A geometric phase for superconducting qubits under the decoherence effect

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We propose a relaxation rate or dissipative cavity-based parameters that can be used as indicators of the stationary limit of a mixed state geometric phase. We perform our considerations for the system of a superconducting qubit in an open transmission line or interacting with a dissipative cavity. This system is very useful for performing an effective quantum computation by exhibiting the long collapse time of the geometric phase. It is shown that the geometric phase in the stationary limit does not depend on interaction time if the decay time exceeds an upper bound.

Keywords: superconducting-qubit, meometric phase, mixed state, decoherence

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1. Introduction

The profound interplay and interrelation of geometry and physics has been the focus in both fields since the creation of the general theory of relativity, in which the quantities responsible for the geometry of space–time are determined by the physical properties of the matter living in this space and *vice versa*. The phase is a fundamental notion in quantum mechanics,[1,2] and the study of geometric phases (GPs) is an attempt to understand quantum mechanics better. The notion of gauge symmetry or equivalence class plays a basic role in the analysis of GPs.[3–5] It has been shown recently that the gauge symmetry without gauge fields is identified with the hidden local gauge symmetry inherent in the second quantization, namely, the arbitrariness of the phase choice of coordinates in the functional space.[6,7] The GP is defined as a path integral and, via the Stokes theorem, can be converted into a surface integral. Therefore, it behaves like a geometric area. A quantity like an area is less dependent on the details of time evolution and therefore is less affected by changes of environmental conditions or an imperfect control, and hence, is typically more robust. This is the key attribute that makes GPs attractive for the implementations of fault-tolerant quantum computation. Several propositions have been presented to realize this objective, e.g. in NMR experiments,[8] ion traps,[9] neutral atoms in cavity QED,[10] quantum dots[11] or Josephson junction devices.[12] The performance of holonomic quantum gates under various conditions has been studied recently. There have been many proposals to extend the notion of the GP for open systems as, e.g., based on the state purification, quantum trajectories, and quantum interferometry (kinematic approach). It can be applied to low-dimensional open systems for which the reduced dynamics is known. One of the most 'popular' guidelines for the description of reduced dynamics is the Lindblad form of the master equation. This approach has been successfully applied in reducing decoherence or dynamics of quantum nonlocal correlations. In this context the GPs gained by time-evolving quantum systems attract considerable attention. They are present either in adiabatically[13] or arbitrarily[14] evolving systems.

The discovery of the GP is one of the most interesting and intriguing findings of the last few decades.[15–21] These investigations provided a deeper understanding of the concept of phase in quantum mechanics. Also, the GP can be considered as a basic tool of geometric quantum computing and can be controlled in implementing the operation of a universal quantum logic gate.[20,21] Sjöqvist *et al.*[22] have discovered a mixed state GP restricted to unitary evolving density operators in the experimental context of one-particle interferometry. On the other hand, the GPs for mixed states proposed in Refs. [23]–[25] are generically different in the unitary case and match only under very special conditions such as in the limit of pure states. Uhlmann[23,24] addressed the issue of a mixed state GP in the presence of the decoherence effect. This has

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been achieved by lifting the density operator to a pure state by attaching an ancilla and letting the combined system be parallel transported along a specific purified path.

Through a simple two-level system, a possibility to control the geometric Pancharatnam phase of a quantum system on a much more sensitive scale than the population dynamics has been proposed.[17] This effect paves the way to many applications. In interferometry the detection of a sensitive quantity can be used to improve the stability of the interferometer itself, leading to very high spatial or temporal resolution. This effect can also be used for the improvement of previous techniques based on interferometry such as Ramsey spectroscopy, wave packets, and neutron interferometry. However the calculation of the GP for open quantum systems in the presence of decoherence is very complicated, and numerical techniques must be used in some cases.

In this present work, the GP induced by decoherence with a superconducting charge qubit in open space or in a dissipative cavity is theoretically investigated. The outline of the paper is as follows. In the next section, the charge qubit model is described, the corresponding energy eigenvalues and eigenvectors are found, and GP of the superconducting charge qubit is identified. Section 3 examines in detail the GPs of the present system and its subsystems. The paper ends with the conclusions.

