Elliptic holes in octagonal quasicrystals*

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The stress potential function theory for the plane elasticity of octagonal quasicrystals is developed. By introducing stress functions, a large number of basic equations involving the elasticity of octagonal quasicrystals are reduced to a single partial differential equation. Furthermore, we develop the complex variable function method (Lekhnitskii method) for anisotropic elasticity theory to that for quasicrystals. With the help of conformal transformation, an exact solution for the elliptic hole of quasicrystals is presented. The solution of the Griffith crack problem, as a special case of the results, is obtained. As a consequence, the phonon stress intensity factor is derived analytically.

Keywords: quasicrystals, plane elasticity, elliptic hole, stress potential

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1. Introduction

In 1982, Israeli scientist Dan Shechtman,[1] who was awarded the 2011 Nobel Prize in chemistry, discovered quasicrystals, a new structure with a long-range quasiperiodic translational order and a long-range orientational order. Since then, quasicrystals have been the subject of intensive theoretical and experimental studies. The physical properties, such as the structural, electronic, magnetic, optical, and thermal properties, of quasicrystals have been investigated extensively. Scientists have discovered natural quasicrystals in a rock sample from Russia's Koryak mountains.[2] This discovery could redefine the field of mineralogy and expand our understanding of how quasicrystals form, leading to new applications.

The mechanical properties have been the subject of numerous studies since its discovery. Among them, the theoretical research on elasticity and defects has made great progress.[3–6] According to these theories, many methods[7–10] were developed to solve the relevant elasticity and defect problems. It is well known that an elliptic hole problem may be used as the basis of a crack problem in elastic analysis. The mechanics of elliptic hole in planar quasicrystals has already been investigated. With the help of conformal transformation, the exact solution for the elliptic hole of two-dimensional quasicrystals of point group 10 was presented.[11] The solution of the Griffith crack problem as a special case of the results was also derived. The elliptic hole subjected to uniform loading at infinity in one-dimensional quasicrystals was considered by means of the generalized Stroh formalism.[12] Based on the complex potential method, plane problems of cubic quasicrystal media containing an elliptic hole subjected to uniform remote loadings were solved.[13] The explicit solutions for the coupled fields were given in the closed form.

The Griffith crack problem of the octagonal quasicrystals was considered and the exact solutions were obtained by the Fourier transform method in Ref. [14]. Their method and results are so complicated that it is difficult to deal with some other problems. Due to complicity of the problems, the elasticity and fracture theory of octagonal quasicrystals have not been perfect as yet. Therefore, it is necessary to develop a new method to solve more practical problems. The stress potential function theory for plane elasticity of octagonal quasicrystals is developed. To illustrate the utility of our method, we consider an octagonal quasicrystal solid with an elliptic hole, which penetrates through the medium along the periodic direction. The edge of the elliptic hole is subjected to the uniform pressure.

2. General solution

Assume that the atom arrangement along the z direction is periodic and along the xy plane is quasiperiodic. According to quasicrystalline elastic theory,[4] we have the equations of deformation geometry

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad w_{ij} = \frac{\partial w_i}{\partial x_j}, \tag{1}$$

the equilibrium equations (if the body force is neglected)

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \frac{\partial H_{ij}}{\partial x_j} = 0, \tag{2}$$

and the generalized Hooke's law

$$\sigma_{xx} = L(\varepsilon_{xx} + \varepsilon_{yy}) + 2M\varepsilon_{xx} + R(w_{xx} + w_{yy}), \tag{3a}$$

$$\sigma_{yy} = L(\varepsilon_{xx} + \varepsilon_{yy}) + 2M\varepsilon_{yy} - R(w_{xx} + w_{yy}), \tag{3b}$$

$$\sigma_{xy} = \sigma_{yx} = 2M\varepsilon_{xy} + R(w_{yx} - w_{xy}), \tag{3c}$$

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$$H_{xx} = K_1 w_{xx} + K_2 w_{yy} + R(\varepsilon_{xx} - \varepsilon_{yy}), \tag{3d}$$
$$H_{yy} = K_1 w_{yy} + K_2 w_{xx} + R(\varepsilon_{xx} - \varepsilon_{yy}), \tag{3e}$$
$$H_{xy} = (K_1 + K_2 + K_3) w_{xy} + K_3 w_{yx} - 2R\varepsilon_{xy}, \tag{3f}$$
$$H_{yx} = (K_1 + K_2 + K_3) w_{yx} + K_3 w_{xy} + 2R\varepsilon_{xy}, \tag{3g}$$

where $L = C_{12}$, and $M = (C_{11} - C_{12})/2 = C_{66}$. The variables σ_{ij}, ε_{ij}, u_i, and C_{ij} are stress, strain, displacement, and elastic constants of phonon fields, respectively. The variables H_{ij}, w_{ij}, w_i, and K_i are the stress, strain, displacement, and elastic constants of phason fields, respectively. The parameter R is the phonon–phason coupling elastic constant.

