Binary collision approximation for solitary waves in a Y-shaped granular chain*

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We implement a binary collision approximation to study solitary wave propagation in a two-dimensional double Y-shaped granular chain. The solitary wave was transmitted and reflected when it met the interface of the bifurcated branches of the Y-shaped granular chains. We obtain the analytic results of the ratios of the transmitted and reflected speeds to the incident speed of the solitary wave, the maximum force between the two neighbor beads in a solitary wave, and the total time taken by the pulse to pass through each branch. All of the analytic results are in good agreement with the experimental observations from Daraio *et al*. [*Phys. Rev.* E **82** 036603 (2010)]. Moreover, we also discuss the delay effects on the arrival of split pulses, and predict the recombination of the split waves traveling in branches in the final stem of asymmetric systems. The prediction of pulse recombination is verified by our numerical results.

Keywords: Y-shaped granule chain, solitary wave, binary collision approximation

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1. Introduction

The dynamics of pulse propagation in granular chains has drawn considerable attention over the past few years.[1–4] This is partly due to the theoretical challenges it poses, and partly due to related technological applications, such as the design of shock absorbers,[5–9] actuating devices,[10] and sound scramblers.[11,12]

Previous investigations of pulse propagation have focused on one-dimensional (1D) granular chains. Available information about granular chains is obtained mostly from numerical and experimental results. The response to impulse propagation in two-, or three-dimensional systems has been seldom considered. Actually, applications of highly nonlinear impulses involve propagation in higher-dimensional granular realms. Recently, the splitting and recombination of highly nonlinear signals in a branched two-dimensional granular network was studied experimentally and numerically.[13] This study may lead to possible engineering applications in the transmission and transformation of acoustic signals.

Theoretically, the long-wavelength approximation and the binary-collision model have been proposed to deal with the propagation of impulse in granular chains, both of which have provided excellent results.[14] The long-wavelength approximation is only limited to be applicable to mono-disperse or nearly mono-disperse granular chains.[15,16] The binary collision approximation is particularly useful when the continuum approximation becomes more difficult to implement.[17–22] The model has been successfully applied to a number of configurations, such as tapered,[18] decorated,[19] randomly decorated chains,[20] periodic dimmer chains,[21] and a chain of o-rings with and without pre-compression.[22] However, it has not yet been applied to any kind of higher-dimensional granular chains.

In this paper, we implement the binary collision model to investigate pulse propagation in a two-dimensional double Y-shaped granular chain. We concentrate on this particular system because it has been studied both numerically and experimentally. It is a matter of interest for a highly nonlinear solitary wave to pass through a two-dimensional Y-shaped geometrical bifurcation, due to uncertainty of the outcome of this interaction. Here, we make use of the binary collision approximation to investigate the pulse velocity in such Y-shaped chains, which extends the approximation to a new geometry. This extension of the binary collision approximation may provide generalization of a successful theory that is implemented for higher-dimensional chains.

2. Model

A two-dimensional double Y-shaped granular chain with a bifurcation angle of 60° is shown in Fig. 1. This setup allows splitting, redirection, and recombination of the pulses formed in the first stem. Following the experimental setup in Ref. [13], we arrange 132 beads horizontally in the double

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Y-shaped chain, among which 19 uniform beads are in stems and 47 beads are in each branch. For convenience of theoretical analysis, beads in stems are denoted by mass m, radius r, elastic modulus E, and Poisson ratio υ; beads in branch 1 by mass m_1, radius r_1, elastic modulus E_1, and Poisson ratio υ_1; beads in branch 2 by mass m_2, radius r_2, elastic modulus E_2, and Poisson ratio υ_2. We also assign sign numbers to all the grains. As shown in Fig. 1, a bead 19 is situated at the interface connecting bifurcated branches. Beads 66 and 67 are, respectively, the corner of the bifurcating branches.