2. Model of superconducting qubits

Several schemes have been proposed for implementing quantum computer hardware in solid state quantum electronics. These schemes use electric charge, magnetic flux, superconducting phase, electron spin, or nuclear spin as the information bearing degree of freedom.[26] In this communication, we consider a superconducting (SC)-qubit coupled to a superconducting-coplanar transmission line resonator. The Cooper pair box (an SC-qubit) is a mesoscopic SC-island,[27–29] which is connected to a large reservoir through a Josephson junction with Josephson energy $E_{\rm J}$ and capacitance $C_{\rm J}$.[30] It is biased by a classical voltage source $V_{\rm g}$ through a gate capacitance $C_{\rm g}$ to the island. If the SC-gap is larger than both the charging energy $E_{\rm C}=e^2/2C_{\rm t}$ (where $C_{\rm t}=C_{\rm J}+C_{\rm g}$ is the total box capacitance) and temperature and if $4E_{\rm C}\gg E_{\rm J}$ and restricting the gate charge to the range $0\leq N_{\rm g}\leq 1$ (where $N_{\rm g}=C_{\rm g}V_{\rm g}/2e$ is the dimensionless gate charge representing the total polarization charge injected into the island by the voltage source), then devices can be modeled as two-level systems with the Hamiltonian $H_{\rm A}=-\frac{1}{2}(E_{\rm el}\hat{\sigma}_z+E_{\rm J}\hat{\sigma}_x)$ given in Refs. [27]–[29]; where $E_{\rm el}$ is the electrostatic energy and $E_{\rm J}$ is the Josephson coupling energy. The electrostatic energy $E_{\rm el}=4E_{\rm c}(1-N_{\rm g})$ is proportional to the charging energy $E_{\rm C}$ and can be tuned by a gate charge $N_{\rm g}$. The Pauli spin matrices $\hat{\sigma}_z$ and $\hat{\sigma}_x$ are in the charge basis; that is, the basis states correspond to either $|g\rangle$ or $|e\rangle$ excess Cooper-pair charges on the island.

The one-dimensional coplanar waveguide resonator consists of a narrow center conductor and two nearby lateral ground planes.[27–29] Close to its resonance frequency $\omega_{\rm r}=1/\sqrt{LC}$, the resonator can be modeled as a parallel combination of a capacitor C and an inductor L. At the charge degeneracy point (where $N_{\rm g}=1/2$), the Hamiltonian of the interacting qubit and resonator system takes the form

$$\hat{H}=\hbar\omega_{\rm r}\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)+\frac{E_{\rm J}}{2}\hat{\sigma}_z-\frac{eC_{\rm g}}{C_{\rm t}}\sqrt{\frac{\hbar\omega_{\rm r}}{LC}}(\hat{a}^{\dagger}+\hat{a})\hat{\sigma}_x. \quad (1)$$

On the other hand, as is well known, no system can be completely isolated from its environment. In real experimental situations, due to coupling to the environment, the entangled state inevitably loses its purity and becomes mixed. This phenomenon of decoherence is the most dangerous obstacle for all entanglement manipulations. Here, we consider only the sources of decoherence acting only on the qubit. If the cavity mode is assumed to be resonant with the qubit transition frequency, then under the rotating wave approximation the Hamiltonian in the interaction picture, $\hat{H}_{\rm I}$, is given by

$$\hat{H}_{\rm I}=\frac{eC_{\rm g}}{C_{\rm t}}\sqrt{\frac{\hbar\omega_{\rm r}}{LC}}(\hat{a}\hat{\sigma}_{+}+\hat{a}^{\dagger}\hat{\sigma}_{-}). \quad (2)$$

The master equation that describes the time evolution of the system is given by[31–35]

$$\begin{aligned}\frac{{\rm d}\hat{\rho}(t)}{{\rm d}t} &= \kappa_{\rm l}\left(2\hat{a}^{\dagger}\hat{a}\hat{\rho}(t)\hat{a}^{\dagger}\hat{a}-\left(\hat{a}^{\dagger}\hat{a}\right)^2\hat{\rho}(t)-\hat{\rho}(t)\left(\hat{a}^{\dagger}\hat{a}\right)^2\right)\\ &\quad -{\rm i}\eta[\hat{a}\hat{\sigma}_{+}+\hat{a}^{\dagger}\hat{\sigma}_{-},\hat{\rho}(t)].\end{aligned}$$

Here $\eta=(eC_{\rm g}/C_{\rm t})\sqrt{\hbar\omega_{\rm r}/LC}$ and $\kappa_{\rm l}$ is the qubit decay constant and the environment is assumed to be at zero temperature, i.e., no thermal excitation is included. This is simply the master equation for a driven, decaying two-state qubit dressed by coupling to a harmonic-oscillator system.

