_\nu]x^\nu + 2\gamma a_1 x^{-1} = 0, \quad (13)$$

from which we obtain the following recursion relations

$$a_{\nu+2} = \frac{(\gamma+m'+\nu)(\gamma+m'+\nu+1)-L(L+1)}{(\nu+2)(\nu+1+2\gamma)}a_\nu \quad (14)$$

and $2\gamma a_1 = 0$.

We study them in two different cases. The first case is the case where $c=0$, $\gamma=0$, and $L=l'=k+m'$ corresponds to the Hartmann potential. Thus, equation (14) is simplified into

$$a_{\nu+2} = \frac{(m'+\nu)(m'+\nu+1)-l'(l'+1)}{(\nu+2)(\nu+1)}a_\nu, \quad (15)$$

which is nothing but the recursion relation for the universal associated-Legendre polynomials.[13,30] The second case is the case where $c>0$, $\gamma=\gamma_1=[\sqrt{1+4c}+1]/2$, and $a_1=0$ implies that the series can be expanded only in even power, i.e., $y(x)=\sum_{\nu=0}^{\infty}a_{2\nu}x^{2\nu}$. Thus, equation (14) can be modified into

$$a_{2\nu+2} = \frac{(\gamma_1+m'+2\nu)(\gamma_1+m'+2\nu+1)-L(L+1)}{(2\nu+2)(2\nu+1+2\gamma_1)} \times a_{2\nu}, \quad (16)$$

from which we find

$$\left|\frac{a_{2\nu}}{a_{2\nu+2}}\right| = 1+\frac{1-m'}{\nu}+O(1/\nu^2). \quad (17)$$

Since $1-m'=1-\sqrt{q\eta^2\sigma^2+m^2}\le 1$, then infinite series (12) diverges when $x=\pm 1$.[31] Thus, we have to choose the following formula in terms of Eq. (16):

$$L = 2k+m'+\gamma_1 = 2k+\sqrt{q\eta^2\sigma^2+m^2}+\gamma_1, \quad |m|,k=0,1,2,\ldots. \quad (18)$$

From Eqs. (16) and (18), we have $a_{2k}\neq 0$ but $a_{2k+2}=0$. Thus, all succeeding even coefficients must also be equal to zero. If we write it in decreasing powers of the argument x, the first term is $a_{2k}x^{2k}$ and succeeding terms are given by Eq. (16). Substituting Eq. (18) into Eq. (16), after some algebraic manipulations we have

$$a_{2\nu} = -\frac{(\nu+1)(\nu+\gamma_1+1/2)}{(k-\nu)(L+\nu-k+1/2)}a_{2\nu+2}. \quad (19)$$

According to the following two special recursion relations

$$a_{2k-2} = -\frac{k(k+\gamma_1-1/2)}{1(L-1/2)}a_{2k}, \quad (20)$$

$$a_{2k-4} = (-1)^2\frac{k!\Gamma(k+\gamma_1+1/2)\Gamma(L+1/2-2)}{2!(k-2)!\Gamma(L+1/2)\Gamma(k+\gamma_1+1/2-2)} \times a_{2k}, \quad (21)$$

we are able to obtain a more general recursion relation,

$$a_{2k-2\nu} = \frac{(-1)^\nu k!}{\nu!(k-\nu)!}\frac{\Gamma(2k+2\gamma_1+1)}{\Gamma(k+\gamma_1+1)}\frac{\Gamma(L+1)}{\Gamma(2L+1)} \times \frac{\Gamma(k+\gamma_1-\nu+1)}{\Gamma(2k+2\gamma_1-2\nu+1)}\frac{\Gamma(2L-2\nu+1)}{\Gamma(L-\nu+1)} \times a_{2k}. \quad (22)$$