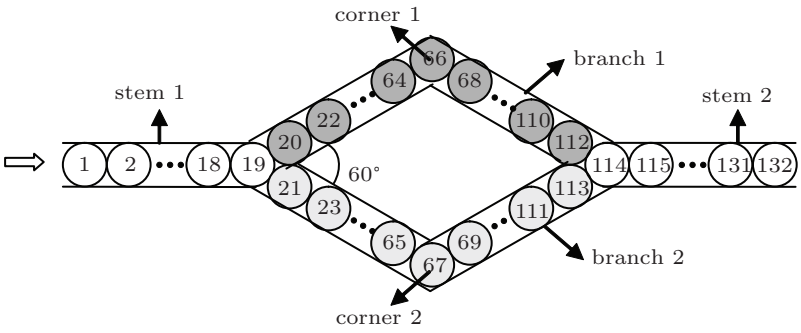


Fig. 1. Setup of the Y-shaped granular chain.

3. Binary collision approximation

3.1. Velocity of the beads in stem 1

The binary collision approximation assumes that the transfer of energy along the chain occurs by means of the succession of two-particle collisions. We use the binary collision approximation to calculate the time required for the pulse to reach the end of the chain. For this purpose, we first calculate the velocity of granule after the collision between two adjacent particles. According to energy and momentum conservation, when two adjacent particles i and $i+1$ collide with each other, we have

$$\begin{cases} m_i V_i = m_i V_i' + m_{i+1} V_{i+1}, \\ \dfrac{1}{2} m_i V_i^2 = \dfrac{1}{2} m_i V_i'^2 + \dfrac{1}{2} m_i V_{i+1}^2, \end{cases} \tag{1}$$

and the velocity of the $(i+1)$-th granular is given by

$$V_{i+1} = \frac{2V_i}{1 + m_{i+1}/m_i}. \tag{2}$$

From Eq. (2), we know that velocity of the latter grain after collisions is unchanged. If the first bead in the stem 1 has an initial velocity of V_0, the second bead then acquires velocity V_0, and will collide with stationary particle 3, and so on.

3.2. Velocity of the beads in the first interface

When the pulse passes through the interface of bifurcated branches, the splitting and reflection of the wave would occur. We assume that after collisions, the velocity of bead 19 at the interface is denoted by V_0', the velocity of bead 20 is denoted by V_1, and the velocity of bead 21 is denoted by V_2. The situation of collisions among beads at the interface is shown in Fig. 2. Using energy and momentum conservation, we obtain

$$\begin{cases} m\boldsymbol{V}_0 = m\boldsymbol{V}_0' + m_1\boldsymbol{V}_1 + m_2\boldsymbol{V}_2, \\ \dfrac{1}{2} m V_0^2 = \dfrac{1}{2} m V_0'^2 + \dfrac{1}{2} m_1 V_1^2 + \dfrac{1}{2} m_2 V_2^2. \end{cases} \tag{3}$$

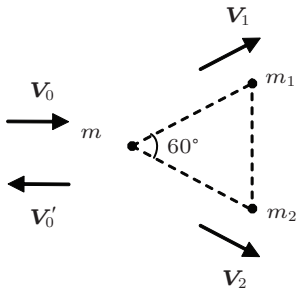


Fig. 2. Collision at the first interface.

Velocities of the three grains involved in the collision are given by

$$\begin{cases} V_0' = \dfrac{3m_1m_2 - mm_1 - mm_2}{3m_1m_2 + mm_1 + mm_2} V_0, \\ V_1 = \dfrac{2\sqrt{3}mm_2}{3m_1m_2 + mm_1 + mm_2} V_0, \\ V_2 = \dfrac{2\sqrt{3}mm_1}{3m_1m_2 + mm_1 + mm_2} V_0. \end{cases} \tag{4}$$