Based on the equations of deformation geometry (1), deformation compatibility equations are expressed as follows:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \quad \frac{\partial w_{xy}}{\partial x} = \frac{\partial w_{xx}}{\partial y},$$
$$\frac{\partial w_{yx}}{\partial y} = \frac{\partial w_{yy}}{\partial x}. \tag{4}$$

Based on generalized Hooke's law (Eqs. (3a)–(3g)), all strain components can be expressed by relevant stress components. Because the expressions are too lengthy, we do not list them here. Now, the deformation compatibility equations (4) can be rewritten in terms of the stresses σ_{ij} and H_{ij}.

If we introduce the stress potential functions $\phi(x,y)$, $\psi_1(x,y)$, and $\psi_2(x,y)$ as follows:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2},$$
$$\sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y},$$
$$H_{xx} = \frac{\partial \psi_1}{\partial y}, \quad H_{xy} = -\frac{\partial \psi_1}{\partial x},$$
$$H_{yx} = -\frac{\partial \psi_2}{\partial y}, \quad H_{yy} = \frac{\partial \psi_2}{\partial x}, \tag{5}$$

with

$$\phi = -c_2 c_3 \nabla^2 \nabla^2 G,$$
$$\psi_1 = \frac{1}{2} R\left(c_3 \frac{\partial^3}{\partial x^2 \partial y}\Pi_1 + c_2 \frac{\partial^3}{\partial y^3}\Pi_1 - c_1 \frac{\partial^3}{\partial x^2 \partial y}\Pi_2\right) G, \tag{6}$$
$$\psi_2 = \frac{1}{2} R\left(c_1 \frac{\partial^3}{\partial x \partial y^2}\Pi_1 - c_2 \frac{\partial^3}{\partial x^3}\Pi_2 - c_3 \frac{\partial^3}{\partial x \partial y^2}\Pi_2\right) G, \tag{7}$$

where

$$\Pi_1 = 3\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}, \quad \Pi_2 = 3\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2},$$
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

and

$$c_1 = \frac{K_3 M + R^2}{K_1 + K_2 + 2K_3} + \frac{K_2 M - R^2}{K_1 - K_2},$$
$$c_2 = \frac{(K_1 + K_2 + K_3)M - R^2}{K_1 + K_2 + 2K_3}, \quad c_3 = \frac{K_1 M - R^2}{K_1 - K_2}.$$

Then, equilibrium equations (2) and the deformation compatibility equations (4) expressed by stresses in practice will be automatically satisfied if

$$\nabla^2 \nabla^2 \nabla^2 \nabla^2 G - 4\varepsilon \nabla^2 \nabla^2 \Lambda^2 \Lambda^2 G + 4\varepsilon \Lambda^2 \Lambda^2 \Lambda^2 \Lambda^2 G = 0, \tag{8}$$

which is the final governing equation based on stress functions of plane elasticity of octagonal quasicrystals, where

$$\varepsilon = \frac{R^2 (L+M)(K_2+K_3)}{[(K_1+K_2+K_3)M - R^2][K_1(L+2M) - R^2]},$$

and

$$\Lambda^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}.$$

It is seen from Eqs. (6) and (7) that if we set $\varepsilon = 0$ (the phonon and phason fields are uncoupled), the stress potential functions $\psi_1(x,y) = \psi_2(x,y) = 0$ and ϕ is a biharmonic function. Therefore, the present solution is reduced to the classical results.

