Fractional Cattaneo heat equation in a semi-infinite medium*

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To better describe the phenomenon of non-Fourier heat conduction, the fractional Cattaneo heat equation is introduced from the generalized Cattaneo model with two fractional derivatives of different orders. The anomalous heat conduction under the Neumann boundary condition in a semi-infinity medium is investigated. Exact solutions are obtained in series form of the H-function by using the Laplace transform method. Finally, numerical examples are presented graphically when different kinds of surface temperature gradient are given. The effects of fractional parameters are also discussed.

Keywords: Caputo fractional derivative, non-Fourier heat conduction, Cattaneo equation, H-function

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1. Introduction

Non-Fourier effects on heat conduction are of both fundamental interest and great potential value in practical engineering.[1] As everyone knows, the traditional Fourier law, which gives the linear relationship between the heat flux and the temperature gradient, is an early empirical law.[2] The Fourier law itself implies an infinite speed of propagation of thermal disturbance. This is obviously unreasonable. Mathematically, this is caused by the parabolic nature of the heat equation. To avoid this, a relaxation of the flux is introduced, leading to the so-called Cattaneo's equation. Such an equation is a hyperbolic type, and characterizes the combined diffusion and wave-like behavior of heat conduction, and predicts a finite speed of heat propagation.[3]

Recently, the subject of fractional calculus has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences, engineering to biological sciences and economics.[4–8] The global dependency or nonlocal property of fractional derivatives is one of the main reasons for its growing popularity.[9] Using phenomenological arguments, Compte and Metzler[10] considered four possible generalizations of the Cattaneo equation, three of which are supported by a different scheme: continuous time random walks (CTRW), nonlocal transport theory, and the delayed flux–force relationship. Mainardi[11,12] and subsequently Atanackovic *et al.*[13] studied the solutions of the generalized Cattaneo equation of the form

$$\partial_t^\beta T + \tau \partial_t^\alpha T = \mathscr{D} \partial_{xx} T \tag{1}$$

in finite and infinite domains, where $0 < \beta \leq \alpha \leq 2$, and $\mathscr{D}$ is the generalized thermal diffusivity. Thus, we are able to recover the heat equation ($\tau = 0$, $\beta = 1$), the Cattaneo equation ($\alpha = 2$, $\beta = 1$), and the wave equation ($\alpha = \beta = 2$). Equation (1) also contains all the cases given by Compte and Metzler.[10] Povstenko[14] showed that all fractional generalizations of the Cattaneo equation in Ref. [10] can be obtained from the time-nonlocal generalization of Fourier law with kernels being the functions of Mittag–Leffler type. The time–space fractional generalizations of the Cattaneo equation are also considered in Refs. [15] and [16]. The physical basis of these fractional models can be found in the fractal or porous structure of the medium.[17] As with CTRW models, the fractional parameter characterizes the degree of heterogeneity with regard to heat transfer, which leads to anomalous behavior in the thermal response of the medium.[18,19] Lewandowska and Kosztołowicz[20,21] used the fractional Cattaneo equation to study the subdiffusive impedance phenomena. Furthermore, this approach has been employed to model subdiffusive heat transfer in geothermal reservoirs.[22] The numerical methods of the related fractional partial differential equations are also investigated.[23,24] Very recently, Ghazizadeh *et al.*,[25] who successfully applied the Levenberg–Marquardt method on the inverse fractional heat transfer problem, showed that the fractional Cattaneo model can give a good temperature prediction matching the experimental data.

As a typical non-Fourier heat conduction phenomenon, pulsed laser heating is now widely used in modern materials science.[26,27] There have been far too many recent works in this area, such as Refs. [28]– [32]. In these works, the classical Cattaneo-type and Jefferys-type models are often used

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as the constitutive relationships.[3,26,27] However, the experiments, which study the non-Fourier heat conduction phenomena in a porous material heated by a microsecond laser pulse, demonstrate that there are still many disagreements between the experimental results and the theoretical analyses with these two models.[29] Consider the successful applications of fractional models in anomalous heat conduction, we study the temperature distribution of the fractional Cattaneo heat equation. The paper is organized as follows. In Section 2, we first introduce the fractional Cattaneo equation from the generalized Cattaneo model with two fractional derivatives of different orders. Then, the exact solution of the fractional Cattaneo heat equation with Nermann boundary conditions in a semi-infinity medium is obtained. Three numerical examples are given and the influences of fractional parameters are discussed in Section 3. The conclusions are presented in Section 4.