3.3. Velocity of the bead on the corner

When the impulse passes through the corner of the bifurcating branch, it can be redirected and reflected. To explore the source, we denote the transmitted velocity of the bead on the corner of branch 1 by $\boldsymbol{V}_1'$. Figure 3 shows the situation of collision on the corner of branch 1. Based on energy and

momentum conservation, we obtain

$$\begin{cases} m_1 \boldsymbol{V}_1 = m_1 \boldsymbol{V}_1'' + m_1 \boldsymbol{V}_1', \\ \dfrac{1}{2} m_1 V_1^2 = \dfrac{1}{2} m_1 V_1''^2 + \dfrac{1}{2} m_1 V_1'^2. \end{cases} \tag{5}$$

The relationship of grain velocity before and after a collision is shown in Fig. 4. Combining the theorem of cosine and energy conservation (Eq. (5)), velocities of beads 66 and 68 are given by

$$V_1'' = \sqrt{3} V_1/2, \quad V_1' = V_1/2. \tag{6}$$

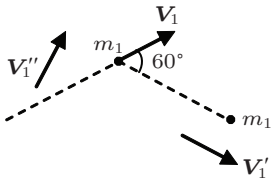


Fig. 3. Collision on the corner 1 in branch 1.

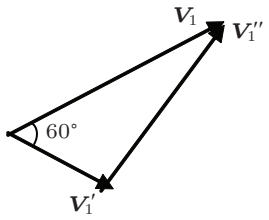


Fig. 4. Relationship of grain velocity before and after collision.

Similarly, we define the velocity of the bead on the corner of branch 2 as $\boldsymbol{V}_2'$. The velocity of bead 69 reads

$$V_2' = V_2/2. \tag{7}$$

3.4. Binary collision approximation for the transfer of pulse in branches

The velocity of granules can be used to calculate the time required for the pulse to move along the chain. When the pulse travels from the interface to the corner in branch 1, according to the binary collision approximation,[17–22] the dynamics of the two colliding beads are governed by

$$\begin{cases} m_1 \ddot{u}_i = -\eta_1 (u_i - u_{i+1})^{3/2}, \\ m_1 \ddot{u}_{i+1} = \eta_1 (u_i - u_{i+1})^{3/2}, \end{cases} \tag{8}$$

where u_i is the displacement of the i-th grain, and the elastic coefficient $\eta_1 = E_1 \sqrt{2r_1}/3(1-\upsilon_1^2)$. Here, we use the Hertzian expression of force between the granules, and neglect the effects of viscous and frictional dissipation.

Define a variable $\delta = u_i - u_{i+1}$ as a relative displacement of the two beads, equation (8) can be rewritten as

$$\ddot{\delta} = \ddot{u}_i - \ddot{u}_{i+1} = -\frac{2\eta_1}{m_1} \delta^{3/2} = -\frac{\eta_1}{\mu} \delta^{3/2}, \tag{9}$$

where $\mu = m_1/2$ is the effective mass of the two beads. At the beginning of the collision, the velocity of the i-th grain is V_1, the velocity of the $(i+1)$-th grain is zero, and the initial velocity of the small system is $\dot{\delta}(0) = V_1$. According to the conservation of energy, we have

$$\frac{1}{2} \mu \dot{\delta}^2(t) + \frac{2}{5} \eta_1 \delta^{5/2}(t) = \frac{1}{2} \mu \dot{\delta}^2(0), \tag{10}$$

which leads to

$$\begin{aligned} \dot{\delta}(t) &= \left[\dot{\delta}^2(0) - \frac{4}{5\mu} \eta_1 \delta^{5/2}(t) \right]^{1/2} \\ &= \left[V_1^2 - \frac{8}{5m_1} \eta_1 \delta^{5/2}(t) \right]. \end{aligned} \tag{11}$$

The pulse is said to reside on the i-th grain until the velocities of the two grains are equal. Obviously, in this case, there is a maximum compression between them. Therefore,

$$\delta_{\max} = \left[\frac{5m_1}{8\eta_1} V_1^2 \right]^{2/5}. \tag{12}$$

The time taken by the pulse on a given particle can be obtained by

$$T_{11} = \int_0^{\delta_{\max}} \frac{\mathrm{d}\delta}{\dot{\delta}} = \frac{2\sqrt{\pi}}{5V_1} \left(\frac{5m_1}{8\eta_1} V_1^2 \right)^{2/5} \frac{\Gamma(0.4)}{\Gamma(0.9)}. \tag{13}$$

The pulse velocity is $v_{s11} = 2r_1/T_{11}$.