The analytic methods for obtaining the exact solution of Eq. (3) in the case of a high-Q cavity ($\kappa_{\rm l}\ll\eta$) obey the dressed-states representation (see Ref. [36]), i.e., representation consisting of the complete set of the Hamiltonian eigenstates is used. The full set of dressed states in the case of exact resonance are

$$|\psi_n^{\pm}\rangle=\frac{1}{\sqrt{2}}(|e,n\rangle\pm|g,n+1\rangle),\quad (n=0,1,2,\ldots). \quad (3)$$

The eigenvalues of $\hat{H}_{\rm I}$ are given by:

$$\begin{aligned}\hat{H}_{\rm I}|\downarrow,0\rangle &= 0,\\ \hat{H}_{\rm I}|\psi_n^{\pm}\rangle &= \pm\mu_n|\psi_n^{\pm}\rangle=\pm\eta\,(n+1)\,|\psi_n^{\pm}\rangle. \quad (4)\end{aligned}$$

Let us assume that the initial state of the system is the product state $\hat{\rho}(0) = \rho^{\mathrm{R}}(0) \otimes \rho^{\mathrm{SC}}(0)$, with the SC-qubit initially in the the mixed

$$\rho^{\mathrm{SC}}(0) = \cos^2(\theta)|e\rangle\langle e| + \sin^2(\theta)|g\rangle\langle g|. \tag{5}$$

While the initial state of the resonator is given by

$$\rho^{\mathrm{R}}(0) = |\alpha\rangle\langle\alpha| = \sum_{n,m=0}^{\infty} b_n(\alpha) b_m^*(\alpha) |n\rangle\langle m|, \tag{6}$$

and $b_n(\alpha) = \alpha^n \sqrt{\mathrm{e}^{-|\alpha|^2}/n!}$.

The solution of the master equation (3) under the above initial conditions can be written as

$$\begin{aligned}\hat{\rho}(t) = \frac{1}{2} \sum_{n,m=0}^{\infty} \sum_{\mathrm{r,s}=\pm} \sum_{k,\ell=0}^{1} & \exp\Big\{-\kappa_1 t\Big[(n-m)^2 + \frac{1}{2}\Big]\Big\} \\ & \times \exp\big\{-\mathrm{i}t(r\mu_m - s\mu_n)\big\} \\ & \times \big\{\langle\psi_m^{(r)}|\hat{\rho}(0)|\psi_n^{(s)}\rangle\big\{1 + \delta_{nm}\delta_{\mathrm{rs}}[\cosh(\kappa_1 t) - 1]\big\} \\ & + \delta_{nm}\delta_{\mathrm{rs}}\langle\psi_m^{(-r)}|\hat{\rho}(0)|\psi_n^{(-s)}\rangle \sinh(\kappa_1 t)\big\} \\ & \times |1-\ell, m+\ell\rangle\langle 1-k, n+k|. \end{aligned} \tag{7}$$

The decoherence effect on the dynamical evolution of the present system will be discussed through the phase-damping constant κ_1.

3. Geometric phase

The different generation of the GP from pure states to mixed states of quantum systems has turned out to be non-unique.[37–40] For a two-level quantum system, described by the total density operator $\rho(t)$, the GP can be written as

$$\begin{aligned}\phi_G(t) \;=\; & -\arg\Big\{\sum_{j=1}^{N} \sqrt{\lambda_j(0)\lambda_j(t)}\langle\xi_j(0)|\xi_j(t)\rangle \\ & \times \exp\Big(-\int_0^t \Big\langle \xi_j(\tau)\Big|\frac{\mathrm{d}\xi_j(\tau)}{\mathrm{d}\tau}\Big\rangle \mathrm{d}\tau\Big)\Big\}. \end{aligned} \tag{8}$$