Thus, equation (12) can be written as

$$y(x)=\sum_{\nu=0}^{k}a_{2k-2\nu}x^{2k-2\nu}. \quad (23)$$

If we choose the expansion coefficient of the highest term in series

$$a_{2k} = \frac{1}{2^L k!}\frac{\Gamma(k+\gamma_1+1)}{\Gamma(2k+2\gamma_1+1)}\frac{\Gamma(2L+1)}{\Gamma(L+1)}, \quad (24)$$

then substituting Eqs. (24) and (22) into Eq. (23), we obtain

$$y(x) = \sum_{\nu=0}^{k}\frac{(-1)^\nu}{2^L\nu!(k-\nu)!}\frac{\Gamma(k+\gamma_1-\nu+1)}{\Gamma(2k+2\gamma_1-2\nu+1)} \times \frac{\Gamma(2L-2\nu+1)}{\Gamma(L-\nu+1)}x^{2k-2\nu}. \quad (25)$$

Thus, we may write down the polar angular wave functions (9) as follows:

$$\begin{aligned} H_{Lm'}(x) &= N_{Lm'}\mathrm{P}_L^{m'}(\gamma_1,x) \\ &= N_{Lm'}(1-x^2)^{m'/2}x^{\gamma_1}y(x) \\ &= N_{Lm'}(1-x^2)^{m'/2}x^{\gamma_1}\sum_{\nu=0}^{k}\frac{(-1)^\nu}{2^L\nu!(k-\nu)!} \\ &\quad\times\frac{\Gamma(k+\gamma_1-\nu+1)}{\Gamma(2k+2\gamma_1-2\nu+1)}\frac{\Gamma(2L-2\nu+1)}{\Gamma(L-\nu+1)} \\ &\quad\times x^{2k-2\nu}, \end{aligned} \quad (26)$$

where the $N_{Lm'}$ is calculated by the normalization condition $\int_{-1}^{+1}|H_{Lm'}(x)|^2\mathrm{d}x=1$ as

$$N_{Lm'} = 2^{\gamma_1}\sqrt{\frac{k!(2L+1)}{2\Gamma(L-k-\gamma_1+1)}\frac{\Gamma(2k+2\gamma_1+1)}{\Gamma(k+\gamma_1+1)}\frac{\Gamma(L-k+1)}{\Gamma(2L-2k+1)}}. \quad (27)$$

The $\mathrm{P}_L^{m'}(\gamma_1,x)$ defined in Eq. (26) is called the super-universal associated-Legendre polynomials. The cases where $\gamma_1=0,1$ and $\gamma_1>1$ correspond to $c=0$ and $c>0$, respectively.

We discuss a special case $c=0$, i.e., $\gamma_1=0,1$. First, when $c=0$ and $\gamma_1=0$, the corresponding L, normalization constant $N_{Lm'}$, and $\mathrm{P}_L^{m'}(\gamma_1,x)$ reduce to

$$L = 2k+m' = 2k+\sqrt{m^2+q\eta^2\sigma^2} = l',$$

$$|m|, k = 0, 1, 2, \ldots, \tag{28}$$

$$N_{l'm'} = \sqrt{\frac{(2l'+1)}{2}\frac{(l'-m')!}{\Gamma(l'+m'+1)}}, \tag{29}$$

$$\begin{aligned} \mathrm{P}_{l'}^{m'}(0,x) &= (1-x^2)^{m'/2} \\ &\times \sum_{v=0}^{k} \frac{(-1)^v \Gamma(2l'-2v+1)}{2^{l'} v!(l'-m'-2v)!\Gamma(l'-v+1)} \\ &\times x^{2k-2v}, \end{aligned} \tag{30}$$

respectively. Based on these formulae, the polar angular wave functions (26) are simplified into

$$\begin{aligned} H_{l'm'}(x) &= N_{l'm'}\mathrm{P}_{l'm'}(0,x) \\ &= \sqrt{\frac{(2l'+1)}{2}\frac{(l'-m')!}{\Gamma(l'+m'+1)}}(1-x^2)^{m'/2} \\ &\times \sum_{v=0}^{k} \frac{(-1)^v \Gamma(2l'-2v+1)}{2^{l'} v!(l'-m'-2v)!\Gamma(l'-v+1)} \\ &\times x^{2k-2v}, \end{aligned} \tag{31}$$