If both of the grains in the Y-shaped chain are the same, we can obtain the velocity of the beads in the first interface from Eq. (4), i.e., $V_0 = 1$, $V_1 = 2\sqrt{3}/5$, and $V_0' = 1/5$. Using the same method as formulated by Eqs. (8)–(13), we can easily obtain the maximum compression and the time taken by the pulse on a particle with the incident, transmitted, and reflected solitary waves separately. As for the incident solitary wave, the maximum compression between the two grains is $\delta_{\max}^{\mathrm{i}} = (5mV_0^2/8\eta)^{2/5}$, and the time required for the pulse on a given grain is

$$T_i = \frac{2\sqrt{\pi}}{5V_0} \left(\frac{5m}{8\eta} V_0^2 \right)^{2/5} \frac{\Gamma(0.4)}{\Gamma(0.9)}.$$

For the transmitted solitary wave, the maximum compression between the two grains is $\delta_{\max}^{\mathrm{s}} = (5mV_1^2/8\eta)^{2/5}$, and the time taken by the pulse on a given grain is

$$T_{\mathrm{s}} = \frac{2\sqrt{\pi}}{5V_1} \left(\frac{5m}{8\eta} V_1^2 \right)^{2/5} \frac{\Gamma(0.4)}{\Gamma(0.9)}.$$

For the reflected solitary wave, the maximum compression between the two grains is $\delta_{\max}^{\mathrm{r}} = (5mV_0'^2/8\eta)^{2/5}$, and the time taken by the pulse on a given grain is

$$T_{\mathrm{r}} = \frac{2\sqrt{\pi}}{5V_0'} \left(\frac{5m}{8\eta} V_0'^2 \right)^{2/5} \frac{\Gamma(0.4)}{\Gamma(0.9)}.$$

Then, the incident speed of the solitary wave is $v_i = 2r/T_i$, and the maximum force is $F_{\rm mi} = \eta\left(5mV_0^2/8\eta\right)^{3/5}$; the transmitted speed is $v_{\rm s} = 2r/T_{\rm s}$, and the maximum force is $F_{\rm ms} = \eta\left(5mV_1^2/8\eta\right)^{3/5}$; the reflected speed is $v_{\rm r} = 2r/T_{\rm r}$, and the maximum force is $F_{\rm mr} = \eta\left(5mV_0'^2/8\eta\right)^{3/5}$.

Similarly, when the pulse passes through the corner in branch 1, the residence time spent by the pulse is

$$T_{12} = \frac{2\sqrt{\pi}}{5V_1'}\left(\frac{5m_1}{8\eta_1}V_1'^2\right)^{2/5}\frac{\Gamma(0.4)}{\Gamma(0.9)}, \tag{14}$$

where V_1' is the velocity of the incoming grain at the beginning of the collision. The velocity of the pulse behind the corner of branch 1 is $v_{\rm s12} = 2r_1/T_{12}$.