2. Mathematical modeling of the heating process

2.1. Fractional Cattaneo heat equation

Among many constitutive models of heat flux, the constitutive relationship proposed by Cattaneo and Vernotte

$$\boldsymbol{q}+\tau\partial_t\boldsymbol{q}=-\lambda\,\mathrm{grad}\,T \tag{2}$$

is one of the most widely used.[3,26,27] Here, $\boldsymbol{q}$, T, τ, and λ are the heat flux vector, the temperature, the relaxation time, and the thermal conductivity, respectively. If the first term on the left of Eq. (2) is rewritten in the form: $\boldsymbol{q}=\partial_t^0\boldsymbol{q}$, then we encounter two different derivatives in time. Thus, replacing the two classical integer derivative with respect to time by the fractional operators, one can obtain the following fractional generalization of the classical Cattaneo model (2):

$$\partial_t^{\beta-1}\boldsymbol{q}+\tau\partial_t^{\alpha-1}\boldsymbol{q}=-\lambda\,\mathrm{grad}\,T, \tag{3}$$

where $0<\beta\leq\alpha\leq 2$, and τ and λ are nonnegative constants. The physical dimensions of τ and λ are $[\tau]=\mathrm{s}^{\alpha-\beta}$ and $[\lambda]=\mathrm{W}/(\mathrm{m}\cdot\mathrm{K}\cdot\mathrm{s}^{\beta-1})$, respectively (the dimensions of τ and λ can be obtained from the definition (4) by dimensional analysis). The fractional derivative or integral in Eq. (3) is defined as

$$\partial_t^{\gamma}f(x,t)\overset{\mathrm{def}}{=}\begin{cases}\dfrac{1}{\Gamma(1-\gamma)}\displaystyle\int_0^t\frac{f'(x,t')}{(t-t')^{\gamma}}\,\mathrm{d}t',\ 0<\gamma<1,\\ \dfrac{1}{\Gamma(-\gamma)}\displaystyle\int_0^t\frac{f(x,t')}{(t-t')^{1+\gamma}}\,\mathrm{d}t',\ \gamma<0,\end{cases} \tag{4}$$

which is referred to as the Caputo fractional derivative.[4] It should be noted that all four fractional generalizations of the Cattano constitutive relationship given by Povstenko[14] can be viewed as special cases of Eq. (3). Through the Laplace transform of Eq. (3) (neglecting the initial value $\boldsymbol{q}(0+)$), we can rewrite Eq. (3) as a time-nonlocal constitutive relationship

$$\boldsymbol{q}(t)=-\frac{\lambda}{\tau}\int_0^t(t-t')^{\alpha-2}E_{\alpha-\beta,\alpha-1}\Big(-\frac{(t-t')^{\alpha-\beta}}{\tau}\Big)\times\mathrm{grad}\,T(t')\,\mathrm{d}t', \tag{5}$$

in which $E_{\mu,\nu}(z)=\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+\nu)}$, is the Mittag–Leffler function.[4] Combining Eq. (3) with a law of conservation of energy

$$\rho c\partial_t T=-\,\mathrm{div}\,\boldsymbol{q}, \tag{6}$$

with ρ being the mass density and c the specific heat capacity, one obtains the fractional Cattaneo heat equation as

$$\partial_t^{\beta}T+\tau\partial_t^{\alpha}T=\mathscr{D}\triangle T,\ \ \mathscr{D}=\lambda/\rho c, \tag{7}$$

where $\mathscr{D}$ is the generalized thermal diffusivity of dimension $[\mathscr{D}]=\mathrm{m}^2/\mathrm{s}^{\beta}$.