where $l'-m'=2k$. They correspond to the polar angular wave functions of even power of the variable x in the Hartmann potential. Second, when $c=0$, $\gamma_1=1$, the quantum number L, normalization constant $N_{Lm'}$, and $\mathrm{P}_L^{m'}(\gamma_1,x)$ reduce to

$$L = 2k+m'+1 = 2k+1+\sqrt{m^2+q\eta^2\sigma^2} = l', \tag{32}$$

$$|m|, k = 0, 1, 2, \ldots,$$

$$N_{l'm'} = \sqrt{\frac{2(2l'+1)(l'-m')!}{\Gamma(l'+m'+1)}}, \tag{33}$$

$$\begin{aligned} \mathrm{P}_{l'm'}(1,x) &= (1-x^2)^{m'/2} x \sum_{v=0}^{k} \frac{(-1)^v}{2^{l'} v!(k-v)!} \\ &\times \frac{\Gamma(k+1-v+1)}{\Gamma(2k+2-2v+1)} \\ &\times \frac{\Gamma(2l'-2v+1)}{\Gamma(l'-v+1)} x^{2k-2v} \\ &= \frac{1}{2}(1-x^2)^{m'/2} \sum_{v=0}^{k} \frac{(-1)^v}{2^{l'} v!(l'-m'-2v)!} \\ &\times \frac{\Gamma(2l'-2v+1)}{\Gamma(l'-v+1)} x^{2k+1-2v} \end{aligned} \tag{34}$$

respectively. Finally, the polar angular wave functions become

$$\begin{aligned} H_{l'm'}(x) &= N_{l'm'}\mathrm{P}_{l'm'}(1,x) \\ &= \sqrt{\frac{(2l'+1)}{2}\frac{(l'-m')!}{\Gamma(l'+m'+1)}}(1-x^2)^{m'/2} \\ &\times \sum_{v=0}^{k} \frac{(-1)^v \Gamma(2l'-2v+1)}{2^{l'} v!(l'-m'-2v)!\Gamma(l'-v+1)} \\ &\times x^{2k+1-2v}, \end{aligned} \tag{35}$$

where $l'-m'=2k+1$. They correspond to the polar angular wave functions of the odd power of the variable x in the Hartmann potential. That is to say, the super-universal associated-Legendre polynomial reduces to universal associated-Legendre polynomial for $c=0$. The values of the quantum number L are taken as

$$\begin{aligned} L &= 2k+m'+\gamma_1 \\ &= \left\{ \begin{array}{l} 2k+\sqrt{m^2+q\eta^2\sigma^2}+[1+\sqrt{1+4c}]/2, \\ \quad c>0,\ |m|,\ k=0,1,2\ldots, \\ 2k+\sqrt{m^2+q\eta^2\sigma^2} \\ 2k+1+\sqrt{m^2+q\eta^2\sigma^2} \end{array} \right\} \\ &\quad = k+\sqrt{m^2+q\eta^2\sigma^2} = l', \\ &\quad c=0,\ |m|,\ k=0,1,2\ldots. \end{aligned} \tag{36}$$

3. Exact solutions of the bound states

We now study Eq. (4). Substitution of Eq. (6) into Eq. (4) leads to

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} + \left(2E + \frac{2\eta\sigma^2}{r} - \frac{L(L+1)}{r^2}\right)u(r) = 0, \tag{37}$$

which is nothing but the Columbic potential. For the bound states $E<0$, taking $\rho=\tau r$, $\tau=\sqrt{-8E}$, $s=2\eta\sigma^2/\tau = \eta\sigma^2/\sqrt{-2E}$, then equation (37) can be rewritten as

$$\frac{\mathrm{d}^2u(\rho)}{\mathrm{d}\rho^2} + \left(\frac{s}{\rho} - \frac{1}{4} - \frac{L(L+1)}{\rho^2}\right)u(\rho) = 0. \tag{38}$$

