Approximate derivative-dependent functional variable separation for quasi-linear diffusion equations with a weak source*

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By using the approximate derivative-dependent functional variable separation approach, we study the quasi-linear diffusion equations with a weak source $u_t = (A(u)u_x)_x + \epsilon B(u,u_x)$. A complete classification of these perturbed equations which admit approximate derivative-dependent functional separable solutions is listed. As a consequence, some approximate solutions to the resulting perturbed equations are constructed via examples.

Keywords: quasi-linear diffusion equation, approximate derivative-dependent functional separable solution, approximate generalized conditional symmetry

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1. Introduction

It is well known that the symmetric method plays a significant role in reducing and constructing exact solutions of nonlinear partial differential equations (NPDEs).[1–6] Over the past decades, there have been several methods proposed to seek exact solutions of NPDEs, for example, the nonclassical method, the generalized conditional symmetry (GCS) method, the nonlocal symmetry method, the potential symmetry method,[7–10] etc. Moreover, much progress has been made in the study on variable separation for NPDEs. Some scholars employed several methods to construct exact solutions, such as the classical method, R matrices method, differential geometry method, the ansatz-based method, the formal variable separation approach, and the multi-linear variable separation approach.[11–16] On the scope of the symmetry group and the ansatz of the solution form, two types of approaches based on GCS were presented. They are respectively the functional variable separation approach (FVSA) and the derivative-dependent functional variable separation approach (DDFVSA), and the latter is the extension of the former.[17–19] Both methods are widely used to classify and solve exact solutions for several wide classes of evolution equations.

Another vital fact, with the development of modern science, technology, and engineering, the study of NPDEs with perturbation is becoming more and more important. The numerical and perturbation methods were used to deal with them in the beginning.[20] During the past decades, some approximate methods based on symmetry group theory and perturbation theory were presented so as to seek approximate solutions and to investigate the properties of perturbed equations, for instance, the approximate symmetry method, the approximate conditional symmetry method, the approximate potential symmetry method, the approximate generalized conditional symmetry method (AGCS), and the approximate homotopy symmetry method.[21–30]

Recently, we defined a new concept of the approximate functional separable solutions (AFSSs), and presented the approximate functional variable separation approach (AFVSA) based on AGCS, which was applied to the porous medium equation with a perturbed nonlinear source.[31] Meanwhile, the approximate derivative-dependent functional variable separation approach (ADDFVSA) was established, and was applied to the generalized diffusion equations with perturbation.[32] In the present paper, we apply ADDFVSA to the perturbed equations

$$u_t = (A(u)u_x)_x + \epsilon B(u,u_x), \tag{1}$$

where $A(u)$ and $B(u,u_x)$ are respectively the diffusion coefficient and the source term. They are smooth functions of the indicated variables, ϵ is a small parameter.

In the case that Eq. (1) with $\epsilon = 1$ and $B(u,u_x) \equiv B(u)$, has a wide range of applications in combustion theory, plasma physics, biology, chemical reactions, and mechanics of porous media. A number of methods have been used to construct its exact solutions and find its symmetry reductions.[7–10]

The layout of the paper is organized as follows. In Section 2, we present a complete classification to the quasi-linear diffusion equations with weak source which admit approximate derivative-dependent functional separable solutions (ADDFSSs). In Section 3, some ADDFSSs to the resulting

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perturbed equations are obtained by examples. Section 4 is devoted to the conclusions and discussion.

