Analytical model of an acoustic diode comprising a superlattice and a nonlinear medium*

Gu Zhong-Ming(顾仲明), Liang Bin(梁 彬)†, and Cheng Jian-Chun(程建春)

Key Laboratory of Modern Acoustics, MOE, Institute of Acoustics, Department of Physics, Nanjing University, Nanjing 210093, *China*

(Received 1 May 2012; revised manuscript received 18 June 2012)

We give an analytical analysis to the acoustic propagation in an acoustic diode (AD) model formed by coupling a superlattice (SL) with a nonlinear medium. Analytical solutions of the acoustic transmission are obtained by studying the propagations in the SL and the nonlinear medium separately with the conventional transfer-matrix method and a perturbation technique. Compared with the previous numerical method, the proposed approach contributes a better physical insight into the intrinsic mechanism of acoustic rectification and helps us to predict the performance of an AD within the effective rectifying bands in a simple way. This is potentially significant for the practical design and fabrication of AD devices.

Keywords: acoustic diode, transfer-matrix method

PACS: 43.25.+y, 43.35.+d **DOI:** 10.1088/1674-1056/22/1/014303

1. Introduction

The electric diode is the first man-made device enabling a rectifying current flux and it contributes significantly to the advancement of modern society in various aspects. Inspired by its invention, much important work has been done in an attempt to control other forms of energy in a similar way.[1–6] Li *et al.* proposed a theoretical model of a thermal diode that can rectify thermal flux.[1,2] Nesterenko *et al.* carried out the theoretical and experimental investigation of a one-way device for solitary waves.[3] Recently, Cheng *et al.* presented both theoretically and experimentally the first acoustic diode (AD) to rectify acoustic waves (another important form of classical wave with a much longer research history than that of electricity) by simply coupling a superlattice (SL) with a nonlinear medium.[4–6] In their work, the transmission of an AD was predicted theoretically by using an extended transfer-matrix method combined with a perturbation technique,[4,6] which together can be used to calculate the nonlinear acoustic propagation in a multilayered structure with nonlinearity.[7] This numerical method is applicable to individual AD models but lacks the generality of analytical solutions. An AD model comprising an SL and a nonlinear medium that is in fact simply structured cannot give an analytical description of the parameter-dependence of its rectifying property, which is academically significant for obtaining a physical insight into the intrinsic mechanism of acoustic rectification. In practice, on the other hand, it is also particularly important to develop a simple and fast method to approximately predict the efficiency of an AD in an analytical manner, which may remarkably facilitate the design and fabrication of the AD device.

In this study, we give an analytical analysis of the acoustic propagation in an AD model and present a simple method that enables a quick estimation of its performance. The propagations of the fundamental wave (FW) and the second harmonic wave (SHW) in a nonlinear medium are investigated by employing a perturbation technique, and their respective transmissions in the SL are predicted using a conventional transfer-matrix method.[8,9] The proposed method takes advantage of analytical solutions and helps us to predict the performance of an AD simply within the rectifying bands. This is both academically and practically significant for the design and application of ADs.

2. Numerical model

Consider the propagation of a longitudinal wave of frequency ω in the theoretical model of an AD shown in Fig. 1. The SL is formed by alternately laminating media I and II in a periodic manner, the total number of the periods is N, and medium III is chosen as a medium with particularly strong acoustic nonlinearity. For medium j $(j = \mathrm{I}, \mathrm{II}, \mathrm{III})$, the parameters d_j, ρ_j, c_j, and Γ_j are employed to represent the thickness, the mass density, the longitudinal wave velocity, and the equivalent nonlinearity parameter, respectively. The nonlinearities of media I and II are assumed negligible as compared with the extremely large value of Γ_{III}, i.e., $\Gamma_{\mathrm{I}} = \Gamma_{\mathrm{II}} = 0$. The material parameters of medium III are assumed to be identical to those of medium I, except for the value of Γ_{III}. In practice, the validity of this assumption can be guaranteed by employing a bubbly soft medium as medium III, in which the natural frequency of each bubble is large enough so that the quasi-static condition is

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10904068, 10834009, 11174138, and 11174139), the Research Fund for the Doctoral Program (for new scholar) of Higher Education of China (Grant No. 20100091120039), and the Priority Academic Program Development of Jiangsu Higher Education Institutions, China.