Taking the physically acceptable solution $u(\rho) = \rho^{L+1}\mathrm{e}^{-\rho/2}f(\rho)$ and substituting this into the above equation yield

$$\begin{aligned} \rho\frac{\mathrm{d}^2f(\rho)}{\mathrm{d}\rho^2} + [2(L+1)-\rho]\frac{\mathrm{d}f(\rho)}{\mathrm{d}\rho} \\ - [L+1-s]f(\rho) = 0, \end{aligned} \tag{39}$$

whose solutions are the confluent hypergeometric functions $\mathrm{F}(\lambda,\gamma;\rho)$ with the parameters $\lambda=[(L+1)-s]$ and $\gamma=2(L+1)$. For the bound states, however, only the polynomial solutions are acceptable, which requires $(L+1)-s=-n_r$, $n_r=0,1,2,\ldots$. Thus, we are able to obtain the eigenvalues

$$E = -\frac{(\eta\sigma^2)^2}{2(n_r+L+1)^2} = -\frac{(\eta\sigma^2)^2}{2N^2}, \tag{40}$$

where

$$\begin{aligned} N &= n_r+L+1 \\ &= \left\{ \begin{array}{l} n_r+2k+\sqrt{m^2+q\eta^2\sigma^2}+[3+\sqrt{1+4c}]/2, \\ \quad c>0, n_r, |m|, k=0,1,2\ldots, \\ n_r+k+\sqrt{m^2+q\eta^2\sigma^2}+1 = n_r+l'+1, \\ \quad c=0, n_r, |m|, k=0,1,2\ldots. \end{array} \right. \end{aligned} \tag{41}$$

The corresponding radial eigenfunctions are proportional to the generalized Laguerre polynomials

$$\mathrm{u}_{NL}(r) = \left[\frac{\eta\sigma^2 n_r!}{N^2\Gamma(N+L+1)}\right]^{1/2}\left(\frac{2\eta\sigma^2 r}{N}\right)^{L+1}$$

$$\times \exp\left(-\frac{\eta\sigma^2 r}{N}\right) \mathrm{L}_{n_r}^{2L+1}\left(\frac{2\eta\sigma^2 r}{N}\right). \quad (42)$$

As discussed above, the special case $c=0$ corresponds to the Hartmann potential. In this case, the super-universal associated-Legendre polynomials reduce to universal associated-Legendre polynomials

$$\begin{aligned} L &= k+\sqrt{m^2+q\eta^2\sigma^2}=l', \\ N &= n_r+l'+1=n', \\ E &= -\frac{(\eta\sigma^2)^2}{2(n')^2}, \end{aligned} \quad (43)$$

$$\begin{aligned} \mathrm{u}_{n'l'}(r) &= \left[\frac{\eta\sigma^2 n_r!}{(n')^2\Gamma(n'+l'+1)}\right]^{1/2}\left(\frac{2\eta\sigma^2 r}{n'}\right)^{l'+1} \\ &\times \exp\left(-\frac{\eta\sigma^2 r}{n'}\right)\mathrm{L}_{n_r}^{2l'+1}\left(\frac{2\eta\sigma^2 r}{n'}\right), \end{aligned} \quad (44)$$

which coincide with the results by solving the Hartmann potential directly.[13] Furthermore, when $q=c=0$ and $\eta\sigma^2=Z$, the super-universal associated-Legendre polynomials reduce to associated-Legendre polynomials

$$L = k+m=l,\ N=n_r+l+1=n,\ E=-\frac{Z^2}{2n^2}, \quad (45)$$

$$\begin{aligned} \mathrm{u}_{nl}(r) &= \left[\frac{Zn_r!}{n^2\Gamma(n+l+1)}\right]^{1/2}\left(\frac{2Zr}{n}\right)^{l+1} \\ &\times \exp\left(-\frac{Zr}{n}\right)\mathrm{L}_{n_r}^{2l+1}\left(\frac{2Zr}{n}\right), \end{aligned} \quad (46)$$

which are consistent with the results of bound states for the Coulomb potential.

