On certain new exact solutions of the Einstein equations for axisymmetric rotating fields

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We investigate the Einstein field equations corresponding to the Weyl–Lewis–Papapetrou form for an axisymmetric rotating field by using the classical symmetry method. Using the invariance group properties of the governing system of partial differential equations (PDEs) and admitting a Lie group of point transformations with commuting infinitesimal generators, we obtain exact solutions to the system of PDEs describing the Einstein field equations. Some appropriate canonical variables are characterized that transform the equations at hand to an equivalent system of ordinary differential equations and some physically important analytic solutions of field equations are constructed. Also, the class of axially symmetric solutions of Einstein field equations including the Papapetrou solution as a particular case has been found.

Keywords: Einstein equations, symmetry analysis, exact solutions

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1. Introduction

From the beginning of the theory of general relativity, there has been a sustained search for new exact solutions to Einstein's equations. The exact solutions to Einstein's field equations play very important roles in the discussion of physical problems. The literature abounds with the many different techniques that have been invoked in an effort to obtain new exact solutions to Einstein's field equations.[1]

Here, we consider the Weyl–Lewis–Papapetrou form for an axisymmetric rotating field as

$$\mathrm{d}s^2 = f\,\mathrm{d}t^2 - 2k\,\mathrm{d}\phi\,\mathrm{d}t - l\,\mathrm{d}\phi^2 - \exp(\mu)(\mathrm{d}\rho^2 + \mathrm{d}z^2), \quad (1)$$

where f, k, l, and μ are all functions of ρ and z and the coordinates ρ, z, and ϕ corresponds to cylinderical polar coordinates. The external gravitational field is described by the symmetric Ricci tensor $R_{\mu\nu}$ which obeys the exterior Einstein vacuum equations

$$R_{\mu\nu} = 0. \quad (2)$$

We label the coordinates as before: $(x^0, x^1, x^2, x^3) = (t, \rho, z, \phi)$. Three of Eq. (2) can be written as follows:

$$\begin{aligned}
2\exp(\mu)D^{-1}R_{00} &= (D^{-1}f_\rho)_\rho + (D^{-1}f_z)_z \\
&\quad + D^{-3}f(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0, \\
-2\exp(\mu)D^{-1}R_{03} &= (D^{-1}k_\rho)_\rho + (D^{-1}k_z)_z \\
&\quad + D^{-3}k(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0, \\
-2\exp(\mu)D^{-1}R_{33} &= (D^{-1}l_\rho)_\rho + (D^{-1}l_z)_z \\
&\quad + D^{-3}l(f_\rho l_\rho + f_z l_z + k_\rho^2 + k_z^2) = 0, \quad (3)
\end{aligned}$$

where $D^2 = fl + k^2$.

From Eq. (3) one gets the following equation

$$\exp(\mu)D^{-1}(lR_{00} - 2kR_{03} - fR_{33}) = D_{\rho\rho} + D_{zz} = 0. \quad (4)$$

Thus the function D satisfies the two-dimensional Laplace equation in the variables ρ and z. It follows that D can be considered as the real part of an analytic function $\Sigma(\rho + \imath z)$ of $(\rho + \imath z)$. Let E be the imaginary part of $\Sigma(\rho + \imath z)$, that is,

$$\Sigma(\rho + \imath z) = D(\rho, z) + \imath E(\rho, z). \quad (5)$$

The Cauchy–Riemann equations imply

$$D_\rho = E_z, \quad D_z = -E_\rho. \quad (6)$$

Consider now the transformation from (ρ, z) to $(\bar\rho, \bar z)$ given by

$$\bar\rho = D(\rho, z), \quad \bar z = E(\rho, z). \quad (7)$$

Because of Eq. (6) we have

$$(\mathrm{d}\bar\rho)^2 + (\mathrm{d}\bar z)^2 = (D_\rho^2 + D_z^2)(\mathrm{d}\rho^2 + \mathrm{d}z^2). \quad (8)$$