2. Classification of Eq. (1) which admits the ADDFSSs

Consider a k-th-order differential system $[E]$, which is perturbed up to the first order in the small parameter ϵ, viz.

$$\begin{aligned} & E^{\beta}(x,u,u_{(1)},\ldots,u_{(k)};\epsilon) \\ \equiv\ & E_0^{\beta}(x,u,u_{(1)},\ldots,u_{(k)})+\epsilon E_1^{\beta}(x,u,u_{(1)},\ldots, \\ & u_{(k)})=0,\ \ \beta=1,\ldots,q, \end{aligned} \tag{2}$$

where $x=(x^1,x^2,\ldots,x^n)$, $u=(u^1,u^2,\ldots,u^m)$, E_i^{β} are smooth functions in their arguments, ϵ is a small parameter, $u_{(i)}$ $(i=1,\ldots,k)$ is the collection of the i-th-order partial derivatives, and

$$D_i=\frac{\partial}{\partial x^i}+u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}}+u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}}+\cdots,\quad i=1,\ldots,n,$$

denotes the operator of the total derivative with respect to x^i. Let

$$V=\eta\frac{\partial}{\partial u}\equiv\eta(x,t,u;\epsilon)\frac{\partial}{\partial u} \tag{3}$$

be an evolutionary vector field and η its characteristic.

Definition 1 The approximate solution $u=u(x,t;\epsilon)$ of Eq. (2) is said to be an ADDFSS if it satisfies Eq. (2) and the ansatz

$$\begin{aligned} & f(u,u_x)+\epsilon g(u,u_x) \\ =\ & \psi(x)+\phi(t)+\epsilon(\omega(x)+\theta(t))+O(\epsilon^2), \end{aligned} \tag{4}$$

where $f(u,u_x)$ and $g(u,u_x)$ are smooth functions of u and u_x, and $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ are some functions of x and t.

If $f(u,u_x)$ and $g(u,u_x)$ are independent of u_x in Eq. (4), the approximate solution is the following AFSS

$$f(u)+\epsilon g(u)=\psi(x)+\phi(t)+\epsilon(\omega(x)+\theta(t))+O(\epsilon^2).$$

In particular, with regard to any perturbed (1+1)-dimensional nonlinear evolution equations, we have a definition as follows:

Definition 2 The evolutionary vector field (3) is said to be an AGCS of the perturbed nonlinear evolution equation

$$u_t=K(x,t,u;\epsilon) \tag{5}$$

if and only if

$$V^{(k)}(u_t-K(x,t,u;\epsilon))|_{[W]\cap[E]}=O(\epsilon^2), \tag{6}$$

whenever $u_t=K(x,t,u;\epsilon)$, where $V^{(k)}$ denotes the k-th-order prolongation to Eq. (3), K and η are differentiable functions of t, x, and u, u_x, u_{xx}, …, $[W]$ indicates the set of all differential consequences of $\eta=O(\epsilon^2)$ with respect to x, that is, $D_x^j\eta=O(\epsilon^2)$, $j=0,1,2,\ldots$, and $[E]$ denotes the solution manifold of Eq. (5).

Considering to the results in Ref. [32], we have the following theorem.

Theorem 1 Equation (5) possesses an ADDFSS (4) if and only if it admits the AGCS

$$\begin{aligned} V\ =\ & \eta\frac{\partial}{\partial u} \\ \equiv\ & [((f_{uu}+\epsilon g_{uu})u_t+(f_{uu_x}+\epsilon g_{uu_x})u_{xt})\,u_x \\ & +((f_{uu_x}+\epsilon g_{uu_x})u_t+(f_{u_xu_x}+\epsilon g_{u_xu_x})u_{xt})\,u_{xx} \\ & +(f_{u_x}+\epsilon g_{u_x})u_{xxt}+(f_u+\epsilon g_u)u_{xt}]\frac{\partial}{\partial u}. \end{aligned} \tag{7}$$

Now we apply the ADDFVS approach to deal with the classification problem of Eq. (1) that admits AGCS Eq. (3). By Definition 2 and Theorem 1, we know that the classification of Eq. (1), which possesses the ADDFSSs, is equivalent to obtaining the AGCS of Eq. (3) satisfying

$$\begin{aligned} & V^{(2)}(u_t-(A(u)u_x)_x-\epsilon B(u,u_x))|_{[W]\cap[E]} \\ =\ & [D_t\eta-\eta(A''u_x^2+A'u_{xx}+\epsilon B_u) \\ & -D_x\eta(2A'u_x+\epsilon B_{u_x})-AD_x^2\eta]|_{[W]\cap[E]}, \end{aligned} \tag{8}$$

where the prime denotes the corresponding order derivative with respect to u.