4. Exact solutions of the continuous states

As is well known, the double ring-shaped Coulomb potential is a non-central field potential, but it is independent of the azimuth angle φ. Therefore, the solutions of the Schrödinger equation for the continuous states must be axially symmetric about the z axis, i.e., the direction of the incident particles. Due to this fact, the continuous states can be discussed by the partial-wave method.

For continuous states $E>0$, we take $\kappa=\sqrt{2E}>0$. Thus, we may modify Eq. (37) as

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2}+\left(\kappa^2+\frac{2\eta\sigma^2}{r}-\frac{L(L+1)}{r^2}\right)u(r)=0, \quad (47)$$

with a boundary condition being $u(0)=0$. From the asymptotic behavior of radial wave functions of the continuous states as $r\to 0$, we take the wave functions of the form

$$u(r)=A(\kappa r)^{L+1}\,\mathrm{e}^{\mathrm{i}\kappa r}f(r). \quad (48)$$

Substituting this into Eq. (47), we have

$$r\frac{\mathrm{d}^2f(r)}{\mathrm{d}r^2}+(2L+2+2\mathrm{i}\kappa r)\frac{\mathrm{d}f(r)}{\mathrm{d}r} + \left[2\mathrm{i}\kappa(L+1)+2\eta\sigma^2\right]f(r)=0. \quad (49)$$

Introducing a new variable $z=-2\mathrm{i}\kappa r$, equation (49) is then re-expressed as

$$\begin{aligned} &z\frac{\mathrm{d}^2f(z)}{\mathrm{d}z^2}+(2L+2-z)\frac{\mathrm{d}f(z)}{\mathrm{d}z} \\ &-(L+1-\mathrm{i}\eta\sigma^2/\kappa)f(z)=0, \end{aligned} \quad (50)$$

whose analytical solutions with $r\to 0$ are the confluent hypergeometric functions

$$f(r)=\mathrm{F}(L+1-\mathrm{i}\eta\sigma^2/\kappa,2L+2;-2\mathrm{i}\kappa r). \quad (51)$$

Thus, the radial wave functions of the continuous states are expressed as

$$\begin{aligned} u_{\kappa L}(r) &= A_{\kappa L}(\kappa r)^{L+1}\,\mathrm{e}^{\mathrm{i}\kappa r} \\ &\times \mathrm{F}(L+1-\mathrm{i}\eta\sigma^2/\kappa,2L+2;-2\mathrm{i}\kappa r). \end{aligned} \quad (52)$$

We now study the asymptotic form of the above expression for large r and calculate the normalization constant $A_{\kappa L}$ of radial wave functions and the phase shifts δ_l'. The asymptotic expression of the confluent hypergeometric functions is given by[31]

$$\begin{aligned} \mathrm{F}(\lambda,\gamma,z) &\to \frac{\Gamma(\gamma)}{\Gamma(\lambda)}\mathrm{e}^z z^{\lambda-\gamma} \\ &+\frac{\Gamma(\gamma)}{\Gamma(\gamma-\lambda)}\mathrm{e}^{\pm\mathrm{i}\pi\lambda}z^{-\lambda},\ |z|\to\infty, \end{aligned} \quad (53)$$

where "+" and "−" correspond to $-\pi/2<\arg z<3\pi/2$ and $-3\pi/2<\arg z<\pi/2$, respectively. When $z=-2\mathrm{i}kr=|z|\,\mathrm{e}^{-\mathrm{i}\pi/2}$, equation (53) is then re-expressed as