2.2. Temperature distribution

In this work, we consider the temperature distribution in a semi-infinite medium ($0\leq x<\infty$), which is initially at a uniform temperature. The boundary surface temperature gradient is given by a time-dependent function. It will be assumed that heat can only enter or leave the body through the surface at $x=0$ and that any thermal process begins at time $t=0$. This yields the following one-dimensional fractional Cattaneo heat equation:

$$\partial_t^{\beta}T+\tau\partial_t^{\alpha}T=\mathscr{D}\partial_{xx}T,\ \ 0<x<\infty,\ \ t>0. \tag{8}$$

Furthermore, it is assumed that temperature and the time derivative of temperature are initially zero throughout the region. It is also assumed that the temperature far from the surface will be neglected. These assumptions lead to the following initial and boundary conditions:

$$T(x,0)=\partial_t T(x,0)=0,\ \ 0\leq x<\infty, \tag{9}$$

$$T(\infty,t)=0,\ \ t\geq 0, \tag{10}$$

$$-\lambda\partial_x T(0,t)=\partial_t^{\beta-1}q(0,t)+\tau\partial_t^{\alpha-1}q(0,t). \tag{11}$$

For simplicity, let

$$q_{\mathrm{w}}(t)=\frac{1}{\lambda}\big(\partial_t^{\beta-1}q(0,t)+\tau\partial_t^{\alpha-1}q(0,t)\big). \tag{12}$$

The exact solution of the above initial-boundary values problem can be possible by adapting the discretization method for solving the inverse Laplace transform. This method is widely used to obtain the analytical solutions of fractional differential equations.[4,33–35] By applying the Laplace transform $\overline{h}(s)=\displaystyle\int_0^{\infty}h(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$ with respect to the time variable t in Eq. (8) and from the initial and boundary conditions (9)–(12), we obtain

$$\frac{\mathrm{d}^2\overline{T}}{\mathrm{d}x^2}=\frac{s^{\beta}+\tau s^{\alpha}}{\mathscr{D}}\,\overline{T}, \tag{13}$$

$$\overline{T}(\infty,s)=0,\quad \frac{\mathrm{d}\overline{T}(0,s)}{\mathrm{d}x}=-\overline{q}_{\mathrm{w}}(s), \tag{14}$$

where $\overline{T}(x,s)$ and $\overline{q}_{\mathrm{w}}(s)$ are the Laplace transforms of $T(x,t)$ and $q_{\mathrm{w}}(t)$, respectively. Thus, from Eqs. (13) and (14), the solution in the transformed domain can be finally written as

$$\overline{T}(x,s)=\overline{q}_{\mathrm{w}}(s)\frac{\sqrt{\mathscr{D}}}{\sqrt{s^{\beta}+\tau s^{\alpha}}}\exp\Big(-\frac{x}{\sqrt{\mathscr{D}}}\sqrt{s^{\beta}+\tau s^{\alpha}}\Big). \tag{15}$$

Based on the convolution theorem of Laplace transform,[36] the solution in the time domain takes the form

$$T(x,t)=\int_0^t G(x,t')q_{\mathrm{w}}(t-t')\,\mathrm{d}t', \tag{16}$$

where the Laplace transform of $G(x,t)$ is given by

$$\overline{G}(x,s)=\frac{\sqrt{\mathscr{D}}}{\sqrt{s^{\beta}+\tau s^{\alpha}}}\exp\Big(-\frac{x}{\sqrt{\mathscr{D}}}\sqrt{s^{\beta}+\tau s^{\alpha}}\Big). \tag{17}$$

In order to determine the inverse Laplace transform of Eq. (17), it is necessary to write it in the following form when $\tau\neq 0$:

$$\overline{G}(x,s)=\frac{\sqrt{\mathscr{D}}}{\sqrt{\tau s^{\alpha}}}\frac{\mathrm{e}^{-r\sqrt{z}}}{\sqrt{z}}, \tag{18}$$

where

$$r=\frac{x\sqrt{\tau s^{\alpha}}}{\sqrt{\mathscr{D}}},\quad z=1+\tau^{-1}s^{\beta-\alpha}. \tag{19}$$