Equation (8) shows that the form of the metric (1) is unaltered by the transformation (7), since we can define a new function $\bar\mu$ given by

$$\exp(\bar\mu) = \exp\mu({D_\rho}^2 + {E_\rho}^2)^{-1}. \quad (9)$$

We can assume that all the functions $f, k, l, \bar\mu$ have been expressed in terms of the variables $(\bar\rho, \bar z)$ obtained by substituting for (ρ, z) after solving for the latter from Eq. (7). Having expressed all functions in terms of $(\bar\rho, \bar z)$, we can drop the bars so that because of Eq. (7) we are left with the following algebraic relation in f, k, and l as

$$D^2 = fl + k^2 = \rho^2. \quad (10)$$

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The above procedure was first used by Weyl (1917) for the axially symmetric static metric (with $k=0$) and generalized to the present case by Lewis (1932). We drop the bar from f_i as well, so that the rest of the nontrivial Einstein equations (with the use of Eq. (10)) can be written as follows:

$$\begin{aligned} 2R_{11} &= -\mu_{\rho\rho} - \mu_{zz} + \rho^{-1}\mu_\rho + \rho^{-2}(f_\rho l_\rho + k_\rho^2) = 0, \\ 2R_{12} &= \rho^{-1}\mu_z + \frac{1}{2}\rho^{-2}(f_\rho l_z + f_z l_\rho + 2k_\rho k_z) = 0, \\ 2R_{22} &= -\mu_{\rho\rho} - \mu_{zz} - \rho^{-1}\mu_\rho + \rho^{-2}(f_z l_z + k_z^2) = 0. \end{aligned} \quad (11)$$

Because of Eq. (10), only two of Eq. (11) are independent. It is more convenient to use the function u and v instead of f and k respectively defined by $u = f^{-1}$ and $v = f^{-1}k$. By eliminating l and k from Eq. (3), the Einstein field equations reduce to the following system of equations

$$u\left(u_{\rho\rho} + u_{zz} + \frac{u_\rho}{\rho}\right) - u_\rho^2 - u_z^2 - \frac{(v_\rho^2 + v_z^2)}{\rho^2} = 0, \quad (12)$$

$$u\left(v_{\rho\rho} + v_{zz} - \frac{v_\rho}{\rho}\right) - 2u_\rho v_\rho - 2u_z v_z = 0, \quad (13)$$

$$u^2 w_\rho = u u_\rho + \frac{1}{2}\rho(u_\rho^2 - u_z^2) - \frac{1}{2\rho}(v_\rho^2 - v_z^2), \quad (14)$$

$$u^2 w_z = u u_z + \rho u_\rho u_z - \frac{1}{\rho} v_\rho v_z. \quad (15)$$

So, we have four equations (12)–(15) for the determination of three unknowns u, v, and w. Thus, we solve Eqs. (12) and (13) for u and v and w can be obtained trivially from Eqs. (14) and (15), once u and v are known.

Symmetries in general relativity have been the subject of much study in recent years, partly because of the considerable simplification of Einstein's equations resulting from the assumption of one or more symmetries, partly because of the interest in the geometric significance of the symmetries, which are described by vector fields of certain geometrical objects on the manifold, and partly because of the possible physical significance of the existence of these symmetries. We will examine the symmetries of Einstein equations (12) and (13) by using the Lie classical method[2–5] and explore the reduction by using symmetries.

The detailed plan of the paper is as follows: Section 2 is devoted to generating various symmetries of the Einstein equation by the Lie classical method and an optimal system comprising basic vector fields is identified. Section 3 contains the reduced ordinary differential equations (ODEs) and their exact solutions. In Section 4, the extension of Papapetrou class of solutions is obtained. Some conclusions are drawn in Section 5.

2. Symmetry analysis

In this section, we will perform Lie group method for Eqs. (12) and (13). For a detailed study of Lie group theory the interested reader is referred to the well-known books.[2–5] Once one has determined the symmetry group of a system of differential equations, a number of applications become available. Some important recent contributions for exact solutions of partial differential equations with the help of the well-known Lie classical method are presented in Refs. [6]–[16].