Using expressions $D_x^i\eta=0$, $i=0,1,2,\ldots$, and Eq. (1), and excluding the higher order derivative of u in Eq. (8), we obtain an expression of the independent derivative of u, which is

$$\begin{aligned} & D_t\eta|_{[W]\cap[E]} \\ =\ & \epsilon[(h_1u_{xx}+h_2)u_{xxx}^2+(h_3u_{xx}^3+h_4u_{xx}^2+h_5u_{xx}+h_6)u_{xxx} \\ & +h_7u_{xx}^4+h_8u_{xx}^3+h_9u_{xx}^2+h_{10}u_{xx}+h_{11}] \\ & +(h_{12}u_{xx}+h_{13})u_{xxx}^2 \\ & +(h_{14}u_{xx}^3+h_{15}u_{xx}^2+h_{16}u_{xx}+h_{17})u_{xxx} \\ & +h_{18}u_{xx}^4+h_{19}u_{xx}^3+h_{20}u_{xx}^2+h_{21}u_{xx}+h_{22} \\ =\ & O(\epsilon^2), \end{aligned} \tag{9}$$

where

$$\begin{aligned} & h_i=h_i(u,u_x)=O(\epsilon),\ \ i=1,\ldots,11, \\ & h_j=h_j(u,u_x)=O(\epsilon^2),\ \ j=12,\ldots,22. \end{aligned} \tag{10}$$

Since the decomposing expressions for h_i $(i=1,\ldots,22)$ are very complicated, we omit them here.

By solving the over-determined system of differential equations (10), we obtain the classification theorem as follows:

Theorem 2 Suppose $A(u) \neq 0$ and $B(u,u_x) \neq$ constant, the perturbed equation

$$u_t = (A(u)u_x)x + \epsilon B(u,u_x)$$

admits the ADDFSS of Eq. (4) if and only if it is equivalent to one of the following equations, up to the first order in ϵ:

1)

$$u_t = (A(u)u_x)_x + \epsilon B(u,u_x), \tag{11}$$
$$f(u,u_x) = b, \ \ b \neq 0, \tag{12}$$
$$g(u,u_x) \neq \text{constant}; \tag{13}$$

2)

$$u_t = (A(u)u_x)_x + \epsilon B(u,u_x), \tag{14}$$
$$f(u,u_x) = f(u), \tag{15}$$
$$g(u,u_x) = F_1(u); \tag{16}$$

3) $c_1 \neq 0$,

$$u_t = (A(u)u_x)_x + \epsilon B(u,u_x), \tag{17}$$
$$f(u,u_x) = f(u), \tag{18}$$
$$g(u,u_x) = c_1 f'(u)\sqrt{A(u)}u_x + F_2(u); \tag{19}$$

4) $ac_1c_2 \neq 0$,

$$u_t = au_{xx} + \epsilon\left[\frac{a(c_1^2H''(u) - c_3c_2^2)}{c_1^2c_2}u_x^2 + c_4u_x + c_5u + c_6\right], \tag{20}$$
$$f(u,u_x) = c_1u_x + c_2u + c_7, \tag{21}$$
$$g(u,u_x) = \frac{1}{2}c_3u_x^2 + \left(\frac{c_1H'(u)}{c_2} + c_8\right)u_x + H(u); \tag{22}$$

5) $c_1c_2 \neq 0$,

$$u_t = -\frac{u_{xx}}{u} + \frac{u_x^2}{u^2} + \epsilon\left(\frac{1}{2}\frac{c_1c_2c_3u_x}{u} + \frac{c_4u^2}{c_2} + c_5u\right), \tag{23}$$
$$f(u,u_x) = \frac{1}{c_1}\ln\left(\frac{c_2u^2}{u_x}\right) + c_6, \tag{24}$$
$$g(u,u_x) = -c_7\ln\left(\frac{c_2u^2}{u_x}\right) + \frac{c_2c_3u}{u_x} + c_8; \tag{25}$$

