Nonautonomous solitons in the continuous wave background of the variable-coefficient higher-order nonlinear Schrödinger equation*

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We reduce the variable-coefficient higher-order nonlinear Schrödinger equation (VCHNLSE) into the constant-coefficient (CC) one. Based on the reduction transformation and solutions of CCHNLSE, we obtain analytical soliton solutions embedded in the continuous wave background for the VCHNLSE. Then the excitation in advancement and sustainment of soliton arrays, and postponed disappearance and sustainment of the bright soliton embedded in the background are discussed in an exponential system.

Keywords: higher-order nonlinear Schrödinger equation, soliton solution, continuous wave background, postponed disappearance and sustainment of soliton

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1. Introduction

The standard nonlinear Schrödinger equation (NLSE) and its relative are ubiquitous and significant models of mathematical physics. Its solitonic solutions, such as optical solitons forming due to the interplay between group-velocity dispersion (GVD) and self-phase modulation (SPM),[1] have been witnessed as good information carriers for long distance optical communication. When short pulse is considered (to nearly 50 fs), the third-order dispersion (TOD) and the higher order nonlinear effects such as the self-steepening (SS)[2] and self-frequency shift (SFS)[3] cannot be neglected. Thus, the NLSE is extended to the higher order nonlinear Schrödinger equation (HNLSE).[4]

In realistic optical fibers, the concept of soliton control has been introduced and is an important development.[5–9] In order to maintain a well-ordered transmission of solitons in a communication system, one usually introduces a slow change of the fiber parameters in the longitudinal direction or a device amplifying the insulation of heat production. Due to the nonuniformity of the core medium, various effects such as loss (or gain), dispersion, SPM, SS, and SFS are influenced. Here, we consider the following variable-coefficient (VC) HNLSE:[10]

$$\begin{aligned} &u_z + a_1(z)u_t + a_2(z)u + \mathrm{i}a_3(z)u_{tt} + a_4(z)u_{ttt} \\ &+ \mathrm{i}a_5(z)|u|^2u + a_6(z)(|u|^2u)_t + a_7(z)u(|u|^2)_t = 0, \end{aligned} \quad (1)$$

where $u(z,t)$ is the complex envelope of the electrical field, and z and t are the propagation distance and time, respectively. The term proportional to a_1 results from the group velocity, a_2 is the heating-insulating amplification or loss, and a_3–a_7 represent GVD, TOD, SPM, SS, and SFS coefficients, respectively. Note that the physical quantity u is not defined in the frame of the group velocity owing to the existence of the term $a_1(z)$. If the coefficients $a_1(z)$–$a_7(z)$ are all constants, then equation (1) can be transformed into the equation defined in the frame of the group velocity in Ref. [4] via the transformation $t = t - a_1 z$. However, we have not found similar transformation to change Eq. (1) into the equation in Ref. [11] due to these variable coefficients $a_1(z)$–$a_7(z)$.

In a previous work,[10] we have obtained bright and dark soliton solutions for Eq. (1) by Darboux transformation and direct method, respectively. Recently, the similarity transformation method (STM)[12–14] for reducing various VCNLSE into constant-coefficient (CC) NLSE has been constructed. Now some interesting and significant issues arise: Can the STM be extended into VCHNLSE? Do other localized structures exist, such as soliton solutions on a continuous wave (CW) background discussed in the CCHNLSE?[4] In this paper, we intend to answer the above questions.

2. Reduction of VCHNLSE into CCHNLSE

In order to deduce the exact analytical solutions for Eq. (1), we construct the transformation

$$u(z,t) = A(z)U[T(z,t), Z(z)]\exp[\mathrm{i}\phi(z,t)], \quad (2)$$

where the amplitude, phase, mapping variable, and effective propagation distance of the pulse read

$$\begin{aligned} \rho &= \rho_0 \exp\left[-\int_0^z a_2(s)\,\mathrm{d}s\right], \\ \phi &= p\left[t + \frac{p\rho_0^2(k\alpha_1 - 2p\alpha_3)}{k^2\alpha_4}\int_0^z \right. \\ &\quad \left. \times \exp\left[-2\int_0^s a_2(\zeta)\,\mathrm{d}\zeta\right] a_6(s)\,\mathrm{d}s \right. \end{aligned} \quad (3)$$

