Folded localized excitations in the (2+1)-dimensional modified dispersive water-wave system*

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By using a mapping approach and a linear variable separation approach, a new family of solitary wave solutions with arbitrary functions for the (2+1)-dimensional modified dispersive water-wave system (MDWW) is derived. Based on the derived solutions and using some multi-valued functions, we obtain some novel folded localized excitations of the system.

Keywords: mapping approach, modified dispersive water-wave system, solitary wave solution, folded localized excitation

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1. Introduction

Many dynamical problems in physics and other fields are usually characterized by the nonlinear evolution of partial differential equations known as governing equations. Searching for an analytical exact solution to a nonlinear system has long been an important and interesting topic in nonlinear science, and various methods for obtaining exact solutions of nonlinear systems have been proposed, for example, the bilinear method, the standard Painlevé truncated expansion, the method of coalescence of eigenvalue or wavenumbers, the homogenous balance method, the hyperbolic function method, the Jacobian elliptic method, the variable separation method, the (G'/G)-expansion method,[1–13] and the mapping method.[14,15]

Previously, much effort has been focused on the single-valued localized excitations, such as solitoffs, dromions, rings, lumps, breathers, instantons, peakons, compactons, localized chaotic, and fractal patterns.[16–29] However, there are various complicated phenomena in the real world like the usual bubbles on a fluid surface or ocean waves, which are folded and obviously cannot be described by single-valued functions. In Ref. [30], Tang and Lou introduced some multi-valued localized excitations to describe folded solitary waves and defined a new type of soliton–foldon. Actually, the simple foldons in lower dimensions can be equivalently called the loop solitons, which can be found in many (1+1)-dimensional integrable models[31,32] and have been applied in some physical branches like quantum theory, string theory, and particle physics.[33,34] In this paper, by using some multi-valued functions, we find new folded localized excitations in the (2+1)-dimensional modified dispersive water-wave system (MDWW)

$$\begin{aligned} &u_{ty}+u_{xxy}-2v_{xx}-(u^2)_{xy}=0,\\ &v_t-v_{xx}-2(uv)_x=0. \end{aligned} \tag{1}$$

The MDWW system is used to model nonlinear and dispersive long gravity waves travelling in two horizontal directions in shallow water with uniform depth, and can also be derived from the celebrated Kadomtsev–Petviashvili (KP) equation with a symmetry constraint.[35]

As is well known, to search for the solitary wave solutions for a nonlinear physical model, we can apply different approaches. One of the most efficient methods for finding soliton excitations of a physical model is the so-called mapping approach with variable coefficients. The basic ideal of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with independent variables $\boldsymbol{x}=(x_0=t,x_1,x_2,\ldots,x_m)$ and dependent variable u in the form

$$P(u,u_t,u_{x_i},u_{x_ix_j},\ldots)=0, \tag{2}$$

where P is in general a polynomial function of its arguments, and the subscripts denote the partial derivatives, the solution can be assumed to be

$$\begin{aligned} u=A(\boldsymbol{x})+\sum_{i=1}^{n}\{&B_i(\boldsymbol{x})\phi^i[q(\boldsymbol{x})]\\ &+C_i(\boldsymbol{x})\phi^{i-1}[q(\boldsymbol{x})]\sqrt{\sigma\phi[q(\boldsymbol{x})]+\phi^2[q(\boldsymbol{x})]}\}, \end{aligned} \tag{3}$$

with

$$\phi'=\sigma\phi+\phi^2, \tag{4}$$

where σ is a constant, and the prime denotes the differentiation with respect to q. To determine u explicitly, we may substitute Eqs. (3) and (4) into the given NPDE and collect coefficients of polynomials of ϕ, then eliminate each coefficient to derive a set of partial differential equations of A, B_i, C_i, and q, and solve the system of partial differential equations to obtain A, B_i, C_i, and q. Finally, as Eq. (4) is known to possess

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the solution

$$\phi = \frac{-\sigma \mathrm{e}^{\sigma q + c\sigma}}{\mathrm{e}^{\sigma q + c\sigma} - 1}, \qquad \sigma \neq 0 \tag{5}$$

by using some appropriate transformation for Eq. (5), we can get the following solitary wave solutions for Eq. (4):

$$\phi = \begin{cases} -\frac{1}{2}\sigma\left[1 + \tanh\left(\frac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \\ -\frac{1}{2}\sigma\left[1 + \coth\left(\frac{1}{2}\sigma q\right)\right], & \sigma \neq 0. \end{cases} \tag{6}$$

Substituting A_i, B_i, C_i, q and Eq. (5) into Eq. (4), we obtain the exact solutions to the given NPDE.