Finally, substituting Eq. (6) into Eqs. (13) and (14), we obtain the total time taken by the pulse to pass through branch 1 as

$$t_1 = 23(T_{11} + T_{12}) = 23(1 + 2^{1/5})T_{11}. \tag{15}$$

Using the same method to analyze the transfer of the pulse in branch 2, we can easily derive the time that the pulse requires to pass the granule ahead of and behind corner 2 as

$$\begin{cases} T_{21} = \dfrac{2\sqrt{\pi}}{5V_2}\left(\dfrac{5m_2}{8\eta_2}V_2^2\right)^{2/5}\dfrac{\Gamma(0.4)}{\Gamma(0.9)}, \\ T_{22} = \dfrac{2\sqrt{\pi}}{5V_2'}\left(\dfrac{5m_2}{8\eta_2}V_2'^2\right)^{2/5}\dfrac{\Gamma(0.4)}{\Gamma(0.9)}, \end{cases} \tag{16}$$

where the elastic coefficient $\eta_2 = E_2\sqrt{2r_2}/3(1-\upsilon_2^2)$. The speeds of the pulse moving ahead and behind the corner of branch 2 are $v_{\rm s21} = 2r_2/T_{21}$ and $v_{\rm s22} = 2r_2/T_{22}$.

Substituting Eq. (7) into Eq. (16), we also derive the total time taken by the pulse to pass branch 2 as

$$t_2 = 23(T_{21} + T_{22}) = 23(1 + 2^{1/5})T_{21}. \tag{17}$$

According to the experimental setup, the grains have equal diameter, but different materials. Because of the symmetric branches, the pulse simultaneously arrives at the interface of two branches. If $t_1 < t_2$, the split pulse travels in branch 1 faster than the pulse in branch 2, and the pulse in branch 1 will first arrive at the second interface of the two branches. A delayed effect on the pulse arrival will be observed. When $t_1 = t_2$, the two split pulses recombine at the second interface of the two branches. The condition of the recombination of the two split pulses is given by

$$\frac{m_1^2}{\eta_1^2 V_1} = \frac{m_2^2}{\eta_2^2 V_2}. \tag{18}$$

Substituting Eq. (4) into Eq. (18), and using the expressions of the two elastic coefficients η_1 and η_2, we obtain the condition of the recombination as

$$\xi_1 = \xi_2, \tag{19}$$

where $\xi_i = m_i^3(1-\upsilon_i^2)^2/E_i^2 r_i$.

4. Numerical results

In order to verify the correctness of the binary collision approximation, we compare our theoretical results with the experimental results in Ref. [13]. We solve the coupled nonlinear equations by using a fifth-order Runge–Kutta numerical scheme.[23] For the convenience of computation, we define a dimensionless quantity $\tau \equiv mV_0^2/(rD^{3/2})$, where $D = 3(1-\upsilon^2)/2E$. The dimensionless mass, radius, and elastic constant are united in the stems.

It is observed experimentally that the two identical single solitary waves are transmitted in the branches, and the reflected wave propagates backward from the junction in stem 1. For a uniform system composed of all stainless steel particles, the speed of the incident solitary wave v_i, the speed of the transmitted solitary wave $v_{\rm s}$, and the speed of the reflected solitary wave $v_{\rm r}$ are, respectively, calculated with our binary collision approximation. We obtain that $v_{\rm s}/v_i \approx 0.929$, and $v_{\rm r}/v_i \approx 0.725$. The corresponding velocity ratios form the experimental data in Ref. [13] were $v_{\rm s}/v_i \approx 0.894$ and $v_{\rm r}/v_i \approx 0.734$. Our theoretical results are in good agreement with the experimental results. The ratio of the velocity of beads in the reflected wave to the velocity of beads in the incident wave is 0.16298, which roughly coincides with the theoretical result ($V_1/V_0 = 1/5$). The maximum force $F_{\rm mi}$ between the two neighbor beads in the incident solitary wave of stem 1, the maximum force $F_{\rm ms}$ in the two branches before the corner, and the maximum force $F_{\rm mr}$ in the reflected wave in stem 1 are, respectively, computed from the binary collision approximation, $F_{\rm mi}/F_{\rm ms} \approx 1.553$ and $F_{\rm mi}/F_{\rm mr} \approx 6.899$. Additionally, the experimental results are $F_{\rm mi}/F_{\rm ms} \approx 2$, and $F_{\rm mi}/F_{\rm mr} \approx 8.4$. Such higher experimental values could be induced by the elastic interaction of the solitary wave between the guide wall and dissipation of grains.[13]