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$$-\int_0^z a_1(s)\,\mathrm{d}s\Big] + \phi_0, \tag{4}$$

$$T = k\Big[t + p\left(\frac{2\alpha_1}{\alpha_4} - \frac{3p\alpha_3}{k^2\alpha_4}\right)\rho_0^2\int_0^z \times \exp\left[-2\int_0^s a_2(\zeta)\,\mathrm{d}\zeta\right]a_6(s)\,\mathrm{d}s - \int_0^z a_1(s)\,\mathrm{d}s\Big] + t_0, \tag{5}$$

$$Z = -\frac{\rho_0^2 k}{\alpha_4}\int_0^z \exp\left[-2\int_0^s a_2(\zeta)\,\mathrm{d}\zeta\right]a_6(s)\,\mathrm{d}s, \tag{6}$$

then the VCHNLSE (1) is transformed into the CCHNLSE[14]

$$U_Z = \mathrm{i}(\alpha_1 U_{TT} + \alpha_2|U|^2U) + \alpha_3 U_{TTT} + \alpha_4(|U|^2U)_T + \alpha_5 U(|U|^2)_T. \tag{7}$$

Note that the SS parameter $a_6(z)$, gain/loss parameter $a_2(z)$, and group velocity parameter $a_1(z)$ influence the form of the soliton, such as the amplitude, phase, and effective propagation distance.

Further, the constraints of system parameters are given by

$$\begin{aligned} &a_6(z) : a_7(z) : a_5(z) : a_4(z) : a_3(z) \\ &= 1 : \frac{\alpha_5}{\alpha_4} : \left(\frac{k\alpha_2}{\alpha_4} - p\right) : \frac{\rho_0^2\alpha_3}{k^2\alpha_4} \\ &\times \exp\left[-2\int_0^z a_2(s)\,\mathrm{d}s\right] : \frac{\rho_0^2(k\alpha_1 - 3p\alpha_3)}{k^2\alpha_4} \\ &\times \exp\left[-2\int_0^z a_2(s)\,\mathrm{d}s\right]. \end{aligned} \tag{8}$$

3. Nonautonomous soliton solutions on a continuous wave background

The one-to-one correspondence (2) in combination with Eqs. (3)–(6) under condition (8) also allow us to obtain abundant solutions, such as bright and dark soliton solutions, and W-shaped and M-shaped soliton solutions in Ref. [10]. Here we focus on nonautonomous soliton solutions in a continuous wave (CW) background.

Employing the transformation (2) together with Eqs. (3)–(6) and Darboux transformation (DT) with non-zero seed solution,[14] and when $\alpha_4 = -\alpha_5$, $\alpha_2 = 2\mu^2\alpha_1$, and $\alpha_4 = 6\mu^2\alpha_3$ with the real constant μ, one can obtain the soliton solution in a CW background

$$u = \rho(z)\left(A_\mathrm{c} + A_\mathrm{s}\frac{a\cosh\theta + \cos\varphi}{\cosh\theta + a\cos\varphi} + \mathrm{i}A_\mathrm{s}\frac{b\sinh\theta + c\sin\varphi}{\cosh\theta + a\cos\varphi}\right)\exp[\mathrm{i}\phi(z,t) + \mathrm{i}\psi(Z,T)], \tag{9}$$

where

$$\begin{aligned} \theta &= \mathrm{i}(M - M^*)T - (M\beta + M^*\beta^*)Z - \theta_0, \\ \varphi &= (M + M^*)T - \mathrm{i}(M\beta - M^*\beta^*)Z - \varphi_0, \\ \psi(Z,T) &= \omega_\mathrm{c}T + \kappa Z, \\ a &= \mu A_\mathrm{c}(L + L^*)/(\mu^2A_\mathrm{c}^2 + |L|^2), \\ b &= \mathrm{i}\mu A_\mathrm{c}(L - L^*)/(\mu^2A_\mathrm{c}^2 + |L|^2), \\ c &= (\mu^2A_\mathrm{c}^2 - |L|^2)/(\mu^2A_\mathrm{c}^2 + |L|^2), \end{aligned}$$