6) $ac_4 \neq 0$,

$$u_t = au_{xx} + \epsilon\left(\frac{1}{2}H(u)u_x^2 + c_1u_x + c_2u + c_3\right), \tag{26}$$
$$f(u,u_x) = -\frac{1}{c_4}\ln\left[\frac{(1\pm\mathrm{i})u_x}{2}\right] + c_5, \tag{27}$$
$$g(u,u_x) = -\frac{1}{2ac_4}\int^u H(\xi)\,\mathrm{d}\xi + \frac{c_6u + c_7}{u_x} + c_8\ln(u_x) + c_9; \tag{28}$$

7) $ac_1 \neq 0$,

$$u_t = au_{xx} + \epsilon\left[\frac{ac_1(H'(u) - c_4)u_x^2}{c_2u + c_3} + c_5u_x + c_6u + c_7\right], \tag{29}$$
$$f(u,u_x) = -\frac{1}{c_1}\ln(u_x + c_2u + c_3) + c_8, \tag{30}$$
$$g(u,u_x) = \int^u \frac{c_4 - H'(\xi)}{c_2\xi + c_3}\,\mathrm{d}\xi + \frac{H(u)}{u_x + c_2u + c_3} + c_9\ln(u_x + c_2u + c_3) + c_{10}; \tag{31}$$

where $F_1(u)$, $F_2(u)$, $f(u)$, and $H(u)$ are arbitrary differentiable functions of u, $K'(u) = \mathrm{d}K(u)/\mathrm{d}u$ $(K = f, H)$, $H''(u) = \mathrm{d}^2H(u)/\mathrm{d}u^2$, and a, b, c_i $(i \in N)$ are arbitrary constants in their sets of definition when not specified, and $\mathrm{i} = \sqrt{-1}$.

3. Construction of ADDFSSs for the resulting equations

By utilizing the ADDFVSA, we construct some ADDFSSs of the resulting equations via the following two examples.

Example 1 Since it is difficult to obtain ADDFSSs from Eqs. (14), (15), and (16) for general $A(u)$, $B(u,u_x)$, $f(u)$, and $F_1(u)$, we take

$$A(u) = a, \ f(u) = k_2\ln(u) + k_1,$$
$$B(u,u_x) = \frac{1}{2}(u + c_1)u_x^2 + u_x + u(c_3\ln(u) - c_2),$$
$$F_1(u) = \frac{1}{12}\frac{k_2\left[u^2 + 3c_1u + (c_1^3 + 12ac_4)u^{-1}\right]}{a} + k_3\ln(u) + k_4,$$

then equation (14) enjoys the following ADDFSSs

$$\begin{aligned} u \ =& \ \exp\left(\frac{\psi(x) + \phi(t) - k_1}{k_2}\right) \\ &- \epsilon\left\{\frac{1}{12a}\left[\exp\left(\frac{3(\psi(x) + \phi(t) - k_1)}{k_2}\right)\right.\right. \\ &\left. + 3c_1\exp\left(\frac{2(\psi(x) + \phi(t) - k_1)}{k_2}\right) + c_1^2\right] \\ &- \frac{1}{k_2}[k_2(\omega(x) + \theta(t)) - k_3(\psi(x) + \phi(t)) - k_2k_4 + k_1k_3] \\ &\left.\times \exp\left(\frac{(\psi(x) + \phi(t) - k_1)}{k_2}\right)\right\}, \end{aligned}$$

where $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ satisfy

(i) $a\lambda > 0$

$$\psi(x) = -\sqrt{\frac{\lambda}{a}}x + \frac{k_2}{2}\ln\left[\frac{a}{4\lambda}\left(k_5\exp\left(\frac{2}{k_2}\sqrt{\frac{\lambda}{a}}x\right) + k_6\right)^2\right],$$
$$\phi(t) = \frac{\lambda t}{k_2} + k_7,$$
$$ak_2^2\omega''(x) + \left[a\left(2k_2\omega'(x) - k_3\psi'(x)\right) + k_2^2\right]\psi'(x) + k_2[c_3(\psi(x) - k_1) + c_2k_2] - \lambda_1 = 0,$$
$$\theta(t) = \frac{1}{2}\frac{c_3\lambda t^2}{k_2} + \left(\frac{\lambda_1}{k_2^2} + c_3k_7\right)t + k_8.$$