For asymmetric systems in which the two branches of the Y-shaped guide were filled with particles of different materials, the stems were always filled with stainless steel particles, the binary collision approximation gives the ratio of the velocity of pulse in the stainless steel branch to the speed pulse traveling in the PTFE branch $v_{\rm s1}/v_{\rm s2} \approx 3.059$. Experimental results indicate that the velocity of the signal in the stainless steel branch is about 4 times higher than the speed of pulse in the PTFE branch. The theoretical results are also in agreement with the experimental results.

To investigate the delay effects on signal arrival, we select asymmetric systems. The stems and branch 1 are filled with 316-typed steels with mass $m = m_1 = 0.45$ g, elastic modulus $E = E_1 = 193$ GPa, and Poisson ratio $\upsilon = \upsilon_1 = 0.30$, while the branch 2 is filled with PTFE beads with mass $m_2 = 0.123$ g, elastic modulus $E_2 = 1.46$ GPa, and Poisson ratio $\upsilon_2 = 0.46$.

All the beads in the system have a radius of 2.38 mm. According to Eq. (19), we calculate the delay factor ξ of the two branches. We find that the factor for the signal traveling in the stainless steel branch $\xi_1 \approx 8.512 \times 10^{-22}$, is smaller than the factor for the pulse in the PTFE branch $\xi_2 \approx 2.280 \times 10^{-19}$. Thus, the signal in the steel branch travels faster than that in the PTFE branch. The conclusion is confirmed by experimental results in Ref. [13] and our simulation results. Figure 5 shows the numerical simulation results of the transmission of split waves in the asymmetric branches of the above system. Numerical simulations indicate that the transfer speed of a pulse in the steel branch is higher than the speed of the signal in the PTFE branch.

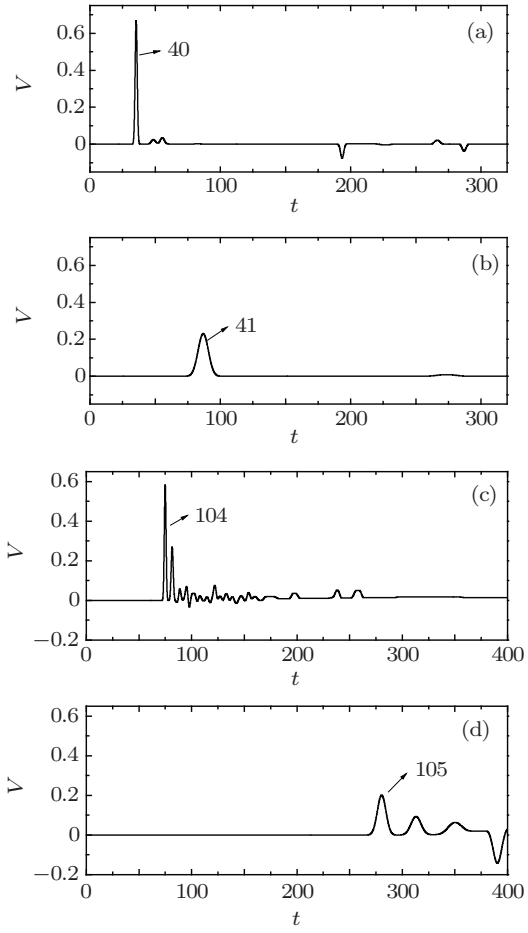


Fig. 5. Numerical results of incident solitary wave propagation in the asymmetric branches. Velocity versus time for (a) bead 40 in branch 1, (b) bead 41 in branch 2, (c) bead 104 in branch 1, and (d) bead 105 in branch 2.