with

$$\begin{aligned} L &= -\mu A_\mathrm{s}/2 - \mathrm{i}(\omega_\mathrm{c} - \omega_\mathrm{s})/2 - \mathrm{i}M, \\ M &= [(\omega_\mathrm{c} - \omega_\mathrm{s} - \mathrm{i}\mu A_\mathrm{s}) + 4\mu^2A_\mathrm{c}^2]^{1/2}/2, \\ \beta &= -\mu A_\mathrm{s}[\alpha_1 + (\omega_\mathrm{c} + 2\omega_\mathrm{s})\alpha_3] + \mathrm{i}(\omega_\mathrm{c} + \omega_\mathrm{s})\alpha_1 \\ &\quad + \mathrm{i}(\omega_\mathrm{c}^2 + \omega_\mathrm{c}\omega_\mathrm{s} + \omega_\mathrm{s}^2 - 2\mu^2A_\mathrm{c}^2 - \mu^2A_\mathrm{s}^2)\alpha_3, \end{aligned}$$

and T, Z, $\rho(z)$, and $\phi(z,t)$ satisfy Eqs. (3)–(6). The parameters A_c, A_s, ω_c, ω_s, κ, θ_0 and φ_0 are all constants.

When $A_\mathrm{s} = 0$, u is reduced to the background $u = \rho A_\mathrm{c}\exp[\mathrm{i}\phi + \mathrm{i}\psi]$. When $A_\mathrm{c} = 0$, u is reduced to the bright soliton

$$u = \rho A_\mathrm{s}\mathrm{sech}\theta\exp[\mathrm{i}\phi + \mathrm{i}\Psi],$$

with

$$\begin{aligned} \theta &= \mu A_\mathrm{s}T - \mu A_\mathrm{s}[2\omega_\mathrm{s}\alpha_1 + (3\omega_\mathrm{s}^2 - \mu^2A_\mathrm{s}^2)\alpha_3]Z - \theta_0, \\ \Psi &= \omega_\mathrm{s}T + [(\mu^2A_\mathrm{s}^2 - \omega_\mathrm{s}^2)\alpha_1 + \omega_\mathrm{s}(3\mu^2A_\mathrm{s}^2 - \omega_\mathrm{s}^2)\alpha_3]Z + \Psi_0. \end{aligned}$$

Since the relation between A_c and A_s is important, we analyze in detail two representative situations: (i) when the power of the CW background exceeds a quarter of the soliton's peak power, i.e., $4A_\mathrm{c}^2 > A_\mathrm{s}^2$; (ii) when the power of the CW background is less than a quarter of the soliton's peak power, namely $A_\mathrm{s}^2 > 4A_\mathrm{c}^2$. Here, for simplicity we take $\omega_\mathrm{c} = \omega_\mathrm{s} = \omega$ in the above expressions.

First, we discuss the first case $4A_\mathrm{c}^2 > A_\mathrm{s}^2$. In this case, the solution (9) can be reduced to the following form:

$$u = \rho(z)\left(A_\mathrm{c} + A_\mathrm{s}\frac{2A_\mathrm{c}\cos\varphi - A_\mathrm{s}\cosh\theta + \mathrm{i}M_\mathrm{R}\sinh\theta}{2A_\mathrm{c}\cosh\theta - A_\mathrm{s}\cos\varphi}\right) \times \exp[\mathrm{i}\phi(z,t) + \mathrm{i}\psi(Z,T)], \tag{10}$$

where

$$\begin{aligned} \theta &= \mu^2A_\mathrm{s}M_\mathrm{R}(\alpha_1 + 3\omega\alpha_3)Z - \theta_0, \\ \varphi &= \mu M_\mathrm{R}T - \mu M_\mathrm{R}[2\omega\alpha_1 + (3\omega^2 - 2\mu^2A_\mathrm{c}^2 - \mu^2A_\mathrm{s}^2)\alpha_3]Z - \varphi_0, \end{aligned}$$

with $M_\mathrm{R} = \sqrt{4A_\mathrm{c}^2 - A_\mathrm{s}^2}$. This solution is periodic in the temporal coordinate and aperiodic in the space variable.