(ii) $a\lambda<0$

$$\psi(x)=\frac{k_2}{2}\ln\left[-\frac{a}{\lambda}\left(k_5\sin\left(\frac{1}{k_2}\sqrt{-\frac{\lambda}{a}}x\right)+k_6\cos\left(\frac{1}{k_2}\sqrt{-\frac{\lambda}{a}}x\right)\right)^2\right],$$

$$\phi(t)=\frac{\lambda t}{k_2}+k_7,$$

$$ak_2^2\omega''(x)+\left[a\left(2k_2\omega'(x)-k_3\psi'(x)\right)+k_2^2\right]\psi'(x)+k_2[c_3(\psi(x)-k_1)+c_2k_2]-\lambda_1=0,$$

$$\theta(t)=\frac{1}{2}\frac{c_3\lambda t^2}{k_2}+\left(\frac{\lambda_1}{k_2^2}+c_3k_7\right)t+k_8,$$

where a, λ, λ_1, c_i, and k_j $(i=1,\ldots,4, j=1,\ldots,8)$ are arbitrary constants when not specified.

Similarly, we take

$$A(u)=\frac{c_1}{u},\ B(u,u_x)=\frac{c_2u}{u_x}+c_3,$$
$$f(u)=s_2\ln(u)+s_1,\ F_1(u)=s_3\ln(u)+s_4,$$

then equation (14) has the following ADDFSSs

$$u=\exp\left(\frac{\psi(x)+\phi(t)-s_1}{s_2}\right)+\frac{\epsilon}{s_2^2}[s_2(\omega(x)+\theta(t))-s_3(\psi(x)+\phi(t))-s_2s_4+s_1s_3]\times\exp\left(\frac{\psi(x)+\phi(t)-s_1}{s_2}\right),$$

where $\psi(x)$, $\phi(t)$, $\omega(x)$, and $\theta(t)$ satisfy

$$\psi(x)=s_2\ln\left[\frac{s_5}{2\mu}\left(\tanh^2\left(\frac{\sqrt{c_1s_5}(x+s_6)}{2c_1s_2}\right)-1\right)\right],$$

$$\phi(t)=s_2\ln\left[\frac{\mu(t+s_7)}{s_2^2}\right],$$

$$s_2^3\left[(c_1\omega''(x)+c_3s_2)\psi'(x)+c_2s_2^2\right]\mathrm{e}^{(s_1-\psi(x))/s_2}-[\mu(s_2\omega(x)-s_3\psi(x)-s_2s_4+s_1s_3)+\mu_1]\psi'(x)=0,$$

$$\theta(t)=s_3\left[\ln\left(\frac{\mu(t+s_7)}{s_2^2}\right)-1\right]+\frac{\mu_1t+s_2s_8\mu}{s_2\mu(t+s_7)},$$

where μ, μ_1, c_i, and s_j $(i=1,\ldots,4, j=1,\ldots,8)$ are arbitrary constants, and $c_1c_2s_2s_5\mu\neq0$.

Example 2 For simplicity, choosing $H(u)=0$ in Eqs. (29), (30), and (31), we obtain the following two cases.