$$\begin{aligned} \mathrm{F}(\lambda,\gamma,z) &\to \frac{\Gamma(\gamma)}{\Gamma(\lambda)}\mathrm{e}^z|z|^{\lambda-\gamma}\,\mathrm{e}^{-\mathrm{i}\pi(\lambda-\gamma)/2} \\ &+\frac{\Gamma(\gamma)}{\Gamma(\gamma-\lambda)}\mathrm{e}^{-\mathrm{i}\pi\lambda/2}|z|^{-\lambda}, \end{aligned} \quad (54)$$

from which, we have

$$\begin{aligned} &\mathrm{F}(L+1-\mathrm{i}\eta\sigma^2/\kappa,2L+2;-2\mathrm{i}\kappa r) \\ &\xrightarrow[r\to\infty]{} \frac{\Gamma(2L+2)}{\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)}\mathrm{e}^{-2\mathrm{i}\kappa r}(2\kappa r)^{-(L+1+\mathrm{i}\eta\sigma^2/\kappa)} \\ &\times \mathrm{e}^{\mathrm{i}\pi(L+1+\mathrm{i}\eta\sigma^2/\kappa)/2} \\ &+\frac{\Gamma(2L+2)}{\Gamma(L+1+\mathrm{i}\eta\sigma^2/\kappa)}\mathrm{e}^{-\mathrm{i}\pi(L+1-\mathrm{i}\eta\sigma^2/\kappa)/2} \\ &\times (2\kappa r)^{-(L+1-\mathrm{i}\eta\sigma^2/\kappa)}. \end{aligned} \quad (55)$$

Defining

$$\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)=\left|\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)\right|\mathrm{e}^{\mathrm{i}\delta_L}, \quad (56)$$

one has

$$\Gamma(L+1+\mathrm{i}\eta\sigma^2/\kappa) = \left|\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)\right| \mathrm{e}^{-\mathrm{i}\delta_L}, \quad (57)$$

where δ_L is a real number. Equation (55) then becomes

$$\begin{aligned} &\mathrm{F}(L+1-\mathrm{i}\eta\sigma^2/\kappa, 2L+2; -2\mathrm{i}\kappa r) \\ \xrightarrow[r\to\infty]{} &\frac{\Gamma(2L+2)\,\mathrm{e}^{-\pi\eta\sigma^2/2\kappa}\,\mathrm{e}^{-\mathrm{i}\kappa r}}{\left|\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)\right|(2\kappa r)^{L+1}} \\ &\times \left[\mathrm{i}\,\mathrm{e}^{-\mathrm{i}[\kappa r+\delta_L-\pi L/2+\eta\sigma^2(\ln 2\kappa r)/\kappa]}\right. \\ &\left. - \mathrm{i}\,\mathrm{e}^{\mathrm{i}[\kappa r+\delta_L-\pi L/2+\eta\sigma^2(\ln 2\kappa r)/\kappa]}\right]. \end{aligned} \quad (58)$$

Substituting Eq. (58) into Eq. (52) leads to

$$\begin{aligned} &\mathrm{u}_{\kappa L}(r) \\ \xrightarrow[r\to\infty]{} &\frac{A_{\kappa L}\Gamma(2L+2)\,\mathrm{e}^{-\pi\eta\sigma^2/2\kappa}}{\left|\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)\right|2^{L+1}} \\ &\times 2\sin\left(\kappa r+\delta_L-\pi L/2+\eta\sigma^2(\ln 2\kappa r)/\kappa\right). \end{aligned} \quad (59)$$

In terms of the following asymptotic behavior:

$$\mathrm{u}_{\kappa l}(r) \xrightarrow[r\to\infty]{} 2\sin\left(\kappa r+\delta_l-l\pi/2+Z(\ln 2\kappa r)/\kappa\right), \quad (60)$$

we know from Ref. [32] that the radial wave functions of the continuous states for the Coulomb potential are normalized on the "$\kappa/2\pi$ scale". Since the double ring-shaped Coulomb potential is the Coulomb potential surrounded by two ring-shaped inverse square potentials, which are short distance potentials, the asymptotic expression of the wave functions for large r will not be affected by the additional contribution of the two ring-shaped inverse squared potentials. In other words, the asymptotic expression of the double ring-shaped Coulomb potential is equal to that of the Coulomb potential when $r\to\infty$, i.e.,

$$\mathrm{u}_{\kappa L}(r) \xrightarrow[r\to\infty]{} 2\sin\left(\kappa r+\delta'_l-\pi l/2+\eta\sigma^2(\ln 2\kappa r)/\kappa\right). \quad (61)$$