Firstly, let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned} \rho^* &= \rho + \varepsilon\xi(\overline{X}, \overline{\sigma}) + O(\varepsilon^2), \\ z^* &= z + \varepsilon\tau(\overline{X}, \overline{\sigma}) + O(\varepsilon^2), \\ \overline{\sigma}^* &= \overline{\sigma} + \varepsilon\overline{\phi}(\overline{X}, \overline{\sigma}) + O(\varepsilon^2), \end{aligned} \quad (16)$$

where $\overline{X} = (\rho, z)$, $\overline{\sigma} = (u, v)$, $\overline{\phi} = (\eta, \phi)$, which leaves Eqs. (12) and (13) invariant. The vector field associated with the above group of transformations can be written as follows:

$$\begin{aligned} V = \xi(\rho, z, u, v)\frac{\partial}{\partial\rho} + \tau(\rho, z, u, v)\frac{\partial}{\partial z} \\ + \eta(\rho, z, u, v)\frac{\partial}{\partial u} + \phi(\rho, z, u, v)\frac{\partial}{\partial v}. \end{aligned} \quad (17)$$

Here, we will obtain the symmetry groups of Eqs. (12) and (13) by using the Lie classical method. We obtained that the symmetries under which the Eqs. (12) and (13) are invariant can be spanned by the following five linearly independent infinitesimal generators

$$\begin{aligned} &V_1 = 2uv\frac{\partial}{\partial u} + (v^2 + \rho^2 u^2)\frac{\partial}{\partial v}, \quad V_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad V_3 = \frac{\partial}{\partial v}, \\ &V_4 = \rho\frac{\partial}{\partial\rho} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}, \quad V_5 = \frac{\partial}{\partial z}. \end{aligned} \quad (18)$$

It is easy to verify that V_j, $j = 1, 2, 3, 4, 5$ is closed under the Lie bracket.

We classify the symmetry algebra of system into conjugacy inequivalent subalgebra under the adjoint action of the symmetry group. Based on the adjoint representations of the vector fields, we obtain the optimal system[4] of Eqs. (12) and (13) as follows:

(i) V_5,

(ii) V_4,

(iii) $V_3 + \lambda V_4$,

(iv) $V_2 + \mu V_4$,

(v) $V_1 + \alpha V_3 + \beta V_4$, (19)

where λ, μ, α, and β are arbitrary constants.

3. Group-invariant solutions

One of the main purposes for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. In the next section, we will obtain the vector fields and the optimal system of Eqs. (12) and (13). In this section, the primary focus is on the

reductions associated with the vector fields (19) and attempt to find some exact solutions.

Case 1 V_5

Corresponding to this vector field, the form of the similarity variable and similarity solution is as follows:

$$\zeta=\rho,\ \ u(\rho,z)=F(\zeta),\ \ v(\rho,z)=G(\zeta).$$

On using these in Eqs. (12) and (13), the following system of reduced ODEs is obtained

$$\Big(F''+\frac{F'}{\zeta}\Big)F-F'^2-\frac{G'^2}{\zeta^2}=0,$$
$$\Big(G''-\frac{G'}{\zeta}\Big)F-2F'G'=0. \tag{20}$$

The solution of the reduced system of ODEs (20) are obtained and and reverting back to the original variables, the solution of system equations (12) and (13) is deduced as:

$$u(\rho,z)=\rho^{c_1}c_2,\quad v(\rho,z)=c_3, \tag{21}$$

and

$$u(\rho,z)=\pm\frac{c_1^2}{\sqrt{-c_2c_1^3\rho\Big(\cosh\Big(\frac{\sqrt{c_1c_2}(\ln\rho-c_4)}{2}\Big)\Big)}},$$
$$v(\rho,z)=-\Big[\frac{c_2c_3+\tanh\Big(\frac{c_4\sqrt{c_1c_2}}{2}-\frac{\ln(\rho)\sqrt{c_1c_2}}{2}\Big)\sqrt{c_1c_2}}{c_2}\Big], \tag{22}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