†Corresponding author. E-mail: liangbin@nju.edu.cn

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satisfied.[4,5] In such cases, the bubble resonance effect is absent and the acoustic attenuation is thus negligible, while the equivalent nonlinearity parameter reaches the maximal value $11\lambda/64\mu$ at an optimal bubble volume fraction $4\mu/3\lambda$, with λ and μ being the Lame coefficients. By choosing a particular soft medium with sufficiently small μ and adjusting the bubble volume fraction β in an optimal manner, a sample of medium III with extremely high nonlinearity parameter can be produced, e.g., the value of $\Gamma_{\rm III}$ can exceed 10^4 in bubbly Plastisol, for which $\lambda/\mu = 2\times10^5$.[5] Meanwhile, the reflection resulting from the acoustic impedance mismatch between medium III and the host medium can be kept at a low level due to the trivial magnitude of β ($\sim 10^{-6}$ for Plastisol). The abbreviations LB and RB refer to the left-most and the right-most boundaries, respectively. In general, the positive and the negative directions of an AD refer to the propagating directions of the acoustic waves that are coming from the side of the nonlinear medium and the SL, respectively (RB and LB in this particular case, respectively). The basic mechanism of an AD can be described briefly as follows.[4] The parameters of the SL must be tuned to yield a stop band at the driving frequency but to yield a pass band at the doubled frequency. The acoustic wave that propagates along the negative direction will hit the SL first. Since the acoustic parameters of media I and III are identical, the transmission property of the AD is virtually identical with that of the SL. When the acoustic wave propagates along the positive direction, however, the acoustic energy will be partially converted to SHW within medium III before the wave enters the SL. The generated SHW can pass through the SL almost totally but the original FW is blocked. As a result, the acoustic energy flux is allowed to pass through the AD model only along the positive direction. It is readily apparent that such a one-way transmission of acoustic waves is achieved by introducing a nonlinearity effect to break the system's symmetry.[4] It is noteworthy that an AD model has a simple 1D structure which is only partially nonlinear: the conversion of acoustic energy from the FW to the SHW occurs within the nonlinear medium III but the SL has a totally linear structure. Therefore, instead of seeking the numerical solution of the acoustic transmission in an AD model by employing the extended transfer-matrix method,[7] the total transmission of acoustic waves can be achieved in a much simpler manner by separately analyzing the acoustic propagations in the SL and those in the nonlinear medium, and an analytical solution will eventually be obtained, as will be shown in the following.

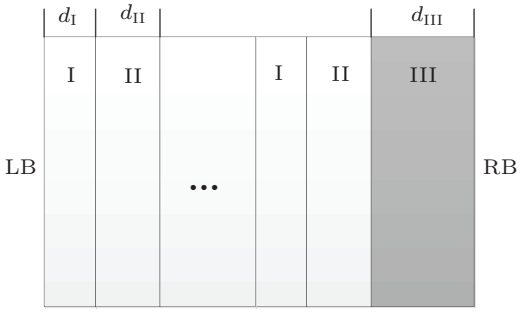


Fig. 1. Schematic illustration of the theoretical model of an AD comprising an SL and a nonlinear medium.

For an AD model with fixed structural parameters, the acoustic rectification will occur only when the driving frequency ω falls within particular frequency ranges referred to as the effective rectifying bands (ERBs), whose locations are completely determined by the band structure of the SL. The dispersion relationship of linear band structure for the SL with a simple 1D structure can be analytically expressed as

$$\begin{aligned}\cos K(d_{\rm I}+d_{\rm II}) \;=\; & \cos(\omega d_{\rm I}/c_{\rm I})\cos(\omega d_{\rm II}/c_{\rm II}) \\ & -\cosh\eta\sin(\omega d_{\rm I}/c_{\rm I})\sin(\omega d_{\rm II}/c_{\rm II}),\end{aligned}$$