We denote by t the total evolutional time, where $\lambda_j(t)$ are $|\xi_j(t)\rangle$ are the eigenvalues and eigenstates of the density matrix of SC-qubit $\rho_{\mathrm{SC}}(t)$. For an SC-qubit system equation, the reduced density matrix can be written as

$$\rho_{\mathrm{SC}}(t) = \mathrm{Tr}_{\mathrm{R}}(\rho(t)) = \sum_{a,b=0}^{1} \rho_{ab}(t)|a\rangle\langle b|, \tag{9}$$

where Tr_{R} means that the trace is taken over the resonator, $\rho_{ab}(t) = \langle a|\rho_{\mathrm{SC}}|b\rangle$ and $a, b = 1, 0$. The eigenvalues of Eq. (9) for $j = (1,2) = (+,-)$ are given by

$$\lambda_{\mathrm{t}}^{\pm} = \frac{1}{2}\left(1 \pm \sqrt{(\rho_{11} - \rho_{00})^2 + 4|\rho_{01}|^2}\right)$$

and their corresponding eigenvectors are $|\xi_{\mathrm{t}}^{\pm}\rangle = \sin\vartheta_{\mathrm{t}}^{\pm}|1\rangle + \cos\vartheta_{\mathrm{t}}^{\pm}|0\rangle$ where

$$\vartheta_{\mathrm{t}}^{\pm} = \tan^{-1}\Big\{\frac{\rho_{10}}{\lambda_{\mathrm{t}}^{\pm} - \rho_{11}}\Big\}.$$

Based on Eq. (8), we present some interesting results for the effect of different parameters on the dynamics properties of SC-qubit in the presence of phase damping or decoherence terms. We recall that the time t has been scaled; one unit of time is given by the inverse of the coupling constant η. Here, our primary aim is to provide that the GP, as a goal feature of quantum evolution, is capable to reflect the information on the character of interaction between an SC-qubit and its environment. In Fig. 1, we display the GP as a function of the scaled time ηt for different values of the phase-damping parameter κ_1/η in the case of $\theta = \pi/4$. Let us notice two important features of our results, that is, the fringe visibility and phase offset. The results presented in Fig. 1 show significant differences in both the absence and presence of the phase damping as a function of excitation time. We begin by discussing the GP when $\kappa_1/\eta = 0$ where there is no exchange energy and information with the environment. In this considered case, the measurement of the GP for the set of initial states is fully encoded in the Hamiltonian of the SC-qubit system. From Fig. 1, we can see that the GP has a different order as a function of scaled time. After an initial change, the GP tends to stabilization with very rapid oscillatory behavior for significantly high values of the scaled time, which indicates that the SC-qubit is trapped by the optical field (see Fig. 1(a)). A great challenge is to investigate GPs in a quantum open system and for a mixed state. This is due to the fact that all realistic systems are coupled to their environment. Let us analyze the influence of the decoherence effect on the GP for different strength regimes of damping. From the numeral results presented in Fig. 1(b), it follows that for the weak damping to the environment (here the scaled dimensionless damping strength $\kappa_1/\eta = 0.01$) the GP oscillations are shown to be increased as the time goes on. However, not all oscillations are shifted by the same amount; in particular, excitation from the two different ground states yields results that are significantly out of phase. To explain this relative phase shift we refer to Fig. 1, in which the amplitude of the ϕ_G decreases as the phase damping parameter increased. The first and most dominant effect is due to the exponential factor of the matrix elements. For strong values of the phase damping (say $\kappa_1/\eta = 0.5$), we see that the GP no longer exists and ϕ_{G} tends to zero for a large interaction time. This case has not been observed for weak damping. It is shown that ϕ_{G} has no oscillations as time goes on when $\kappa_1 t$ exceeds a threshold value $\kappa_1 t_{\mathrm{c}}$. Here we should point out that the effective threshold value of the decay time corresponding

to the transition of the geometric phase from oscillatory behavior to fully linear function is independent of t. In this situation, the GP remains unaffected by the environment exhibiting a collapse time which is due to a high symmetry of the model in this limit. As discussed above, the GP is insensitive to the environment as the time becomes significantly larger. However, it does not mean that the SC-qubit becomes uncoupled from the environment. If we display the behavior of its corresponding von Neumann entropy, will show an variation in the purity system,[41] which clearly indicates that the SC-qubit is affected by the environment. So, in zero temperature, the quantum-mechanical properties are more transparent because the "classical" sources of dissipation and decoherence are frozen and the fluctuations are unavoidable due to vacuum fluctuations of the environment.

From these results, the phase damping may restrain and stabilize the GP amplitude during the time evolution. In this context, the robustness of the GP with respect to environment is a basic condition for an effective quantum computation for an SC-qubit system. Moreover, the results obtained here suggest that one can exploit the GP as a quantifier characterizing a nature of the system damping.