Equation (8) can be rewritten as follows:

$$LG = 0, \tag{9}$$

where

$$L = \frac{\partial^8}{\partial x^8} + \frac{\partial^8}{\partial y^8} + 4(1-4\varepsilon)\frac{\partial^8}{\partial x^6 \partial y^2} + 4(1-4\varepsilon)\frac{\partial^8}{\partial x^2 \partial y^6} + 2(3+16\varepsilon)\frac{\partial^8}{\partial x^4 \partial y^4}.$$

Equation (9) can be solved by means of complex variables. A nontrivial solution follows by setting the characteristic equation zero, namely,

$$\mu^8 + 4(1-4\varepsilon)\mu^6 + 2(3+16\varepsilon)\mu^4 + 4(1-4\varepsilon)\mu^2 + 1 = 0.$$

Since μ cannot be real if the strain energy is positive, μ occurs as four pairs of complex variables conjugate. We assume they are distinct, and let

$$\mathrm{Im}\,\mu_k > 0, \quad \mu_{k+4} = \overline{\mu}_k, \quad (k = 1,2,3,4),$$

where Im stands for the imaginary part, and the overbar denotes the complex conjugate. Once the roots μ_k are known, the general solution of Eq. (9) can be obtained as follows:

$$G = 2\,\mathrm{Re}\sum_{k=1}^{4} G_k(z_k), \quad z_k = x + \mu_k y, \tag{10}$$

where Re denotes the real part of the corresponding complex expression.

By substituting the general solution (10) into the expressions (6) and (7), and Eq. (5), the complex representation of stresses is obtained as follows:

$$\sigma_{xx} = -2c_2 c_3\,\mathrm{Re}\sum_{k=1}^{4}(\mu_k^2 + 2\mu_k^4 + \mu_k^6) g_k'(z_k), \tag{11}$$

$$\sigma_{yy} = -2c_2c_3\,\mathrm{Re}\sum_{k=1}^{4}(1+2\mu_k^2+\mu_k^4)g_k'(z_k), \tag{12}$$

$$\sigma_{xy} = \sigma_{yx} = 2c_2c_3\,\mathrm{Re}\sum_{k=1}^{4}(\mu_k+2\mu_k^3+\mu_k^5)g_k'(z_k), \tag{13}$$

$$H_{xx} = R\mathrm{Re}\sum_{k=1}^{4}[(4c_3-c_2)\mu_k^2+2(3c_2-2c_3)\mu_k^4 - c_2\mu_k^6]g_k'(z_k), \tag{14}$$

$$H_{xy} = -R\mathrm{Re}\sum_{k=1}^{4}[(4c_3-c_2)\mu_k+2(3c_2-2c_3)\mu_k^3 - c_2\mu_k^5]g_k'(z_k), \tag{15}$$

$$H_{yx} = -R\mathrm{Re}\sum_{k=1}^{4}[c_2\mu_k+2(c_3-2c_2)\mu_k^3 - (4c_3-c_2)\mu_k^5]g_k'(z_k), \tag{16}$$

$$H_{yy} = R\mathrm{Re}\sum_{k=1}^{4}[c_2+2(c_3-2c_2)\mu_k^2 - (4c_3-c_2)\mu_k^4]g_k'(z_k), \tag{17}$$

where

$$g_k(z_k) = \frac{\mathrm{d}^5 G_k(z_k)}{\mathrm{d}z_k^5}, \quad g_k'(z_k) = \frac{\mathrm{d}g_k(z_k)}{\mathrm{d}z_k}.$$

We can conclude from the above discussion that the plane elasticity problem of octagonal quasicrystals can be considered as a problem of determining the four complex stress functions $g_k(z_k)$ $(k=1,2,3,4)$, which satisfy the boundary conditions.

3. Elliptic hole problem

In this section, we consider application of the complex variable method for plane elasticity of octagonal quasicrystals to the elliptic hole problem. We consider an octagonal quasicrystal solid with an elliptic hole, which penetrates through the medium along the z-axis direction (periodic direction), and the edge of the elliptic hole $L(x^2/a^2+y^2/b^2=1)$ is subjected to the uniform pressure p (see Fig. 1).

The boundary conditions of this problem can be represented as follows:

$$\begin{aligned}
&\sigma_{xx}\cos(n,x)+\sigma_{xy}\cos(n,y)=T_x,\\
&\sigma_{xy}\cos(n,x)+\sigma_{yy}\cos(n,y)=T_y,\\
&H_{xx}\cos(n,x)+H_{xy}\cos(n,y)=0,\\
&H_{xy}\cos(n,x)+H_{yy}\cos(n,y)=0,
\end{aligned} \tag{18}$$

where n is the direction of external normals to internal contours, $T_x=-p\cos(n,x)$ and $T_y=-p\cos(n,y)$ denote the components of surface traction, p is the magnitude of the pressure, and the phason stresses are assumed to vanish.