The wave functions of the continuous states for the double ring-shaped Coulomb potential are normalized on the "$\kappa/2\pi$ scale", too. In comparison of Eq. (61) with Eq. (59), we may obtain the normalization constant of the continuous states as

$$A_{\kappa L} = \frac{2^{L+1}\left|\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)\right|\mathrm{e}^{\pi\eta\sigma^2/2\kappa}}{\Gamma(2L+2)}, \quad (62)$$

and the phase shifts δ'_l as

$$\begin{aligned} \delta'_l &= \delta_L+\pi(l-L)/2 \\ &= \arg\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)+\pi(l-L)/2. \end{aligned} \quad (63)$$

Substituting Eq. (62) into Eq. (52), we obtain the normalized wave functions of the continuous states on the "$\kappa/2\pi$ scale" as

$$\begin{aligned} \mathrm{u}_{\kappa L}(r) = &\frac{2^{L+1}\left|\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)\right|\mathrm{e}^{\pi\eta\sigma^2/2\kappa}}{\Gamma(2L+2)}(\kappa r)^{L+1} \\ &\times \mathrm{e}^{\mathrm{i}\kappa r}\mathrm{F}(L+1-\mathrm{i}\eta\sigma^2/\kappa, 2L+2; -2\mathrm{i}\kappa r). \end{aligned} \quad (64)$$

We discuss a special case $c=0$, i.e., the Hartmann potential case. In this case, the super-universal associated-Legendre polynomials reduce to universal associated-Legendre polynomials

$$\begin{aligned} L &= k+\sqrt{m^2+q\eta^2\sigma^2} = l', \\ \delta'_l &= \arg\Gamma(l'+1-\mathrm{i}\eta\sigma^2/\kappa)+\pi(l-l')/2, \quad (65) \\ \mathrm{u}_{\kappa l'}(r) &= \frac{2^{l'+1}\left|\Gamma(l'+1-\mathrm{i}\eta\sigma^2/\kappa)\right|\mathrm{e}^{\pi\eta\sigma^2/2\kappa}}{\Gamma(2l'+2)}(\kappa r)^{l'+1} \\ &\quad \times \mathrm{e}^{\mathrm{i}\kappa r}\mathrm{F}(l'+1-\mathrm{i}\eta\sigma^2/\kappa, 2l'+2; -2\mathrm{i}\kappa r). \quad (66) \end{aligned}$$

They coincide with the results obtained by studying the continuous states of the Hartmann potential directly.[15] Furthermore, when $q=c=0$ and $\eta\sigma^2=Z$, the super-universal associated-Legendre polynomials reduce to associated-Legendre polynomials

$$\begin{aligned} L &= k+m=l, \quad \delta_l = \arg\Gamma(l+1-iZ/\kappa), \quad (67) \\ \mathrm{u}_{\kappa l}(r) &= \frac{2^{l+1}\left|\Gamma(l+1-\mathrm{i}Z/\kappa)\right|\mathrm{e}^{\pi Z/2\kappa}}{\Gamma(2l+2)}(\kappa r)^{l+1} \\ &\quad \times \mathrm{e}^{\mathrm{i}\kappa r}\mathrm{F}(l+1-\mathrm{i}Z/\kappa, 2l+2; -2\mathrm{i}\kappa r), \quad (68) \end{aligned}$$

which coincide with the results of scattering states for Coulomb potential.