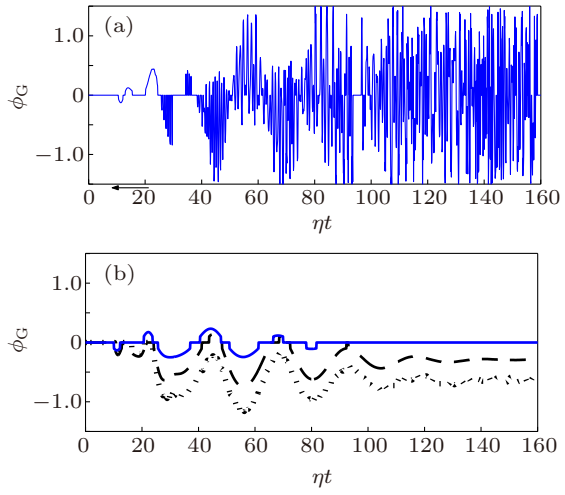


Fig. 1. (color online) Mixed state geometric phase against the scaled time ηt. The parameters are $\alpha = 5$, $\theta = \pi/4$ and for different values of decoherence parameter κ_1/η, where panal 1(a) for $\kappa_1/\eta = 0.0$, and panel 1(b) for $\kappa_1/\eta = 0.01$ (dotted curve), $\kappa_1/\eta = 0.1$ (dashed curve), and $\kappa_1/\eta = 0.5$ (solid curve).

To understand how decoherence affects the exponential enhancement of the ϕ_G it is useful to consider a new value of the initial state parameter θ such as $\theta = 2\pi/3$ as shown in Fig. 2. It is noticed that the GP shows different behaviors. It is shown that the parameter θ plays a prominent role in determining the maximum GP, providing a new way to control the system dynamics. Our calculated mixed state GP is drastically different from the usual pure state GP derived in previous work, since as the decoherence parameter increased, the GP shows asymptotic behavior and does not depend on time.

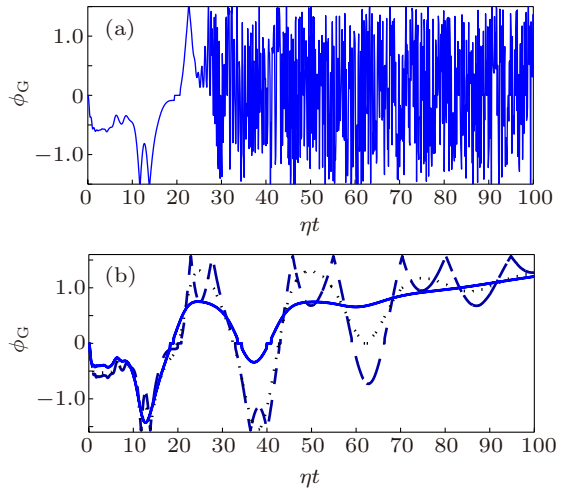


Fig. 2. (color online) Mixed state geometric phase against the scaled time ηt. The parameters are $\alpha = 5$, $\theta = 2\pi/3$ and with different values of decoherence parameter κ_1/η, where (a) $\kappa_1 = 0.0$, (b) $\kappa_1/\eta = 0.1$ (dashed curve), $\kappa_1/\eta = 0.5$ (dotted curve), and $\kappa_1/\eta = 0.8$ (solid curve).

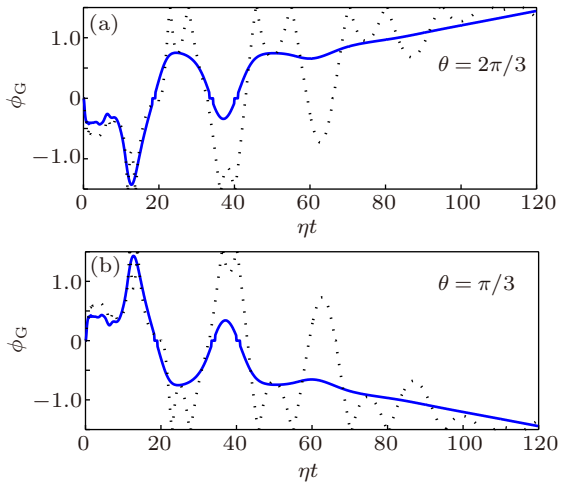


Fig. 3. (color online) Mixed state geometric phase against the scaled time ηt. The parameters are $\alpha = 5$, with different values of κ_1/η and θ, where $\kappa_1/\eta = 0.1$ (dotted curve), and $\kappa_1/\eta = 0.8$ (solid curve).