To expand the Laplace transform (18) as a series expansion, we first give the Taylor series expansion for $g(z)=\mathrm{e}^{-r\sqrt{z}}/\sqrt{z}$ about $z=1$

$$\begin{aligned} g(z) &= rH_{0,1}^{1,0}\big[r\sqrt{z}\big|_{(-1,1)}^{-}\big]\\ &= \sum_{k=0}^{\infty}\frac{g^{(k)}(1)}{k!}(z-1)^k\\ &= \sum_{k=0}^{\infty}\frac{(z-1)^k}{k!}rH_{1,2}^{1,1}\Big[r\,\Big|\,{}^{(0,1/2)}_{(-1,1),(k,1/2)}\Big], \end{aligned} \tag{20}$$

where $H_{p,q}^{m,n}\big[z\,\big|\,{}^{(a_p,A_p)}_{(b_q,B_q)}\big]$ is the Fox H-function.[37] The identity (1.125) and the formula (1.69) for the k-th derivative of H-function in the book of Mathai *et al.*[37] are used in Eq. (20). Then, using the properties of H-function (see Section 1.4 in Ref. [37]) and the Legendre's duplication rule for the Gamma function, we modify the last part in Eq. (20) as follows:

$$\begin{aligned} rH_{1,2}^{1,1}\Big[r\big|_{(-1,1),(k,1/2)}^{(0,1/2)}\Big] &= 2rH_{1,2}^{1,1}\Big[r^2\big|_{(-1,2),(k,1)}^{(0,1)}\Big]\\ &= (-1)^k 2rH_{1,2}^{2,0}\Big[r^2\big|_{(k,1),(-1,2)}^{(0,1)}\Big]\\ &= 2(-1)^k H_{1,2}^{2,0}\Big[r^2\big|_{(k+1/2,1),(0,2)}^{(1/2,1)}\Big]\\ &= \frac{(-1)^k}{\sqrt{\pi}}H_{0,2}^{2,0}\Big[\frac{r^2}{4}\Big|_{(0,1),(k+1/2,1)}^{-}\Big]. \end{aligned} \tag{21}$$

Finally, using Eqs. (18)–(21), we obtain the series expansion for the Laplace transform

$$\begin{aligned} \overline{G}(x,s) &= \frac{\sqrt{\mathscr{D}}}{\sqrt{\pi\tau}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{s^{k(\beta-\alpha)-\alpha/2}}{\tau^k}\\ &\quad\times H_{0,2}^{2,0}\Big[\frac{\tau x^2 s^{\alpha}}{4\mathscr{D}}\Big|_{(0,1),(k+1/2,1)}^{-}\Big]. \end{aligned} \tag{22}$$

Thus, the inverse Laplace transform of Eq. (22) can be evaluated term by term using the inverse Laplace transform of the H-function (Eq. (2.20) in Ref. [37]). Finally, we obtain the solution

$$\begin{aligned} G(x,t) &= \frac{\sqrt{\mathscr{D}}}{\sqrt{\pi\tau}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{t^{k(\alpha-\beta)+\alpha/2-1}}{\tau^k}\\ &\quad\times H_{1,2}^{2,0}\Big[\frac{\tau x^2}{4\mathscr{D}t^{\alpha}}\Big|\,{}^{(k(\alpha-\beta)+\alpha/2,\alpha)}_{(0,1),(k+1/2,1)}\Big]. \end{aligned} \tag{23}$$

Now let us make some analyses for the fractional Cattaneo model.

(i) When $\tau=0$, equation (8) becomes the fractional diffusion-wave equation,[11,12] and Eq. (17) can be simplified as

$$\begin{aligned} \overline{G}(x,s) &= \frac{\sqrt{\mathscr{D}}}{\sqrt{s^{\beta}}}\exp\Big(-\frac{x}{\sqrt{\mathscr{D}}}\sqrt{s^{\beta}}\Big)\\ &= \frac{\sqrt{\mathscr{D}}}{\sqrt{s^{\beta}}}H_{0,1}^{1,0}\Big[\frac{x}{\sqrt{\mathscr{D}}}\sqrt{s^{\beta}}\Big|_{(0,1)}^{-}\Big]. \end{aligned} \tag{24}$$