Note that according to the expression (6), the values of the effective propagation distance Z and the actual propagation distance z are different. Thus, a different evolutional behavior of soliton solutions on a CW background in VCHNLSE will appear compared with that in CCHNLSE.[14] For free Z and T, the soliton solution on a cw background in CCHNLSE is displayed in Fig. 1(a), which shows a train of optical solitons restricted in some domain of Z. However, in applications propagating soliton arrays are required. The realization of this control or demand lies in the relation between the effective propagation distance Z and the original propagation distance z.

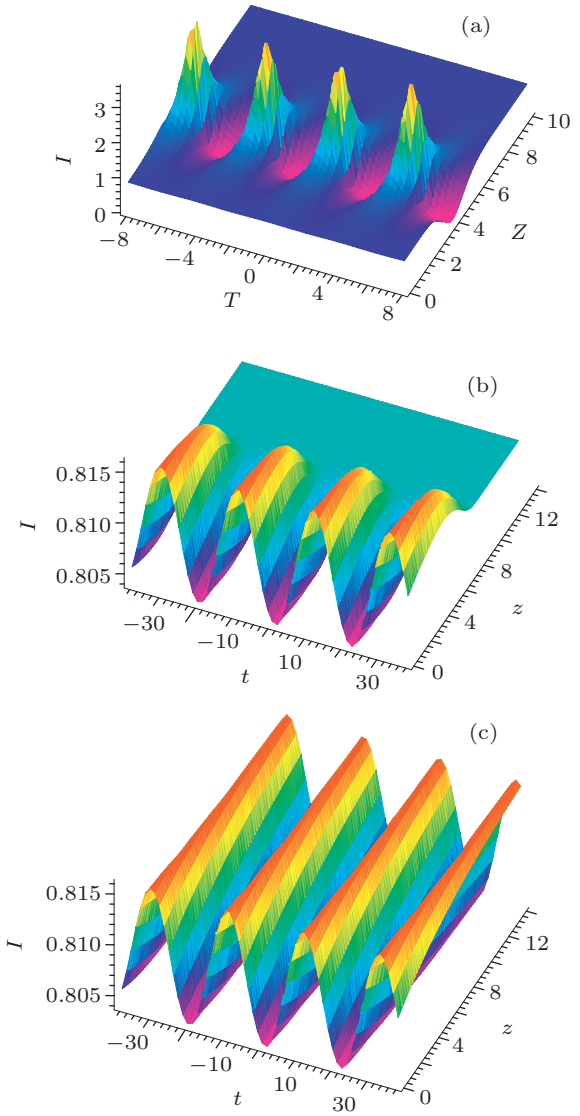


Fig. 1. (color online) (a) Train of optical solitons in Z–T coordinates. (b) and (c) Excitation in advancement and sustainment of soliton arrays, respectively. The parameters are $A_c = 0.9$, $A_s = 1$, $\mu = 1$, $\theta_0 = 6$, $k = \omega = 0.2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 0.02$, $\alpha_4 = 0.12$, $\alpha_5 = -0.12$, $\phi_0 = 0$, $\rho_0 = p = 0.1$, $a_{20} = 0.01$, $a_{60} = 0.1$, $a_{10} = 0.5$, $\delta = -0.1$ and (b) $\sigma = 1$; (c) $\sigma = -1$.