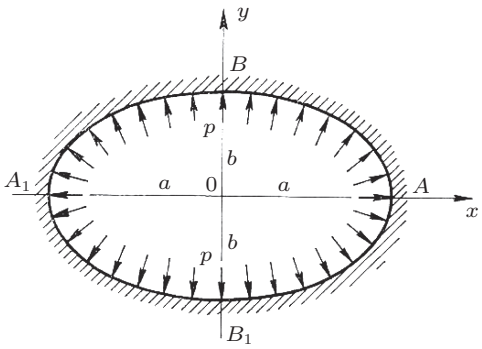


Fig. 1. An elliptic hole penetrates through the octagonal quasicrystals along the z-axis direction (periodic direction).

Similar to Ref. [15] by Lekhnitskii, we use the conformal mapping

$$z_k = \frac{a-\mathrm{i}\mu_k b}{2}\zeta_k + \frac{a+\mathrm{i}\mu_k b}{2}\frac{1}{\zeta_k}, \quad (k=1,2,3,4) \tag{19}$$

to transform the interior of the ellipse on the z_k plane into the exterior of the unit circle on the ζ_k plane. The four complex stress functions are determined as follows:

$$g_k(z_k) = -\frac{p}{c_2c_3}\frac{\Delta_k}{\Delta}\zeta_k^{-1}, \quad (k=1,2,3,4), \tag{20}$$

where

$$\zeta_k = \frac{z_k+\sqrt{z_k^2-a^2-\mu_k^2b^2}}{a-\mathrm{i}\mu_k b}, \quad (k=1,2,3,4), \tag{21}$$

and

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{vmatrix},$$

where

$$\begin{aligned}
&a_{1k}=1+2\mu_k^2+\mu_k^4, \quad a_{2k}=\mu_k+2\mu_k^3+\mu_k^5,\\
&a_{3k}=(4c_3-c_2)\mu_k+2(3c_2-2c_3)\mu_k^3-c_2\mu_k^5,\\
&a_{4k}=c_2+2(c_3-2c_2)\mu_k^2-(4c_3-c_2)\mu_k^4.
\end{aligned}$$

The determinant Δ_k can be obtained by substituting the k array of determinant Δ with vector $((-a)/2,(-\mathrm{i}b)/2,0,0)^{\mathrm{T}}$, where the superscript T represents the transpose. Based on the four complex stress functions (20), the stress field can also be obtained.

The stress of interest to us is that at the edge of hole. To study the effect of the hole on the stress, we give the formula for the hoop phonon stress. At a point on the edge L, let ω be the angle the tangent to the ellipse makes with the x-axis. The unit vectors tangential to the ellipse is $\boldsymbol{m}(\omega)=(\cos\omega,\sin\omega,0)^{\mathrm{T}}$. For the surface traction $\boldsymbol{t}_m$ at a point along the hole boundary of which the normal is $\boldsymbol{m}(\omega)$, we have

$$\boldsymbol{t}_m = \boldsymbol{t}_2\sin\omega+\boldsymbol{t}_1\cos\omega, \tag{22}$$

where $\boldsymbol{t}_1=\{\sigma_{xx},\sigma_{xy}\}^{\mathrm{T}}$, and $\boldsymbol{t}_2=\{\sigma_{xy},\sigma_{yy}\}^{\mathrm{T}}$. Thus, the hoop phonon stress can be expressed as

$$\boldsymbol{\sigma}_{mm} = \boldsymbol{m}(\omega)^{\mathrm{T}}\cdot\boldsymbol{t}_m$$

$$= -4p\,\mathrm{Re}\sum_{k=1}^{4}\frac{\Delta_k}{\Delta}\frac{\mathrm{i}(a\sin\omega+ib\cos\omega)}{ab(\cos\omega+\mu_k\sin\omega)} \times[(1+2\mu_k^2+\mu_k^4)\sin^2\omega - 2(\mu_k+2\mu_k^3+\mu_k^5)\cos\omega\sin\omega + (\mu_k^2+2\mu_k^4+\mu_k^6)\cos^2\omega]. \quad (23)$$

It can be seen from Eq. (23) that the phonon stress concentration is dependent on the angle ω, the characteristic roots μ_k, and the major and minor axes of the ellipse.