By making substitutions of these expressions (21) and (22) for $u(\rho,z)$ and $v(\rho,z)$ in Eqs. (14) and (15) and solving for $w(\rho,z)$, we get

$$w(\rho,z)=\frac{1}{2}c_1^2\ln(\rho)+c_1\ln(\rho)+c_5,$$

and

$$w(\rho,z)=-\frac{1}{2}\ln(\rho)+\frac{c_1c_2}{8}\ln(\rho)+c_6,$$

where c_5 and c_6 are arbitrary constants.

Case 2 V_4

For this vector field, the form of the similarity variable and similarity solution are as follows:

$$\zeta=\frac{\rho}{z},\ \ u(\rho,z)=\frac{F(\zeta)}{z},\ \ v(\rho,z)=G(\zeta).$$

Corresponding to these similarity variables, the reduced system of ODEs are as follows:

$$\Big[F''(1+\zeta^2)+F'\Big(\frac{1}{\zeta}+4\zeta\Big)+2F\Big]F-F'^2$$
$$-(-F'\zeta-F)^2-G'^2\Big(\frac{1}{\zeta^2}+1\Big)=0,$$
$$\Big[G''(1+\zeta^2)+G'\Big(-\frac{1}{\zeta}+2\zeta\Big)\Big]F-2F'G'$$
$$+2\zeta G'(-F'\zeta-F)=0. \tag{23}$$

We can obtain the solution of the system (23) and, further by back substitution to original variables, the exact solution of Eqs. (12) and (13) is given by

$$u(\rho,z)=\frac{\pm c_2\imath}{\sqrt{\rho^2+z^2}},\quad v(\rho,z)=c_1+\frac{c_2z}{\sqrt{\rho^2+z^2}}, \tag{24}$$

where c_1 and c_2 are arbitrary constants.

Using the above expressions for $u(\rho,z)$ and $v(\rho,z)$ in Eqs. (14) and (15) and solving, we get

$$w(\rho,z)=-\frac{1}{2}\ln(\rho^2+z^2)+c_3,$$

where c_3 is an arbitrary constant.

Case 3 $V_3+\lambda V_4$

For this vector field, the similarity variable and the form of the similarity solution are as follows:

$$\zeta=\frac{\rho}{z},\ \ u(\rho,z)=\frac{F(\zeta)}{z},\ \ v(\rho,z)=\frac{1}{\lambda}\ln(z)+G(\zeta).$$

On using these in Eqs. (12) and (13), the system of reduced ODEs is given by

$$\Big(F''(1+\zeta^2)+F'(\frac{1}{\zeta}+4\zeta)+2F\Big)F-F'^2$$
$$-(-F'\zeta-F)^2-\frac{G'^2}{\zeta^2}-\Big(-G'+\frac{1}{\lambda\zeta}\Big)^2=0,$$
$$\Big(G''(1+\zeta^2)+G'(-\frac{1}{\zeta}+2\zeta)-\frac{1}{\lambda}\Big)F-2F'G'$$
$$-2\Big(-\zeta G'+\frac{1}{\lambda}\Big)(-F'\zeta-F)=0. \tag{25}$$

After solving this system of ODEs (25), we obtain the solution of Eqs. (12) and (13) as follows:

$$u(\rho,z)=\frac{\pm\imath\Big(\operatorname{arctanh}\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)-c_1\lambda\Big)}{\lambda\sqrt{(\rho^2+z^2)}},$$
$$v(\rho,z)=\frac{1}{\lambda}\ln(\rho)-\frac{c_1z}{\sqrt{\rho^2+z^2}}$$
$$+\frac{z}{\lambda\sqrt{\rho^2+z^2}}\operatorname{arctanh}\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)+c_2, \tag{26}$$

where c_1 and c_2 are arbitrary constants.