For example, we consider this case in the following soliton control system with $a_6 = a_{60}\exp(\sigma z), a_1 = a_{10}\exp(\delta z)$, and constant gain/loss $a_2 = a_{20}$. Based on the relation (8), GVD, TOD, and nonlinearity parameters all have an exponential form. The presented system is similar to the one given by Eqs. (15)–(17) in Ref. [15]. In this system, from Eq. (6), we have $Z = -k\rho_0^2(\alpha_4(2a_{20}-\sigma))^{-1}\{1-\exp[-(2a_{20}-\sigma)z]\}$. For positive k and α_4, when $a_{20} < \sigma/2$, one has $Z \geq z$ and $Z \to \infty$ with $z \to \infty$. In this case, the excitation in advancement of soliton arrays is found in Fig. 1(b), where soliton arrays are excited at initial distance $z = 0$, and then disappear at $z = 7$, whose position coincides with that of disappearance in Fig. 1(a). If $a_{20} > \sigma/2$, one has $Z < z$ and $Z \to Z_{\max} = k\rho_0^2(\alpha_4(2a_{20}-\sigma))^{-1}$. In this case, soliton arrays are excited at initial distance $z = 0$, and then sustain all the time (see Fig. 1(c)).

Next, we analyze the second case $A_s^2 > 4A_c^2$, where solution (9) is rewritten as

$$u = \rho(z)\left(M_I\frac{M_I\cos\varphi + iA_s\sin\varphi}{A_s\cosh\theta - 2A_c\cos\varphi} - A_c\right) \times \exp[i\phi(z,t) + i\psi(Z,T)], \quad (11)$$

where

$$\theta = -\mu M_I T + \mu M_I[2\omega\alpha_1 + (3\omega^2 - 2\mu^2A_c^2 - \mu^2A_s^2)\alpha_3]Z - \theta_0$$

and $\varphi = \mu^2A_sM_I(\alpha_1 + 3\omega\alpha_3)Z - \varphi_0$ with $M_I = \sqrt{A_s^2 - 4A_c^2}$. This solution represents the bright soliton embedded in the CW background in the presence of higher-order effects. The main characteristic of this solution is the periodic oscillation of the optical field amplitude, along the propagation direction of the fiber. For free Z and T, the soliton solution on a CW background in CCHNLSE is displayed in Fig. 2(a), which shows the bright soliton that propagates on a CW

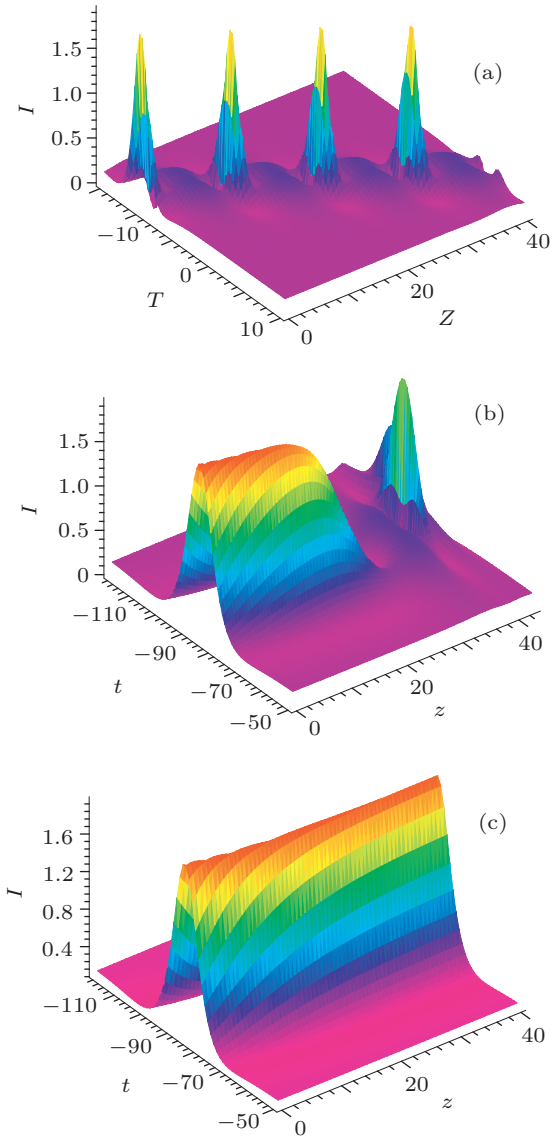


Fig. 2. (color online) (a) Bright soliton embedded in the CW background in Z–T coordinates; (b) and (c) postponed disappearance and sustainment of the bright soliton embedded in the background, respectively. The parameters are the same as that in Fig. 1 except for $A_c = 0.4$.