Using the inverse Laplace transform of the Fox H-function on Eq. (24), we obtain

$$\begin{aligned} G(x,t) &= \sqrt{\mathscr{D}}t^{\beta/2-1}H_{1,1}^{1,0}\Big[\frac{x}{\sqrt{\mathscr{D}t^{\beta}}}\Big|\,{}^{(\beta/2,\beta/2)}_{(0,1)}\Big]\\ &= \sqrt{\mathscr{D}}t^{\beta/2-1}W\Big(-\frac{x}{\sqrt{\mathscr{D}t^{\beta}}};-\frac{\beta}{2},\frac{\beta}{2}\Big), \end{aligned} \tag{25}$$

where $W(z;\mu,\nu)=\sum_{k=0}^{\infty}\dfrac{z^k}{k!\Gamma(\mu k+\nu)}$, is the Wright function.[4] Furthermore, if $\tau=0$ and $\beta=1$, then Eq. (8) is the classical heat equation. In addition, using the series expansion of H-function or the series definition of the Wright function, we can easily prove that

$$G(x,t)=\sqrt{\frac{\mathscr{D}}{\pi t}}\exp\Big(-\frac{x^2}{4\mathscr{D}t}\Big), \tag{26}$$

which is just the result of the heat equation given in Refs. [7] and [26].

(ii) When $\alpha=2$ and $\beta=1$, equation (8) is the classical Cattaneo equation, and the solution has been given in Ref. [26], namely,

$$G(x,t)=\sqrt{\frac{\mathscr{D}}{\tau}}\,\mathrm{e}^{-t/2\tau}I_0\Big(\frac{1}{2\tau}\sqrt{t^2-\frac{\tau}{\mathscr{D}}x^2}\Big)u\Big(t-x\sqrt{\frac{\tau}{\mathscr{D}}}\Big). \tag{27}$$

Here, $I_0(\cdot)$ is the modified Bessel function of order zero and $u(\cdot)$ denotes the unit step function. Our solution (23) for $\alpha=2$ and $\beta=1$ can be viewed as a new form in this case.

3. Numerical examples and discussion

Based on the solutions derived in the previous section, numerical calculations are carried out to investigate the temperature distributions when the surface temperature gradient, $q_w(t) = q_w f(t)$, is given. We discuss three kinds of surface temperature gradient which are widely used in the numerical simulation of pulsed laser heating.[26–32] The influences of the model parameters are also studied. For simplicity, we take $q_w = 1.0$, $\mathscr{D} = 1.0$, and $\tau = 0.1$ in all numerical examples.

Example 1 In this example, we assume that the surface temperature gradient is constant (i.e., $f(t) = u(t)$), and the fractional parameters in Eq. (8) are taken as $\alpha = 2.0$ and $\beta \in [1.0, 2.0]$, which correspond to the GCE II in Refs. [10] and [14].

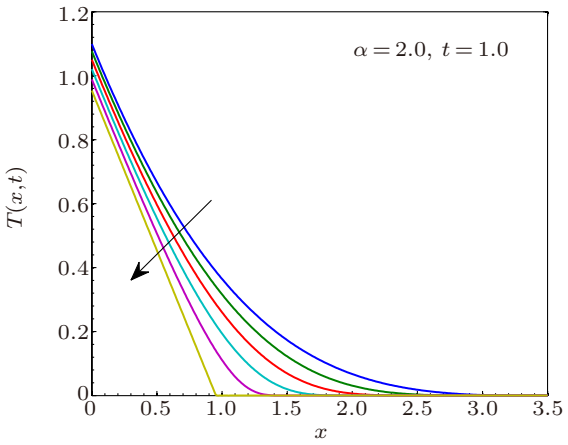


Fig. 1. (color online) Temperature $T(x,t)$ versus x with β=1.0, 1.2, 1.4, 1.6, 1.8, 2.0. The arrow indicates plots with increasing β.