Consequently, $w(\rho,z)$ is given by:

$$w(\rho,z)=\int\frac{\Big(c_1\lambda z-z\operatorname{arctanh}\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)+\sqrt{\rho^2+z^2}\Big)\mathrm{d}z}{\Big(\rho^2+z^2\Big)\Big(\operatorname{arctanh}\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)-c_1\lambda\Big)}$$
$$+\int\Big\{-(\rho^2+z^2)\Big(-\operatorname{arctanh}\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)+c_1\lambda\Big)$$
$$\times\int\Big[2\sqrt{\rho^2+z^2}\rho^2z\Big(\operatorname{arctanh}\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)\Big)^2$$
$$-\rho^2\big(4z\sqrt{\rho^2+z^2}c_1\lambda+z^2+\rho^2\big)\operatorname{arctanh}\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)$$
$$+z((2{c_1}^2\lambda^2+1)\rho^2+z^2)\sqrt{\rho^2+z^2}+\rho^2c_1\lambda\big(\rho^2+z^2\big)\Big]$$
$$\times\Big(-\operatorname{arctanh}\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)+c_1\lambda\Big)^{-2}\big(\rho^2+z^2\big)^{-5/2}\mathrm{d}z$$

$$+\arctanh\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)\rho^2+\sqrt{\rho^2+z^2}t-\rho^2c_1\lambda\Big\}$$
$$\times\left(\rho^2+z^2\right)^{-1}\Big(-\arctanh\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)+c_1\lambda\Big)^{-2}\mathrm{d}\rho$$
$$+c_3,$$

where c_3 is an arbitrary constant.

Case 4 $V_2+\mu V_4$

The similarity variable and the form of similarity solution are as follows:

$$\zeta=\frac{\rho}{z},\ \ u(\rho,z)=F(\zeta)z^{(1-\mu)/\mu},\ \ v(\rho,z)=z^{1/\mu}G(\zeta).$$

The reduced ODEs in this case is as follows:

$$\Big\{F''(1+\zeta^2)+F'\Big[\frac{1}{\zeta}+2\zeta-\frac{2\zeta(1-\mu)}{\mu}\Big]+F\frac{(1-\mu)^2}{\mu^2}$$
$$-F\frac{(1-\mu)}{\mu}\Big\}F-F'^2-\Big(-F'\zeta+F\frac{(1-\mu)}{\mu}\Big)^2$$
$$-\frac{G'^2}{\zeta^2}-\Big(-G'+\frac{G}{\mu\zeta}\Big)^2=0,$$
$$\Big[G''(1+\zeta^2)+G'\Big(-\frac{1}{\zeta}+2\zeta-\frac{2\zeta}{\mu}\Big)+G\Big(\frac{1}{\mu^2}-\frac{1}{\mu}\Big)\Big]F$$
$$-2\Big(-\zeta G'+\frac{G}{\mu}\Big)\Big(-F'\zeta+F\frac{(1-\mu)}{\mu}\Big)-2F'G'=0. \quad (27)$$

For $\mu=1$, we get final solution of Eqs. (12) and (13) as

$$u(\rho,z)=c_1\arctan\Big(\frac{z}{\sqrt{-\rho^2-z^2}}\Big)+c_2,$$
$$v(\rho,z)=c_1\sqrt{\rho^2+z^2}, \quad (28)$$

where c_1 and c_2 are arbitrary constants.

Making substitutions of these expressions for $u(\rho,z)$ and $v(\rho,z)$ in Eqs. (14) and (15) and solving for $w(\rho,z)$, we get

$$w(\rho,z)=\ln\Big[\arctan\Big(\frac{z}{\sqrt{(-\rho^2-z^2)}}\Big)c_1+c_2\Big]+c_3,$$

where c_3 is an arbitrary constant.