Case 1 $c_2\neq0$

$$u_t=au_{xx}+\epsilon\left(-\frac{ac_1c_4u_x^2}{c_2u+c_3}+c_5u_x+c_6u+c_7\right),\ a\neq0, \quad (32)$$

$$f(u,u_x)=-\frac{1}{c_1}\ln(u_x+c_2u+c_3)+c_8,\ c_1\neq0, \quad (33)$$

$$g(u,u_x)=\frac{c_4\ln(c_2u+c_3)}{c_2}+c_9\ln(u_x+c_2u+c_3)+c_{10}; \quad (34)$$

Case 2 $c_2=0, c_3\neq0$

$$u_t=au_{xx}+\epsilon\left(-\frac{ac_1c_4u_x^2}{c_3}+c_5u_x+c_6u+c_7\right),\ a\neq0, \quad (35)$$

$$f(u,u_x)=-\frac{1}{c_1}\ln(u_x+c_3)+c_8,\ c_1\neq0, \quad (36)$$

$$g(u,u_x)=\frac{c_4}{c_3}u+c_9\ln(u_x+c_3)+c_{10}. \quad (37)$$

By the ADDFVSA, some special ADDFSSs of Eqs. (32) and (35) are given.

When $c_3=c_4=c_7=0$ in Case 1, we get the following ADDFSSs of Eq. (32)

$$u=\exp(c_1(c_8-\phi(t))-c_2x)\int\mathrm{e}^{c_2x-c_1\psi(x)}\mathrm{d}x+R(t)\mathrm{e}^{-c_2x}-c_2^{-1}c_3-\epsilon\Big[c_1\exp(c_1(c_8-\phi(t))-c_2x)\times\left(\int\omega(x)\mathrm{e}^{c_2x-c_1\psi(x)}\mathrm{d}x+c_1c_9\int\psi(x)\mathrm{e}^{c_2x-c_1\psi(x)}\mathrm{d}x+(c_1c_9\phi(t)+\theta(t)-c_1c_8c_9-c_{10})\times\int\mathrm{e}^{c_2x-c_1\psi(x)}\mathrm{d}x\right)-K(t)\mathrm{e}^{-c_2x}\Big],$$

where $\psi(x)$, $\phi(t)$, $R(t)$, $\omega(x)$, $\theta(t)$, and $K(t)$ satisfy

i) $\alpha\neq0$

$$a(c_1\psi(x)+c_2)\mathrm{e}^{c_2x-c_1\psi(x)}-\alpha\int\mathrm{e}^{c_2x-c_1\psi(x)}\mathrm{d}x+\beta=0,$$

$$\phi(t)=\frac{(\alpha-ac_2^2)t}{c_1}+b_1,$$

$$R(t)=\mathrm{e}^{ac_2^2t}\left[b_2-\frac{\beta\exp(-\alpha t+c_1(c_8-b_1))}{\alpha}\right],$$

$$c_2\left(ac_1\omega'(x)-ac_1c_2c_9-c_5\right)\mathrm{e}^{c_2x-c_1\psi(x)}+c_1c_2\beta(c_1c_9\psi(x)+\omega(x))-[c_1c_2\alpha(c_1c_9\psi(x)+\omega(x))-c_2(c_1c_9c_2^2a+c_2c_5-c_6)+\gamma]\int\mathrm{e}^{c_2x-c_1\psi(x)}\mathrm{d}x+c_1c_2\alpha\left[\int\omega(x)\mathrm{e}^{c_2x-c_1\psi(x)}\mathrm{d}x+c_1c_9\int\psi(x)\mathrm{e}^{c_2x-c_1\psi(x)}\mathrm{d}x\right]-c_1c_2\beta(c_1c_8c_9+c_9+c_{10})+\mu=0,$$

$$\theta(t)=\frac{\gamma t}{c_1c_2}+b_3,$$

$$K(t)=\Bigg[\left(\frac{\beta(\gamma-c_1c_2c_9(ac_2^2-\alpha))\exp(-\alpha t+c_1(c_8-b_1))}{c_2\alpha}-b_2(c_2c_5-c_6)\right)t+\left(\frac{\beta\gamma}{c_2\alpha^2}+\frac{c_1b_3\beta}{\alpha}-\frac{\mu}{c_2\alpha}-\frac{\beta(c_2c_5-c_6+c_1c_9(ac_2^2-b_1c_1\alpha-\alpha))}{\alpha^2}\right)\times\exp(-\alpha t+c_1(c_8-b_1))+b_4\Bigg]\mathrm{e}^{ac_2^2t}.$$