background restricted in some domain of Z. This feature illustrates the characteristic breather behavior of the soliton as it propagates on a CW background. The soliton's maximum intensity is $I = |u|^2 = \rho^2\left(A_c^2 + A_s M_I^2/(A_s - 2A_c\cos\varphi)\right)$ at $\theta = 0$, while the minimum intensity is $I = |u|^2 = \rho^2\left(A_c^2 - A_c^2 M_I^2\cos^2\varphi/(A_s^2 - 4A_c^2\cos^2\varphi)\right)$ at $\theta = \operatorname{arccosh}(A_s/A_c\cos\varphi - 2A_c/A_s\cos\varphi)$.

As discussed above, via modulating parameters we can also realize postponement and sustainment of the bright soliton embedded in the background. As shown in Fig. 2(b), for the above-mentioned exponential system with positive k and α_4, when $a_{20} < \sigma/2$, the first soliton keeps its shape for about 30 propagation distances and finishes its propagation, thus the completion of the first soliton is postponed. Note that the green spike at the very end of the graph is the beginning of the second soliton. When $a_{20} > \sigma/2$, the bright soliton embedded in the background maintains its shape in the form of maximum intensity forever, where its amplitude and width never change after a short propagation distance from the initial condition. This property implies that the injection of a CW pulse can be used for parametric control of soliton timing.

4. Conclusions

In conclusion, we reduce the VCHNLSE into the CCHNLSE via the STM. With the help of the similarity transformation and the corresponding solutions of CCHNLSE, we obtain analytical soliton solutions embedded in the CW background for the VCHNLSE. Then the propagation behaviors of the soliton arrays and the bright soliton embedded in the background are discussed in the exponential system. When system parameters $a_{20} < \sigma/2$, the excitation in advancement of soliton arrays and the postponed disappearance of the bright soliton embedded in the background are found. If $a_{20} > \sigma/2$, the sustainment of the soliton arrays and the bright soliton embedded in the background can be realized.

References

[1] Haus H A, Wong W S1996 *Rev. Mod. Phys.* **68** 423
[2] de Oliveria J R and de Moura M A 1998 *Phys. Rev.* E **57** 4751
[3] Mitschke F M and Mollenauer L F 1986 *Opt. Lett.* **11** 659
[4] Li S Q, Li L, Li Z H and Zhou G S 2004 *J. Opt. Soc. Am.* B **21** 2089
[5] Wang H and Li B 2011 *Chin. Phys.* B **20** 040203
[6] Zhang X F, Zhang P, He W Q and Liu X X 2011 *Chin. Phys.* B **20** 020307
[7] Liu X B and Li B 2011 *Chin. Phys.* B **20** 114219
[8] Ma Z Y and Ma S H 2012 *Chin. Phys.* B **21** 030507
[9] Xu S L, Liang J C and Li Z M 2011 *Chin. Phys.* B **20** 114214
[10] Dai C Q and Zhang J F 2006 *J. Phys.* A: *Math. Gen.* **39** 723
[11] Dai C Q, Zhou G Q and Zhang J F 2012 *Phys. Rev.* E **85** 016603
[12] Chen Y M, Ma S H and Ma Z Y 2012 *Chin. Phys.* B **21** 050510
[13] Dai C Q, Wang Y Y and Zhang J F 2010 *Opt. Lett.* **35** 1437
[14] Dai C Q, Chen R P and Wang Y Y 2012 *Chin. Phys.* B **21** 030508
[15] Hao R Y, Li L, Li Z H and Zhou G S 2004 *Phys. Rev.* E **70** 066603