4. Solution to a crack

When the elliptic hole degenerates into a crack, let $b=0$. Then, equation (20) can be simplified into

$$g_k(z_k) = -\frac{p}{c_2c_3a}\frac{\Delta_k}{\Delta}(z_k-\sqrt{z_k^2-a^2}), \quad (k=1,2,3,4). \quad (24)$$

Thus, the phonon and phason stresses can be expressed with a complex variable z_k (see Appendix A for details).

With the usual definition, the phonon field intensity factor is given by

$$K_{\mathrm{I}} = \sqrt{2\pi}\lim_{x\to a}\sqrt{x-a}\,\sigma_{yy}(x,0). \quad (25)$$

Substituting Eqs. (12) and (24) into Eq. (25) leads to

$$K_{\mathrm{I}} = \sqrt{\pi a}\,p. \quad (26)$$

The above result shows that the phonon stress intensity factor is identical with the corresponding result in conventional linear elasticity fracture mechanics.

5. Conclusion

The final governing equation based on stress potential function are obtained, then the elliptic hole problem of octagonal quasicrystals is solved. Specially, the solution of the Griffith crack problem is also derived. The method adopted here can be used to solve other more complicated defect problems of octagonal quasicrystals.

Moreover, we extend the Lekhnitskii method for the anisotropic elasticity theory to the plane elasticity theory of octagonal quasicrystals, which contributes in developing and enriching the theory as well as the method to deal with plane elasticity of quasicrystals. The present paper shows again that the stress potential and complex variable function method are powerful in solving complicated stress boundary value problem of quasicrystals.

Appendix A

The phonon and phason stress fields of the crack can be given by

$$\sigma_{xx} = \frac{2p}{a}\,\mathrm{Re}\sum_{k=1}^{4}\frac{\Delta_k}{\Delta}(\mu_k^2+2\mu_k^4+\mu_k^6)\left(1-\frac{z_k}{\sqrt{z_k^2-a^2}}\right), \quad (A1)$$

$$\sigma_{yy} = \frac{2p}{a}\,\mathrm{Re}\sum_{k=1}^{4}\frac{\Delta_k}{\Delta}(1+2\mu_k^2+\mu_k^4)\left(1-\frac{z_k}{\sqrt{z_k^2-a^2}}\right), \quad (A2)$$

$$\sigma_{xy} = \sigma_{yx} = \frac{2p}{a}\,\mathrm{Re}\sum_{k=1}^{4}\frac{\Delta_k}{\Delta}(\mu_k+2\mu_k^3+\mu_k^5) \times\left(\frac{z_k}{\sqrt{z_k^2-a^2}}-1\right), \quad (A3)$$

$$H_{xx} = \frac{Rp}{c_2c_3a}\,\mathrm{Re}\sum_{k=1}^{4}\frac{\Delta_k}{\Delta}[(4c_3-c_2)\mu_k^2 + 2(3c_2-2c_3)\mu_k^4 - c_2\mu_k^6]\left(\frac{z_k}{\sqrt{z_k^2-a^2}}-1\right), \quad (A4)$$

$$H_{xy} = \frac{Rp}{c_2c_3a}\,\mathrm{Re}\sum_{k=1}^{4}\frac{\Delta_k}{\Delta}[(4c_3-c_2)\mu_k + 2(3c_2-2c_3)\mu_k^3 - c_2\mu_k^5]\left(1-\frac{z_k}{\sqrt{z_k^2-a^2}}\right), \quad (A5)$$

$$H_{yx} = \frac{Rp}{c_2c_3a}\,\mathrm{Re}\sum_{k=1}^{4}\frac{\Delta_k}{\Delta}[c_2\mu_k + 2(c_3-2c_2)\mu_k^3 - (4c_3-c_2)\mu_k^5]\left(1-\frac{z_k}{\sqrt{z_k^2-a^2}}\right), \quad (A6)$$

$$H_{yy} = \frac{Rp}{c_2c_3a}\,\mathrm{Re}\sum_{k=1}^{4}\frac{\Delta_k}{\Delta}[c_2 + 2(c_3-2c_2)\mu_k^2 - (4c_3-c_2)\mu_k^4]\left(\frac{z_k}{\sqrt{z_k^2-a^2}}-1\right), \quad (A7)$$

where $b=0$ in the determinant Δ_k.

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