5. Analytical properties of the scattering amplitude

From the general theory of the partial-wave method, the scattering amplitude is defined by

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)\left[\frac{\mathrm{e}^{2\mathrm{i}\delta'_l}-1}{2i\kappa}\right]\mathrm{P}_l(\cos\theta), \quad (69)$$

where l is the angular quantum number. We now discuss the analytical properties of the scattering amplitude, i.e., we discuss the analytical properties of the scattering amplitude in the whole complex κ plane by regarding the scattering amplitude as the function of the energy. To this end, from Eqs. (63) and (69) we discuss the analytical properties of $\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)$. On the basis of the definition of the Gamma function

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \cdots, \quad (70)$$

we know that $z=0,-1,-2,\ldots$ are the first order poles of the $\Gamma(z)$. That is to say, the first order poles of $\Gamma(L+1-\mathrm{i}\eta\sigma^2/\kappa)$ are situated at

$$L+1-\mathrm{i}\eta\sigma^2/\kappa = 0,-1,-2,\ldots$$

$$= -n_r, \quad (n_r = 0, 1, 2, \ldots). \tag{71}$$

At the poles of the scattering amplitude, the corresponding energy levels are given by

$$E = \frac{\kappa^2}{2} = -\frac{(\eta\sigma^2)^2}{2(n_r + L + 1)^2} = -\frac{(\eta\sigma^2)^2}{2N^2}, \tag{72}$$

which is exactly the energy equation of the bound states. Substituting Eq. (71) into Eq. (64) yields

$$\begin{aligned} &\mathrm{u}_{NL}(r) \\ &= N'_{NL}\left(\frac{2\eta\sigma^2 r}{N}\right)^{L+1} \\ &\quad \times \exp\left(-\frac{\eta\sigma^2 r}{N}\right) \mathrm{F}\left(-n_r, 2L+2; \frac{2\eta\sigma^2 r}{N}\right), \end{aligned} \tag{73}$$

which implies that $\mathrm{F}\left(-n_r, 2L+2; 2\eta\sigma^2 r/N\right)$ reduces to a polynomial of degree n_r since n_r is a positive integer or zero. Making use of the relation between the confluent hypergeometric function and the generalized Laguerre polynomial $\mathrm{L}_n^{\alpha} \propto \mathrm{F}(-n, \alpha+1; x)$,[31] we can rewrite Eq. (73) as

$$\begin{aligned} \mathrm{u}_{NL}(r) &= N_{NL}\left(\frac{2\eta\sigma^2 r}{N}\right)^{L+1} \\ &\quad \times \exp\left(-\frac{\eta\sigma^2 r}{N}\right) \mathrm{L}_{n_r}^{2L+1}\left(\frac{2\eta\sigma^2 r}{N}\right), \end{aligned} \tag{74}$$

which is the exact radial wave functions of the bound state. In other words, the energies of particles and the radial wave functions will reduce to those of the bound states at the poles of the scattering amplitude.

6. Concluding remarks

The pending unsolved problem, i.e., double ring-shaped Coulomb potential (2) is not continuous at $\theta \to \pi/2$, is solved by introducing super-universal associated-Legendre polynomials. It is found that the results presented here can go back to those of the Hartmann potential and Coulomb potential respectively when the potential parameters $c = 0$ and $q = c = 0$, respectively. Before ending this work, we are going to make two useful remarks on this work. First, due to the importance of the proposed super-universal associated Legendre polynomials, we shall further study their possible mathematical properties, such as the orthogonal relations, generating functions, the recurrence relations and others. Second, we have recognized that the present double ring-shaped Coulomb potential (2) composed of three terms has caused the symmetry to break from the original free hydrogen atom $\mathrm{O}(4) \supset \mathrm{O}(3) \supset \mathrm{O}(2)$ symmetries to the present O(2) symmetry due to the surrounded two ring-shaped inversed square potentials. That is to say, the latter two terms in Eq. (2) break the symmetry at the O(4) and O(3) levels, but dihedral angles do not change, thus keeping the O(2) symmetry. Just due to this reason, we introduce the so-called super-universal associated Legendre polynomials to study the polar angular differential equation.

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