Finally, we investigate the condition of recombination of the split waves for asymmetric systems. In Ref. [13], the recombination of waves traveling in branches together in the final stem only appears in the symmetric branches where each one of the branches is filled with the same particles. However, the binary collision approximation predicts that the signals in the branches could recombine in the final stem for an asymmetric system, only if the condition (19) is satisfied. To validate condition (19), we suppose that the stems and branch 1 are made of 316-typed steel beads, and branch 2 with beads of mass $m_2 = 0.225$ g, elastic modulus $E_2 = 62.9869$ GPa, Poisson ratio $\upsilon_2 = 0.40$ which may be constructed by a polymer coating. The radius of all the beads is 2.38 mm. Figure 6 shows the numerical results of the transmission of waves in the asymmetric branches. It is evident that the speed of the transmitted pulse in one branch is the same as that in the other branch (see Figs. 6(a) and 6(b)). When the transmitted pulses cross over corners of branches, the pulse speeds are reduced. However, the two pulses on branches can also reach corresponding symmetric granules simultaneously (see Figs. 6(c) and 6(d)).

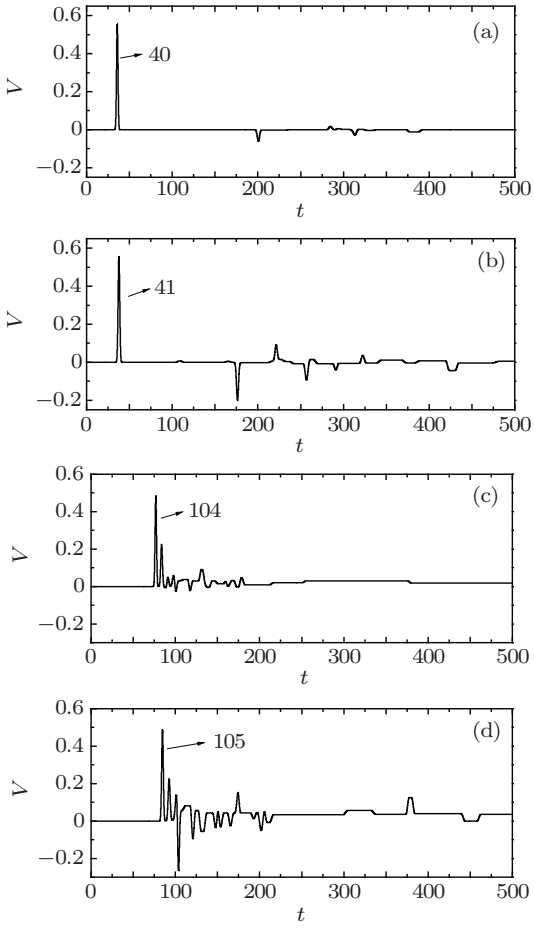


Fig. 6. Numerical results of the propagation of incident solitary waves in the asymmetric branches with the condition of recombination. Velocity versus time for (a) bead 40 in branch 1, (b) bead 41 in branch 2, (c) bead 104 in branch 1, and (d) bead 105 in branch 2.

5. Conclusions

In the paper, we use the binary collision model to investigate pulse propagation in a two-dimensional double Y-shaped granular chain. Such a granular system may set up a bridge between one-dimensional and higher-dimensional granular systems. Some theoretical results which can supplement the experimental results are expected. We obtain the ratios of the transmitted and reflected solitary wave speeds over the incident solitary wave speed, the maximum forces between the

two neighbor beads in solitary wave, and the total time taken by the pulse to pass through each branch. That all of the results come from theoretical analysis is in good agreement with experimental results in Ref. [13]. Furthermore, we analyze the delay effect on the splitting pulse arrival and predict the recombination of the splitting waves traveling in branches together in the final stem for asymmetric systems. The prediction of pulse recombination is also in excellent agreement with the numerical results.

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