2. Solitary wave solutions of the MDWW system

First, let us make a transformation of Eq. (1): $v = u_y$. Substituting this transformation into Eq. (1) yields

$$u_{yt} - u_{xxy} - (u^2)_{xy} = 0. \tag{7}$$

Now we apply the mapping approach to Eq. (7). By using the balancing procedure, ansatz (3) becomes

$$\begin{aligned} u \;=\; & f(x,y,t) + g(x,y,t)\phi(q(x,y,t)) \\ & + h(x,y,t)\sqrt{\sigma\phi(q(x,y,t)) + \phi^2(q(x,y,t))}, \end{aligned} \tag{8}$$

where f, g, h, and q are functions of (x,y,t) to be determined. Substituting Eqs. (8) and (4) into Eq. (7) and collecting coefficients of polynomials of ϕ, then setting each coefficient to zero, we have

$$96ghq_yq_x + 48hq_x^2q_y = 0, \tag{9}$$

$$48gq_yq_x^2 + 48h^2q_yq_x + 48g^2q_yq_x = 0, \tag{10}$$

$$\begin{aligned} & 32gq_yh_x + 120hq_x^2q_y\sigma + 32hgq_{xy} + 32hg_yq_x \\ & + 32fhq_yq_x + 32h_xq_yq_x + 16hq_yq_{xx} \\ & + 32hg_xq_y + 224ghq_yq_x\sigma + 32hq_xq_{xy} - 16hq_yq_t = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} & 16fhq_{xy} + 8h_{xx} + 56hg_xq_y\sigma - 32h\sigma q_yq_t \\ & + 32h\sigma q_yq_{xx} + 56hg_yq_x\sigma + 64hq_xq_{xy}\sigma + 56hgq_{xy}\sigma \\ & + 64h_xq_yq_x\sigma + 56gq_yh_x\sigma + 16hf_xq_y + 98hq_x^2q_y\sigma^2 \\ & + 64fhq_yq_x\sigma + 164ghq_yq_x\sigma^2 + 16hg_{xy} \\ & + 16h_xq_{xy} + 8hq_{xxy} - 8hq_{yt} + 16g_yh_x \\ & + 16fh_xq_y - 8h_tq_y + 16f_yhq_x = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} & 16g^2q_{xy} + 16h^2q_{xy} + 80g^2q_xq_y\sigma + 96h^2q_yq_x\sigma \\ & + 32fgq_yq_x + 96gq_yq_x^2\sigma + 32gq_yg_x \\ & + 32h^2q_yg_x\sigma + 16gq_yq_{xx} + 32g_xq_yq_x + 32gq_xq_{xy} \\ & + 32g_ygq_x - 16gq_yq_t + 16g_yq_x^2 = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} & 16g_yh_x\sigma + 12h_{xx}q_y\sigma + 12hq_{xxy}\sigma - 12h_tq_y\sigma \\ & + 36h_xq_yq_x\sigma^2 + 36hq_xq_{xy}\sigma^2 + 24hf_xq_y\sigma \\ & + 36ghq_yq_x\sigma^2 + 36fhq_yq_x\sigma + 24ghq_{xy}\sigma^2 \\ & + 17hq_x^2q_y\sigma^3 - 18hq_yq_t\sigma^2 + 24gq_yh_x\sigma^2 \\ & + 18hq_yq_{xx}\sigma^2 + 24h_xq_{xy}\sigma - 12hq_{yt}\sigma + 16f_yh_x = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} & 8g_yq_{xx} - 8gq_{yt} - 8g_yq_t + 24h^2q_{xy}\sigma + 48fgq_yq_x\sigma \\ & + 48g_xq_yq_x\sigma + 32g_ygq_x\sigma - 24gq_yq_t\sigma \\ & + 48hh_xq_y\sigma + 56gq_yq_x^2\sigma^2 + 32gq_yg_x\sigma + 32g^2q_yq_x\sigma^2 \\ & + 16f_ygq_x + 24g_yq_x^2\sigma + 16fg_xq_y + 16fg_yq_x + 48gq_xq_{xy}\sigma \\ & + 24gq_yq_{xx}\sigma + 8gq_{xxy} + 8g_{xx}q_y + 16g_{xy}q_x + 16gg_{xy} \\ & + 16g_yg_x + 16fgq_{xy} + 16g^2q_{xy}\sigma - 8g_tq_y = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} & 8h_xq_{xy}\sigma^2 - 4hq_{yt}\sigma^2 + 4hq_xq_{xy}\sigma^2 + 4h_xq_yq_x\sigma^3 \\ & + 8f_yhq_x\sigma^2 + 2hq_yq_{xx}\sigma^2 + 8fh_xq_y\sigma^2 + 8hf_xq_y\sigma^2 \\ & + 8fhq_{xy}\sigma^2 + 4h_{xx}q_y\sigma^2 + 4fhq_yq_x\sigma^3 \\ & + hq_x^2q_y\sigma^4 - 2hq_yq_t\sigma^3 + 4hq_{xxy}\sigma^2 + 16f_yh_x\sigma = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} & 16g_xq_{xy}\sigma + 8g_{xx}q_y\sigma + 8h^2q_{xy}\sigma^2 - 8g_yq_t\sigma + 8g_yq_x^2\sigma^2 \\ & - 8g_tq_y\sigma + 16gq_xq_{xy}\sigma^2 + 8gq_yq_{xx}\sigma^2 + 16fgq_{xy}\sigma \\ & + 16gq_yf_x\sigma + 16fgq_yq_x\sigma^2 + 8h^2q_yq_x\sigma^3 + 8gq_{xxy}\sigma + 16gf_{xy} \\ & + 16g_{xy}q_x\sigma - 8gq_{yt}\sigma - 8g_{yt} + 8g_{xxy} + 16f_yg_x = 0, \end{aligned} \tag{17}$$