where $\eta = \ln(\rho_{\rm I}c_{\rm I}/\rho_{\rm II}c_{\rm II})$, and K is the usual Bloch wavenumber. Figure 2 shows the dispersion relationship of an SL, in which the ERBs are marked as the gray regions, where ω is located in the band gap of the SL, while 2ω falls within the pass band. In the present study, the chosen media I and II are water and glass, respectively, for which the material parameters are $\rho_{\rm I} = 998$ kg/m^3, $c_{\rm I} = 1483$ m/s and $\rho_{\rm II} = 2767$ kg/m^3, $c_{\rm II} = 5784$ m/s. We assume that the whole system is immersed in an infinite matrix of water. Unless otherwise stated, the system parameters are $d_{\rm II} = 0.7d_{\rm I}$, $d_{\rm III} = 1.3d_{\rm I}$, $N = 5$, and $\Gamma_{\rm III} = 10^4$. The pressure of the incident wave is 4 kPa.

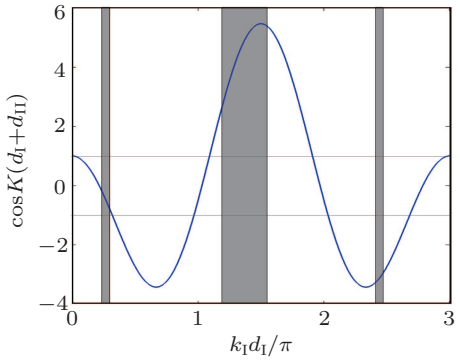


Fig. 2. (color online) Dispersion relationship of the linear band structure of an SL.

When an acoustic wave of a particular frequency propagates in the SL along the two opposite directions, the transmissions are identical due to the linear nature of the structure. The problem of the wave propagation in the SL can be conveniently solved by using a conventional transfer-matrix method.

The transfer matrix $\boldsymbol{M}$ for an individual period can be written as

$$\boldsymbol{M} = \begin{pmatrix} M_1 & M_2 \\ M_2^* & M_1^* \end{pmatrix},$$

where

$$\begin{aligned} M_1 &= \frac{(R_1+R_2)^2}{4R_1R_2}\,\mathrm{e}^{-\mathrm{i}k_\mathrm{I}d_\mathrm{I}-\mathrm{i}k_\mathrm{II}d_\mathrm{II}} \\ &\quad + \frac{(R_1-R_2)(R_2-R_1)}{4R_1R_2}\,\mathrm{e}^{\mathrm{i}k_\mathrm{I}d_\mathrm{I}-\mathrm{i}k_\mathrm{II}d_\mathrm{II}}, \\ M_2 &= \frac{(R_1+R_2)(R_2-R_1)}{4R_1R_2}\,\mathrm{e}^{-\mathrm{i}k_\mathrm{I}d_\mathrm{I}+\mathrm{i}k_\mathrm{II}d_\mathrm{II}} \\ &\quad + \frac{(R_1-R_2)(R_1+R_2)}{4R_1R_2}\,\mathrm{e}^{\mathrm{i}k_\mathrm{I}d_\mathrm{I}+\mathrm{i}k_\mathrm{II}d_\mathrm{II}}, \end{aligned}$$

and R_1 and R_2 are employed to represent the acoustic impedance ratios of media I and II, respectively.

Therefore the transfer matrix $\boldsymbol{M}^N$ for N periods is connected as

$$\boldsymbol{M}^N = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} 1/t & r*/t* \\ r/t & 1/t* \end{pmatrix},$$

where r and t refer to the sound pressure reflection and transmission coefficients, respectively. Then the energy transmission rate of an acoustic wave with frequency ω in the SL can be obtained as

$$T_\mathrm{SL}(\omega) = |t^2| = \frac{1}{|M_{11}|^2}.$$

For the nonlinear acoustic propagation in medium III, the acoustic wave can be expressed as the sum of the FW and the SHW up to the second order approximation, i.e.,

$$p \approx p_1 + p_2, \quad E \approx E_1 + E_2,$$

where p and E refer to the acoustic pressure and the energy of the acoustic wave, respectively, and subscripts 1 and 2 refer to the FW and the SHW, respectively.