Case 5 $V_1+\alpha V_3+\beta V_4$

Corresponding to this case, the following form of the similarity variable and similarity solution is found

$$u(\rho,z)=\pm\frac{v(\rho,z)}{\rho},\ \ v(\rho,z)=\frac{1}{\frac{-2}{\beta}\ln(\rho)+\ln(G(\zeta))}.$$

Substituting these in Eqs. (12) and (13), the reduced ODE is given by

$$-G''G\zeta+\zeta G'^2-G'G-GG''\zeta^3-2\zeta^2G'G+G'^2\zeta^3=0. \quad (29)$$

Solving Eq. (29) and reverting back to the original variables, we thus get the following exact solution of the Eqs. (12) and (13)

$$u(\rho,z)=\frac{\pm1}{\rho\Big(\frac{-2\ln(\rho)}{\beta}+c_2-c_1\arctanh\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)\Big)},$$
$$v(\rho,z)=\frac{1}{\frac{-2\ln(\rho)}{\beta}+c_2-c_1\arctanh\Big(\frac{z}{\sqrt{\rho^2+z^2}}\Big)}, \quad (30)$$

and

$$w(\rho,z)=\frac{-1}{2}\ln(\rho)+c_3,$$

where c_1, c_2, and c_3 are arbitrary constants.

Also, by using the transformation $v(\rho,z)=\pm\rho u(\rho,z)$, in Eqs. (12) and (13), we get

$$u\Big(u_{\rho\rho}-\frac{u_\rho}{\rho}+u_{zz}\Big)-2u_\rho^2-2u_z^2-u^2=0. \quad (31)$$

Thus, the solution to Eqs. (12) and (13) is finally given by

$$u(\rho,z)=\frac{\imath k_5(-\mathrm{J}_1(\sqrt{-k_5}\rho)\mathrm{Y}_0(\sqrt{-k_5}\rho)+\mathrm{J}_0(\sqrt{-k_5}\rho)\mathrm{Y}_1(\sqrt{-k_5}\rho))}{(k_1\mathrm{J}_0(\sqrt{-k_5}\rho)-k_2\mathrm{Y}_0(\sqrt{-k_5}\rho))(k_3\sin(\sqrt{k_5}z)-k_4\cos(\sqrt{k_5}z))},$$
$$v(\rho,z)=\pm\frac{-\imath k_5\rho(-\mathrm{J}_1(\sqrt{-k_5}\rho)\mathrm{Y}_0(\sqrt{-k_5}\rho)+\mathrm{J}_0(\sqrt{-k_5}\rho)\mathrm{Y}_1(\sqrt{-k_5}\rho))}{(k_1\mathrm{J}_0(\sqrt{-k_5}\rho)-k_2\mathrm{Y}_0(\sqrt{-k_5}\rho))(k_3\sin(\sqrt{k_5}z)-k_4\cos(\sqrt{k_5}z))}, \quad (32)$$

where k_1, k_2, k_3, k_4, and k_5 are arbitrary constants.

Using Eqs. (14) and (15), corresponding to these expressions of $u(\rho,z)$ and $v(\rho,z)$, we obtain

$$w(\rho,z)=-\frac{1}{2}\ln(\rho)+k_6,$$

where k_6 is an arbitrary constant.

4. Extension of Papapetrou class of solutions

Following to the approach given in Ref. [17], one can select the function $v(\rho,z)$ satisfying the equation

$$v_{\rho\rho}+v_{zz}-\frac{v_\rho}{\rho}=0 \quad (33)$$

as

$$v(\rho,z)=k\rho\sigma_\rho, \quad (34)$$

where k is a constant.

By using Eq. (34), equation (33) can be written as

$$\sigma_{\rho\rho\rho}+\sigma_{\rho zz}+\left(\rho^{-1}\sigma_\rho\right)_\rho=0. \quad (35)$$

Integrating Eq. (35), we get

$$\sigma_{\rho\rho}+\sigma_{zz}+\frac{\sigma_\rho}{\rho}=\theta(z). \quad (36)$$

Therefore, we introduced an arbitrary function $\theta=\theta(z)$. In the case of $\theta(z)=0$, equation (36) becomes a Laplace equation, which is similar to the expression given in Ref. [17].