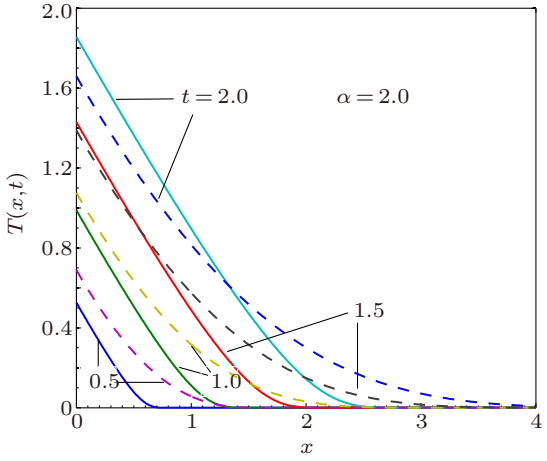


Fig. 2. (color online) Temperature $T(x,t)$ versus x at various times for $\beta = 1.2$ (dashed line) and $\beta = 1.8$ (continuous line).

Figure 1 presents the temperature profile for different values of β as a function of the space variable x at a fixed time. It can be seen clearly that the temperature values of the GCE II range between those of the wave equation ($\beta = 2$) and the Cattaneo equation ($\beta = 1$). As we all know, propagation speeds of thermal disturbances corresponding to the Cattaneo equation and the wave equation are $c_t = \sqrt{\mathscr{D}/\tau}$ and $c_w = \sqrt{\mathscr{D}/(1+\tau)}$, respectively.[3,26] Thus, when the GCE II is used, the speed for heat propagation is between c_t and c_w, and the speed will be lower as the value of β increases. This is consistent with an intermediate process between the Cattaneo equation and the wave equation.[38] The temperature profile for a range of time t as a function of variable x for $\beta = 1.2$ and $\beta = 1.8$ is depicted in Fig. 2. One can see that the boundary surface temperature decreases with increasing β for a small t, whereas increasing β raises the temperature for a large t. Figure 3 demonstrates the boundary surface temperature $T(0,t)$ with various β. This paradoxical behaviour can be seen more clearly. In view of the applications of the boundary surface temperature, this paradoxical behaviour can be explained in terms of the following expression of boundary surface temperature in the Laplace domain:

$$\overline{T}(0,s) = \frac{q_w\sqrt{\mathscr{D}}}{s\sqrt{s^\beta + \tau s^\alpha}}. \tag{28}$$

On the basis of the operational calculus theory, we calculate the asymptotic expressions for $\overline{T}(0,s)$ for a large and small s, respectively. Then, we have the following expressions:

$$T(0,t) \propto t^{\alpha/2}(1 - t^{\alpha-\beta}/\tau + t^{2(\alpha-\beta)}/\tau^2 + \cdots), \tag{29}$$

for small t, and

$$T(0,t) \propto t^{\beta/2}(1 - \tau t^{\beta-\alpha} + \tau^2 t^{2(\beta-\alpha)} + \cdots), \tag{30}$$

for large t. Equations (29) and (30) give a good fit with the numerical results shown in Figs. 2 and 3. It should be noted that equations (29) and (30) can also be obtained from the solution by using the series expansions of the H-function.[37]

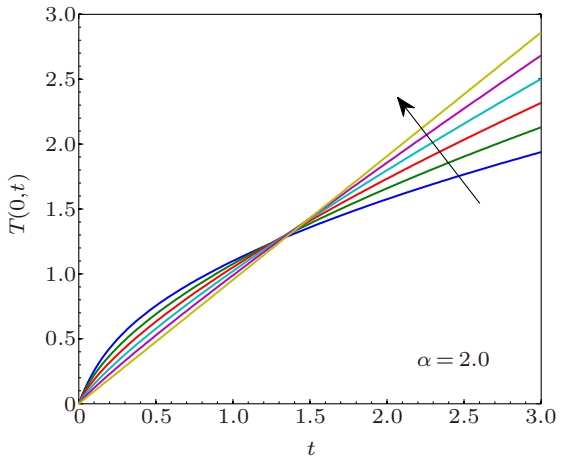


Fig. 3. (color online) Boundary surface temperature $T(0,t)$ versus t with $\beta = 1.0$, 1.2, 1.4, 1.6, 1.8, and 2.0. The arrow indicates plots with increasing β.