In Fig. 3 we consider the following basic question: is it possible for the environment to induce a GP for all values of the mixed state parameter, θ? To answer this question we studied the impact of decoherence for different values of θ due to weak coupling and strong coupling to the environment and display here two cases ($\theta = 2\pi/3$, and $\theta = \pi/3$). The interesting observation of this choice is that $\phi_G(\eta t, 2\pi/3) = -\phi_G(\eta t, \pi/3)$, i.e., there is a symmetric behavior around the x axis. For the decoherence which is always present (though possibly not the dominant channel), we already found that the GP always increases with increasing the decoherence parameter γ: even though, for some values of θ ($0 < \theta \leq \pi/2$), the

GP can be kept under control for a bath which is well-behaved in the opposite direction to the case in which $\pi/2 < \theta \leq \pi$ (see Fig. 3) the existence of many oscillations converging near the initial interaction have been caused by weak decoherence effect. This growth can be slowed down a bit via adapted different schemes for the interaction of the SC-qubit with the field, but not stopped. Since the present model is considered as a prototypical example for a superconducting charge qubits, we expect our results to reflect general features of the general superconducting transitions.

Figure 4 shows the dynamics of ϕ_G as a function of the mixed state parameter θ and for fixed value of the scaled time ηt. It is noticed that ϕ_G has a periodic behavior with period which equals the scaled time ηt. On the other hand, ϕ_G takes maximum values at $\eta t = (2m-1)\pi/2$, $m = 1,2,3\ldots$, while the minimum values appears at $\eta t = 2m\pi$. In Fig. 4 (dashed and solid) curves examine the effect of the decoherence parameter on the evolution of the GP. It is observed that κ_1/η leads to decreasing the oscillation of ϕ_G around the half of periodic, also the amplitude of ϕ_G increases as κ_1/η increases.

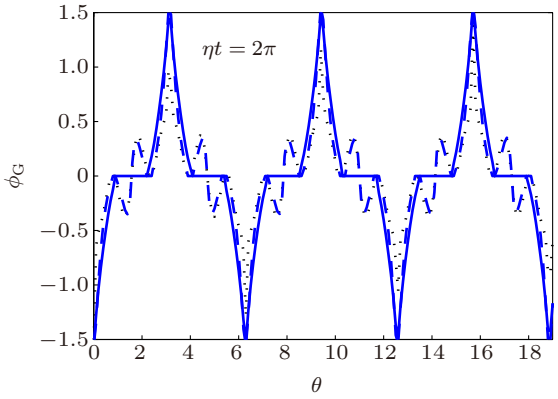


Fig. 4. (color online) Mixed state geometric phase against the mixed state parameter θ, for $t/\eta = 2\pi$, $\alpha = 5$, with different values of κ_1/η, where $\kappa_1 = 0$ (dotted curve), $\kappa_1/\eta = 0.05$ (dashed curve), and $\kappa_1/\eta = 0.1$ (slid curve).

4. Conclusion

We have shown that the geometric phase plays a crucial role in a variety of physical problems and has observable consequences in a wide range of quantum systems by testing the fundamentals of quantum dynamics and details of interactions modeled by Hamiltonians. In particular, we identified a new scheme for a mixed state geometric phase for critical current and charge noise in superconducting devices based on a macroscopic two-level system which is in a quantitative agreement with the predictions of quantum optics for a point-like scatterer interacting with the electromagnetic field in one-dimensional open space. We observed a very close connection between the structures of the mixed-state geometric phase due to the decoherence effect for different values of the mixed state parameter, θ. At strong coupling with the environment we have identified a qualitatively different behavior of the mixed state geometric phase. We have provided a criterion delimitating the parameter range at which these processes dominate the geometric phase. Due to its simplicity and clear geometric structure the identified definition the mixed state geometric phase may be important for quantum information and quantum computing applications, where superconducting charge qubits dynamics serves as a building block of quantum computation schemes.[42] The mixed state geometric phase described in this paper can be used not only for time-independent problems, but also for some problems of spatial propagation of light pulses, such as the coupling between directional waveguide, and multi-mode optical fibers. We expect that similar results, revealing a deeper link with decoherence theory, can be found for multi-level and multi-qubit systems. The present model, describing two-state dynamics, can be generalized to include more complex exactly solvable multi-state models. Therefore, we believe that the effect of the parameter κ_1 proposed in this paper could be a useful tool for controlling the geometric phase.

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