However, for $\theta(z)\neq 0$, we have Poisson-like equation (36) for the function θ. By using Eq. (34) for the $v(\rho,z)$ and Eq. (33), equation (13) becomes

$$-u_\rho(\sigma_{zz}-\theta)+u_z\sigma_{\rho z}=0. \tag{37}$$

From Eqs. (36) and (37), the general solution for the metric function u takes the following form

$$u=u(\bar{\sigma}), \tag{38}$$

where $\bar{\sigma}=\sigma_z-\int\theta(z)\mathrm{d}z$ and $u'=\partial u/\partial\bar{\sigma}$. Therefore, we can rewrite the derivatives of the metric functions u in the following ways

$$\begin{aligned}&u_\rho=u'\sigma_{\rho z},\quad u_{\rho\rho}=u''\sigma_{\rho z}^2+u'\sigma_{\rho\rho z},\quad u_z=u'(\sigma_{zz}-\theta(z)),\\&u_{zz}=u''(\sigma_{zz}-\theta(z))^2+u'(\sigma_{zzz}-\theta(z)_z).\end{aligned} \tag{39}$$

Using Eqs. (39) and (36), we get

$$\begin{aligned}&u_{\rho\rho}+u_{zz}+\frac{u_\rho}{\rho}=u''(\sigma_{\rho z}^2+(\sigma_{zz}-\theta)^2),\\&u_\rho^2+u_z^2=u'^2(\sigma_{\rho z}^2+(\sigma_{zz}-\theta)^2),\\&v_\rho^2+v_z^2=k^2\rho^2(\sigma_{\rho z}^2+(\sigma_{zz}-\theta)^2).\end{aligned} \tag{40}$$

Finally, after inserting the derived expressions into Eq. (12), we obtain

$$uu''-u'^2-k^2=0, \tag{41}$$

which look like the extension of the Papapetrou class of solutions

$$\begin{aligned}u(\rho,z)=\ &\alpha\cosh\left(\sigma_z-\int\theta(z)\mathrm{d}z\right)\\&+\beta\sinh\left(\sigma_z-\int\theta(z)\mathrm{d}z\right),\end{aligned} \tag{42}$$

where the functions θ and σ satisfy Eq. (36) and $k^2=\alpha^2-\beta^2$. Here the parameters α and β have the same meaning as these in the Papapetrou solutions. As a particular case, in the limit $\theta(z)=0$, the solutions (42) reduce to Papapetrou solutions.

In the case of the Papapetrou solutions in the Newtonian limit, that is, at large arguments $r=\sqrt{\rho^2+z^2}\longrightarrow\infty$, the asymptotics look like

$$u(\rho,z)=\alpha^{-1}\left(1+\frac{\beta z}{\alpha r^3}+O(r^{-2})\right), \tag{43}$$

and does not contain a term proportional to r^{-1}. We expect that in the general case when $\theta\neq 0$, it would be possible to construct a solution, which gives flat asymptotics at infinity for massive sources. As an example, one of the possible solutions could be obtained in the following way. Assume that the function $\sigma(\rho,z)=f_1(\rho)f_2(z)$ could be taken as a product of two functions f_1 and f_2. Then from Eq. (36) one can get

$$f_2f_{1\rho\rho}+f_1f_{2zz}+\frac{1}{\rho}f_{1\rho}f_2=\theta(z). \tag{44}$$

Taking the solution for the function f_2 in the form $f_2=f_{2zz}=\theta(z)$, with $\theta(z)=C_0(\exp(z)-\exp(-z))$, for the function f_1, we have the following equation

$$g_{1\rho\rho}+g_1+\frac{1}{\rho}g_{1\rho}=0, \tag{45}$$

where $f_1=1+g_1$ and C_0 is constant.