ii) $\alpha=0$

$$\psi(x)=\frac{1}{c_1}\left[\ln\left(\frac{2ac_2}{\mathrm{e}^{2c_2(x-d_1)}-\beta}\right)+c_2x\right],\ \theta(t)=\frac{\gamma t}{c_1c_2}+d_5,$$

$$\phi(t)=-\frac{ac_2^2t}{c_1}+d_2,\ R(t)=[\beta\mathrm{e}^{c_1(c_8-s_2)}t+d_3]\mathrm{e}^{ac_2^2t},$$

$$\omega(x)=\frac{1}{\mathrm{e}^{2c_2(x-d_1)}-\beta}\left\{c_9\beta\ln\left(\frac{2ac_2}{\mathrm{e}^{2c_2(x-d_1)}-\beta}\right)+\frac{1}{4c_2^3c_1a}\right.$$
$$\times\left[(3ac_1c_2^3c_9-c_6c_2-c_5c_2^2-\gamma)\ln(\beta\,\mathrm{e}^{-2c_2(x-d_1)}\right.$$
$$-1)\,\mathrm{e}^{2c_2(x-d_1)}-(ac_1c_9c_2^3+c_2c_6+c_5c_2^2+\gamma)$$
$$\times\,\mathrm{e}^{2c_2(x-d_1)}\ln\left(\frac{ac_2}{\mathrm{e}^{2c_2(x-d_1)}-\beta}\right)+2c_2\beta(c_1c_9c_2^3a$$
$$+c_6c_2-c_5c_2^2+\gamma)x-ac_1c_2^3(4c_1c_8c_9\beta+c_9\beta$$
$$+4c_{10}\beta-4d_4\,\mathrm{e}^{2c_2(x-d_1)})+\beta(c_2c_6+\gamma)$$
$$\left.\left.+c_2^2(4a\mu+c_5\beta)\right]\right\},$$
$$K(t)=\mathrm{e}^{ac_2^2t}\left[\frac{\beta\,\mathrm{e}^{c_1(c_8-d_2)}(ac_1c_9c_2^3-c_5c_2^2+c_2c_6-\gamma)t^2}{2c_2}\right.$$
$$-\left(\frac{\mathrm{e}^{c_1(c_8-d_2)}(c_1c_2\beta(c_1c_9d_2+d_5)-\mu)}{c_2}\right.$$
$$\left.\left.+d_3(c_2c_5-c_6)\right)t+d_6\right],$$

where α, β, μ, γ, b_i, and d_j $(i=1,\ldots,4, j=1,\ldots,6)$ are arbitrary constants.

If $c_6=0$ in Case 2, the ADDFSSs to Eq. (35) are explicitly given by

$$u=\mathrm{e}^{c_1(c_8-\phi(t))}\int\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x-c_3x+R(t)$$
$$+\epsilon\left\{\frac{c_1c_4\,\mathrm{e}^{2c_1(c_8-\phi(t))}}{c_3}\int\mathrm{e}^{-c_1\psi(x)}\left(\int\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x\right)\mathrm{d}x\right.$$
$$+c_1\left[\left(c_1c_9(\phi(t)-c_8)-\theta(t)+c_{10}+\frac{c_4R(t)}{c_3}\right)\right.$$
$$\times\int\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x-c_4\int x\,\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x$$
$$\left.-c_1c_9\int\psi(x)\,\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x-\int\omega(x)\,\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x\right]$$
$$\left.\times\,\mathrm{e}^{c_1(c_8-\phi(t))}+K(t)\right\},$$