There is a simple relationship between the amplitudes of the FW and the SHW (referred to as $p_{1\mathrm{a}}$ and $p_{2\mathrm{a}}$ respectively)[10]

$$p_{2\mathrm{a}} = \Gamma_\mathrm{III} d_\mathrm{III} s p_{1\mathrm{a}}^2,$$

where $s = k_\mathrm{I}/(4\rho_\mathrm{I}c_\mathrm{I}^2)$, and x is the propagating distance. Then we have

$$E_2 = \frac{\Gamma^2 k_1^2 x^2}{16\rho_\mathrm{I}^2 c_\mathrm{I}^4} E_1^2.$$

When an acoustic wave propagates along the negative direction, the acoustic wave hits the SL first and part of the acoustic energy is reflected by the SL. It is obvious, however, that all the acoustic energy penetrating into medium III will be allowed to pass through the whole AD model regardless of the amount of acoustic energy converted from the FW to the SHW. In other words, the total energy transmission rate of the acoustic wave along the negative direction, denoted as T_-, is identical to the transmission rate of the SL, i.e.,

$$T_- = T_\mathrm{SL}(\omega). \tag{1}$$

On the other hand, as the incident direction is reversed, the acoustic wave has to pass through the nonlinear medium III where the acoustic energy will be partly converted from the FW to the SHW before the wave enters the SL. Then the total energy transmission rate of the wave propagating along the positive direction, denoted as T_+, is the sum of the transmissions of the FW and SHW

$$T_+ = [T_\mathrm{SL}(\omega)E_1 + T_\mathrm{SL}(2\omega)E_2]/E.$$

In such a case, it is readily apparent that the propagation of the incident wave is substantially different as a result of the elimination of the system's symmetry by the nonlinearity. To guarantee the validity of the perturbation technique, the energy of the SHW should be much smaller than that of the FW, i.e., $E_2 \ll E_1$. Then the expression of T_+ can be approximated in a more concise form as

$$T_+ \approx T_\mathrm{SL}(\omega) + T_\mathrm{SL}(2\omega)\Gamma_\mathrm{III}^2 d_\mathrm{III}^2 s^2 E. \tag{2}$$

It is of particular interest to investigate the transmission of the AD model within the ERBs, where the significant phenomenon of acoustic rectification exists. When the driving frequency falls in the ERBs, we have $T_\mathrm{SL}(\omega) \approx 0$ and $T_\mathrm{SL}(2\omega) \approx 1$, and equation (2) can be further simplified as

$$T_+ \approx \Gamma_\mathrm{III}^2 d_\mathrm{III}^2 s^2 E. \tag{3}$$

Equation (2) reveals a simple relationship between the system parameters and the efficiency of the AD model. It is observable that within the ERBs, the total acoustic energy that can be transmitted through the whole model along the positive direction depends only on the structural parameters of the nonlinear medium.

3. Numerical results and discussion

We have carried out a series of numerical simulations to investigate the validity and efficiency of the proposed method, and will show some typical results. Figure 3 illustrates the comparison between the results predicted using the extended transfer-matrix method and the proposed approach. The dashed-dotted and the dashed lines refer to the transmissions of the waves from the LB and the RB, respectively. The solid line refers to the transmission obtained from Eq. (3). The gray regions denote the frequency ranges in which the rectifying condition is satisfied, as predicted from the dispersion relationship of the SL. It is apparent that when the frequency of the incident wave sits in the ERBs, the analytical solutions

given by Eqs. (1) and (3) agree satisfactorily with the exact numerical results predicted by using the extended transfer-matrix method, except for some Gibbs oscillations caused by the finite number of periods contained in the SL.