The last equation (45) is the Bessel one and $f_1=1+\mathrm{J}_0(\rho)$, where $\mathrm{J}_0(\rho)$ is the zero-rank Bessel function. Finally

$$\sigma(\rho,z)=C_0(1+\mathrm{J}_0(\rho))(\exp(z)-\exp(-z)), \tag{46}$$

and hence

$$\bar{\sigma}=C_0\mathrm{J}_0(\rho)(\exp(z)+\exp(-z)), \tag{47}$$

which takes the simple form $\bar{\sigma}=2C_0\mathrm{J}_0(\rho)$ at the plane $z=0$. For the large arguments $\rho\longrightarrow\infty$, the zero-rank Bessel function has the following asymptotic behavior

$$\lim_{\rho\longrightarrow\infty}\mathrm{J}_0(\rho)=\sqrt{\frac{2}{\pi\rho}}\cos\left(\rho-\frac{\pi}{4}\right), \tag{48}$$

and consequently

$$\bar{\sigma}_{|z=0}=2C_0\sqrt{\frac{2}{\pi\rho}}\cos\left(\rho-\frac{\pi}{4}\right)=\sqrt{\frac{1}{\rho}}\cos\left(\rho-\frac{\pi}{4}\right)\equiv x, \tag{49}$$

where we put, for the simplification $C_0=\sqrt{\pi/2}/2$. Using asymptotic properties of hyperbolic functions one could write

$$\begin{aligned}\cosh(x)=\ &1+\frac{1}{2}\frac{1}{\rho}\left(\cos\left(\rho-\frac{\pi}{4}\right)\right)^2+\frac{1}{24}\frac{1}{\rho^2}\left(\cos\left(\rho-\frac{\pi}{4}\right)\right)^4\\&+O(\rho^{-2}),\\\sinh(x)=\ &\rho^{-1/2}\left(\cos\left(\rho-\frac{\pi}{4}\right)\right)+\frac{1}{6}\rho^{-3/2}\left(\cos\left(\rho-\frac{\pi}{4}\right)\right)^3\\&+O(\rho^{-2}).\end{aligned} \tag{50}$$

Substituting the obtained asymptotic expressions (50) in solution (42), we can write the asymptotics of function $u(\rho,z)$, at the plane $z=0$ when $\rho\longrightarrow\infty$, in the following form

$$\begin{aligned}\lim_{\rho\longrightarrow\infty}u_{|z=0}=\alpha\Big[&1+\frac{1}{4}\rho^{-1}+\frac{1}{4}\rho^{-1}\sin(2\rho)\\&+\frac{1}{24}\rho^{-2}\left(\cos\left(\rho-\frac{\pi}{4}\right)\right)^4+\frac{\beta}{\alpha}\rho^{-1/2}\left(\cos\left(\rho-\frac{\pi}{4}\right)\right)\\&\times\frac{\beta}{6\alpha}\rho^{-3/2}\left(\cos\left(\rho-\frac{\pi}{4}\right)\right)^3+O(\rho^{-2})\Big].\end{aligned} \tag{51}$$

We can see from the asymptotics (51) that in contrast to the Papapetrou one (43) there is a term that is proportional to ρ^{-1} and at the same time we have an asymptotically flat solution at the $z=0$ plane.

5. Concluding remarks

We analyze the underlying system of nonlinear partial differential equations which arises in the study of Einstein field equations corresponding to the Weyl–Lewis–Papapetrou form for an axisymmetric rotating field. Our interest in this system lies in discussing new solutions that can be found by means of

Lie point symmetries. Firstly, the Lie classical method is utilized for the purpose of obtaining the group infinitesimals. The basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations. Several new families of exact solutions are found explicitly. We also obtained solution (42), which is an extension of the well-known Papapetrou class of solutions, which can be applied to the physical systems, presenting rotating bounded nonzero masses. We have constructed the solution for the particular function $\theta(z)$, which in some sense demonstrates the existence of ρ^{-1} term in asymptotics.

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