Example 2 On the basis of Example 1, we further study the temperature distribution in which the boundary surface temperature gradient is given by a step function, $f(t) = u(t) - u(t - t_s)$. Here, t_s is the width of the step function, and we take $t_s = 3$ for numerical computation. The GCE I in Refs. [10] and [14] is used, that is, $\beta \in [0.5, 1.0]$ and $\alpha = 2\beta$ in Eq. (8).

Figure 4 depicts the temporal variation of the temperature distribution at different locations inside the medium for $\beta = 0.9$. It can be observed that the temperature rises sharply in the early heating period because of the internal energy gain from the heat source. As the heating continues, the temperature rise becomes gradual due to the enhancement of the heat transfer from the surface region to the medium. Once the heating progress ends ($t > 3$), the temperature first reduces rapidly, and during the cooling period, the temperature decay becomes gradual. For the purpose of comparison, the boundary surface temperatures with different values of β are shown in Fig. 5. It can be seen that the heating progress ($t < 3$) is similar to Example 1. However, the greater β is, the faster the temperature drops when the cooling process begins.

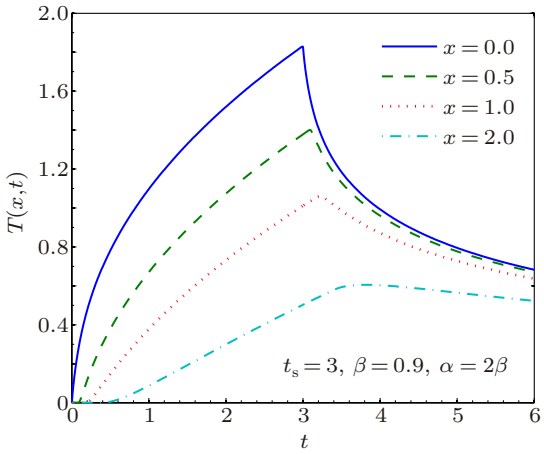


Fig. 4. (color online) Temperature distribution over time at different locations.

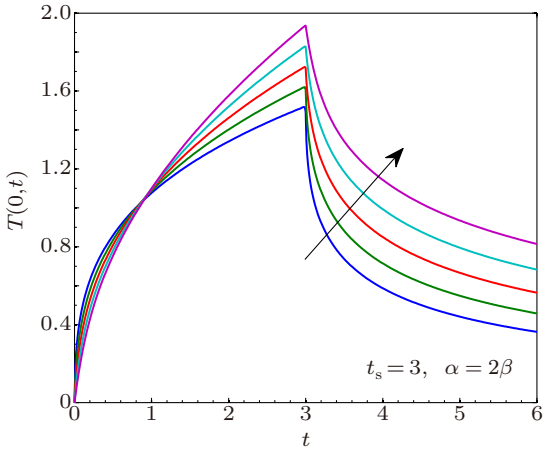


Fig. 5. (color online) Boundary surface temperature $T(0,t)$ versus t with β=0.6, 0.7, 0.8, 0.9, and 1.0. The arrow indicates plots with increasing β.

Example 3 Finally, we study the temperature distribution when the boundary surface temperature gradient is given by an exponential function $f(t) = \exp(-\mu t) - \exp(-\nu t)$, where μ and ν are positive parameters. The GCE III in Refs. [10] and [14] is used, that is, $\beta \in (0, 1.0]$ and $\alpha = 1 + \beta$ in Eq. (8).