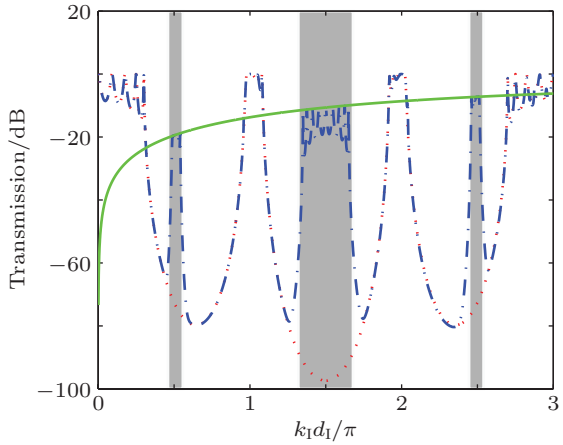


Fig. 3. (color online) The comparison between the transmissions of waves that are coming from the left boundary (dotted line) and the right boundary (dash-dotted line), the solid line refers to the transmission obtained by Eq. (3).

As the frequency of the incident wave falls within the ERBs, it is apparent that the simplified expression of T_+ given by Eq. (3) helps to conveniently estimate the parameter dependence of the transmission efficiency of an AD model. From Eq. (3) we can readily predict the simple relationship between T_+ and the equivalent nonlinearity parameter and the thickness of medium III, i.e., $T_+ \propto (\Gamma_{\rm III})^2$, $T_+ \propto (d_{\rm III})^2$. Figures 4 and 5 illustrate the comparison between the exact solution of the total transmission of energy flux and the solution of the simplified function. In Figs. 4 and 5, three particular values of normalized frequencies located in the ERBs are considered: $kd_{\rm I} = 0.5$, 1.5, and 2.5. Figure 4 (5) clearly shows that the value of T_+ is proportional to the square of $\Gamma_{\rm III}$ ($d_{\rm III}$). This is exactly consistent with the relationship given by Eq. (3). Hence

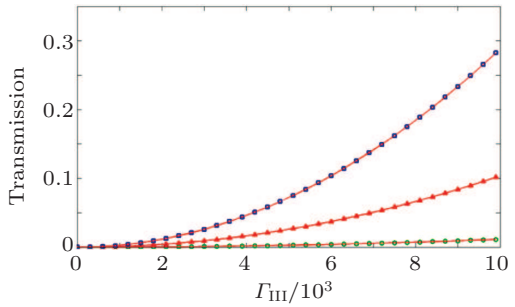


Fig. 4. (color online) Total transmissions of energy flux versus the equivalent nonlinearity parameter of medium III for three cases ($kd_{\rm I} = 0.5$, 1.5, and 2.5) located in the ERBs. The circles, triangles, and squares refer to the three cases (from bottom to top) obtained using Eq. (3), respectively. The three lines refer to the exact solutions for the three cases.

it is obvious that, as the incident frequency falls within the ERBs, the efficiency of an AD model can be enhanced monotonically in three ways: increasing the incident wave pressure, choosing a medium with stronger nonlinearity, and enlarging the AD. It turns out that the proposed approach is valid and helps us to predict the performance for an AD in a very convenient way.

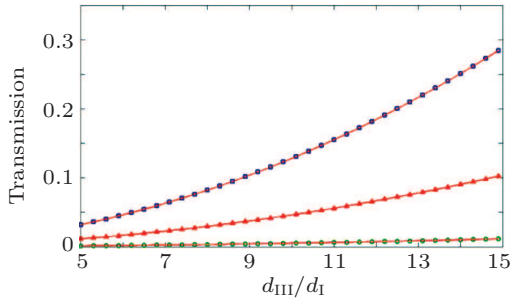


Fig. 5. (color online) Total transmissions of energy flux versus normalized thickness of medium III for three cases: $kd_{\rm I} = 0.5$, 1.5, and 2.5. The circles, triangles, and squares refer to the three cases (from bottom to top) obtained using Eq. (3), respectively. The three lines refer to the exact solutions for the three cases.

4. Conclusion

In summary, an analytical method is proposed to analyze the AD model formed by coupling an SL and a nonlinear medium. Compared with the numerical methods developed in the previous works, this method gives a clearer description of this AD model both physically and mathematically. Within the ERBs whose locations are determined by the dispersion relationship of the SL, the rectifying efficiency of the AD is studied as a function of the system parameters. The numerical simulations indicate that the rectifying ratio can be controlled by choosing the system parameters properly, which is potentially significant for the potential application of ADs.

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