where $\psi(x)$, $\phi(t)$, $R(t)$, $\omega(x)$, $\theta(t)$, and $K(t)$ satisfy

a) $\lambda\neq0$

$$ac_1\,\mathrm{e}^{-c_1\psi(x)}\psi'(x)-\lambda\int\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x+\rho=0,$$
$$\phi(t)=\frac{\lambda t}{c_1}+k_1,\ \ R(t)=-\frac{\rho\exp(-\lambda t+c_1(c_8-k_1))}{\lambda}+k_2,$$
$$c_3(ac_1\omega'(x)-ac_1c_4-c_5)\,\mathrm{e}^{-c_1\psi(x)}+c_1c_3\rho(c_1c_9\psi(x)$$
$$+\omega(x)+c_4x)+c_1c_3\lambda\left(c_1c_9\int\psi(x)\,\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x\right.$$
$$\left.+\int\omega(x)\,\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x+c_4\int xe^{-c_1\psi(x)}\,\mathrm{d}x\right)$$
$$-c_1[c_3\lambda(c_1c_9\psi(x)+\omega(x)+c_4x)+\nu]$$
$$\times\int\mathrm{e}^{-c_1\psi(x)}\,\mathrm{d}x-c_1c_3\rho(c_1c_8c_9+c_9+c_{10})+\mu=0,$$
$$\theta(t)=\frac{\nu t}{c_3}+k_3,$$
$$K(t)=\frac{c_1c_4\rho^2\exp(2(-\lambda t+c_1(c_8-k_1)))}{2c_3\lambda^2}$$
$$+\left[\frac{c_1\rho((c_3c_9\lambda+\nu)t-c_4k_2)-\mu}{c_3\lambda}\right.$$
$$\left.+\frac{c_1\rho(c_3c_9\lambda(c_1k_1+1)+c_3k_3\lambda+\nu)}{c_3\lambda^2}\right]$$
$$\times\exp(-\lambda t+c_1(c_8-k_1))$$
$$-(ac_1c_3c_4+a_5c_3-c_7)t+k_4.$$

b) $\lambda=0,\ \rho\neq0$

$$\psi(x)=\frac{1}{c_1}\ln\left[\frac{a}{\rho(x+s_1)}\right],\ \ \phi(t)=s_2,$$
$$R(t)=\rho\,\mathrm{e}^{c_1(c_8-s_2)}t+s_3,\ \ \theta(t)=\frac{\nu t}{c_3}+s_5,$$
$$\omega(x)=-c_9\ln\left[\frac{a}{\rho(x+s_1)}\right]+\frac{1}{x+s_1}\left[\frac{\nu x^3}{6ac_3}\right.$$
$$+\frac{(c_1s_1\nu+c_3c_5)x^2}{2ac_1c_3}+\left(\frac{c_5s_1}{c_1a}-\frac{\mu}{c_1c_3\rho}\right.$$
$$\left.\left.+c_1c_8c_9+c_4s_1+c_{10}\right)x-c_9s_1+s_4\right],$$
$$K(t)=\frac{c_1\rho(c_4\rho\,\mathrm{e}^{2c_1(c_8-s_2)}-\nu\,\mathrm{e}^{c_1(c_8-s_2)})t^2}{2c_3}$$
$$+[(c_3^{-1}(c_1\rho c_4s_3+\mu)-c_1^2s_2\rho c_9-c_1\rho s_5)$$
$$\times\,\mathrm{e}^{c_1(c_8-s_2)}-ac_1c_4c_3-c_3c_5+c_7]t+s_6,$$

where λ, ρ, μ, ν, k_i, and s_j $(i=1,\ldots,4, j=1,\ldots,6)$ are arbitrary constant.

4. Conclusion and discussion

In summary, by using the ADDFVSA, we have classified the quasi-linear diffusion equations with a weak source which admit ADDFSSs. The ADDFSSs of some resulting equations are derived. In general, these approximate results cannot be obtained by other approximate methods. Other types of perturbed NPDEs will be investigated by means of the ADDFVSA soon, and some new results will be presented.

There are still two interesting questions to be investigated later:

(I) How can the ADDFVSA be applied to other types of evolution equations, such as higher dimensional equations, and the nonlinear system of equations with perturbation?

(II) How can the ADDFVSA be extended so as to obtain more accurate approximate solutions?

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