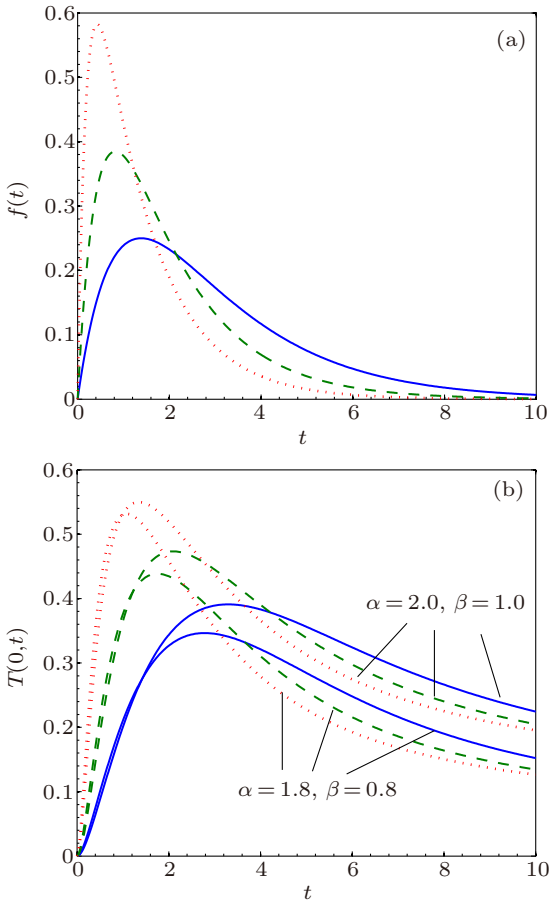


Fig. 6. (color online) (a) Function $f(t)$ and (b) boundary surface temperature $T(0,t)$ verus t for $\mu = 1/2$, $\nu = 1$ (continuous line), $\mu = 2/3$, $\nu = 2$ (dashed line), and $\mu = 5/6$, $\nu = 5$ (dotted line).

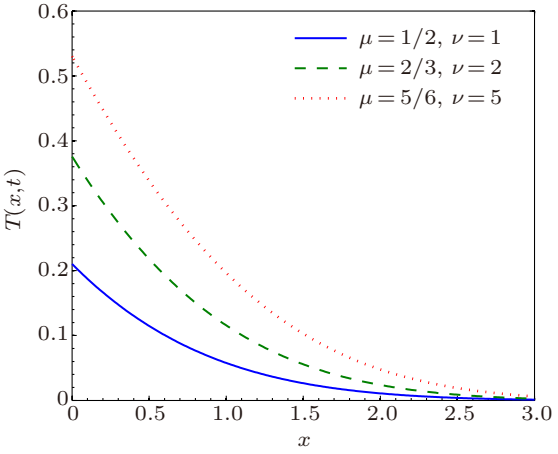


Fig. 7. (color online) Temperature $T(x,t)$ versus x for three exponential pulses at time $t = 1$ when $\alpha = 1.8$ and $\beta = 0.8$.

Figure 6 shows the temporal variation of exponential function $f(t)$ and the corresponding temperature rise at the boundary surface. In the early heating period, the rise of boundary temperature for the Cattaneo model ($\beta = 1$) and the fractional Cattaneo model ($\beta = 0.8$) is almost the same. However, the boundary temperature attains a higher value for the classical Cattaneo model than that corresponding to the fractional Cattaneo model. It is also noted that the temporal variation of temperature at the surface does not follow exactly the

temporal variation of $f(t)$ because of the energy transfer from the surface region to the medium. The temperature distribution inside the medium is also shown in Fig. 7. It can be seen that the boundary gradient distribution with time has a significant effect on the temperature distribution inside the medium.

4. Conclusions

We introduced the fractional Cattaneo constitutive model and the corresponding fractional Cattaneo equation with two fractional derivatives of different orders. An exact solution for the temperature distribution has been derived for the Neumann boundary value problem in a semi-infinite medium. Some special cases are also discussed. The numerical examples indicate that the fractional model parameters α and β ($\leq \alpha$) have an important influence on the temperature field. Furthermore, the fractional derivative of order α plays a major role on the temperature distribution for a short time, whereas the fractional derivative of order β has a strong effect for a long time. The model and results in this paper provide a new theoretical perspective for an in-depth study of the non-Fourier heat conduction. Given the successful applications of the fractional operators to modern mechanics,[38] we believe that the fractional Cattaneo constitutive relationship and the corresponding fractional Cattaneo equation here will be useful in science and engineering.

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