$$8f_{xxy} - 8f_{yt} + 16f_yf_x + 16ff_{xy} = 0. \tag{18}$$

Based on Eqs. (9) and (10), we have

$$g = -\frac{q_x}{2}, \quad h = \pm\frac{q_x}{2}. \tag{19}$$

Substituting Eq. (19) in Eq. (11) yields

$$f = -\frac{1}{4}\frac{2q_{xx} + q_x^2\sigma - 2q_t}{q_x}. \tag{20}$$

Substituting Eqs. (19) and (20) in Eqs. (9)–(18) yields the following single equation:

$$\begin{aligned} & -q_x^2q_{xx}q_{xy}\sigma - 2q_{xy}q_{xx}q_t + 2q_xq_{xxy}q_t \\ & + 2q_xq_{xy}q_{xt} - 2q_xq_{xy}q_{xxx} + q_x^3q_{xxy}\sigma \\ & - 2q_x^2q_{xyt} + 2q_x^2q_{xxxy} - 2q_xq_{xx}q_{xxy} + 2q_{xy}q_{xx}^2 = 0. \end{aligned} \tag{21}$$

Obviously, it is very difficult to obtain a general solution to the above equation. Fortunately, in this special case, one of the special solutions can be expressed as

$$q = \chi(x,t) + \varphi(y), \tag{22}$$

where $\chi \equiv \chi(x,t)$ and $\varphi \equiv \varphi(y)$ are two arbitrary variable separated functions of (x,t) and y, respectively. Based on the solutions of Eq. (4), we can derive the following solitary wave solutions of Eq. (1):

$$\begin{aligned} u_1 \;=\; & \frac{1}{4q_x}\left[-2\chi_{xx} + 2\chi_t + \chi_x^2\sigma\tanh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\right. \\ & \left. + I\chi_x^2\sigma\,\mathrm{sech}\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\right], \end{aligned} \tag{23}$$

$$v_1 \;=\; -\frac{1}{8}\frac{\chi_x\varphi_y\sigma^2\left[-1 + I\sinh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\right]}{\cosh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2}; \tag{24}$$

$$u_2 = \frac{1}{4q_x}\left[-2\chi_{xx}+2\chi_t+\chi_x^2\sigma\coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)+\chi_x^2\sigma\,\mathrm{csch}\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\right], \tag{25}$$

$$v_2 = -\frac{1}{8}\frac{\chi_x\varphi_y\sigma^2}{\cosh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)-1}; \tag{26}$$

$$u_3 = -\frac{1}{4q_x}\left[2\chi_{xx}-2\chi_t-\chi_x^2\sigma\tanh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)+I\chi_x^2\sigma\,\mathrm{sech}\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\right], \tag{27}$$

$$v_3 = \frac{1}{8}\frac{\chi_x\varphi_y\sigma^2\left[1+I\sinh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\right]}{\cosh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)^2}; \tag{28}$$

$$u_4 = -\frac{1}{4q_x}\left[2\chi_{xx}-2\chi_t-\chi_x^2\sigma\coth\left(\frac{1}{2}\sigma(\chi+\varphi)\right)+\chi_x^2\sigma\,\mathrm{csch}\left(\frac{1}{2}\sigma(\chi+\varphi)\right)\right], \tag{29}$$

$$v_4 = \frac{1}{8}\frac{\chi_x\varphi_y\sigma^2}{\cosh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)+1}. \tag{30}$$

3. Novel folded localized excitations for the MDWW system

Due to the arbitrariness of functions $\chi(x,t)$ and $\varphi(y)$ included in the above cases, the physical quantities u and v may possess rich structures. In this section, we mainly discuss some folded localized coherent excitations in the (2+1)-dimensional MDWW system. For simplicity, we only discuss the field v_2 of Eq. (26), namely,

$$V = v_2 = -\frac{1}{8}\frac{\chi_x\varphi_y\sigma^2}{\cosh\left(\frac{1}{2}\sigma(\chi+\varphi)\right)-1}. \tag{31}$$

3.1. Folded excitation

In order to construct interesting folded localized excitations, we introduce some suitable multi-valued functions.[30] For instance,

$$\chi_x = \sum_{j=1}^{M} V_j(\varepsilon - c_j t), \quad x = \varepsilon + \sum_{j=1}^{M} P_j(\varepsilon - c_j t), \tag{32}$$

where c_j $(j = 1,2,\ldots,M)$ are arbitrary constants, and V_j and P_j are localized excitations with the properties $V_j(\pm\infty) = 0$, $P_j(\pm\infty) =$ const. we also treat the function $\varphi(y)$ in this way

$$\varphi_y = \sum_{j=1}^{N} U_j(\zeta), \quad y = \zeta + \sum_{j=1}^{N} Q_j(\zeta). \tag{33}$$

In Fig. 1, four folded localized excitations are presented for the field V determined by Eq. (31). The related functions chosen in the plots are directly given in the figure captions.

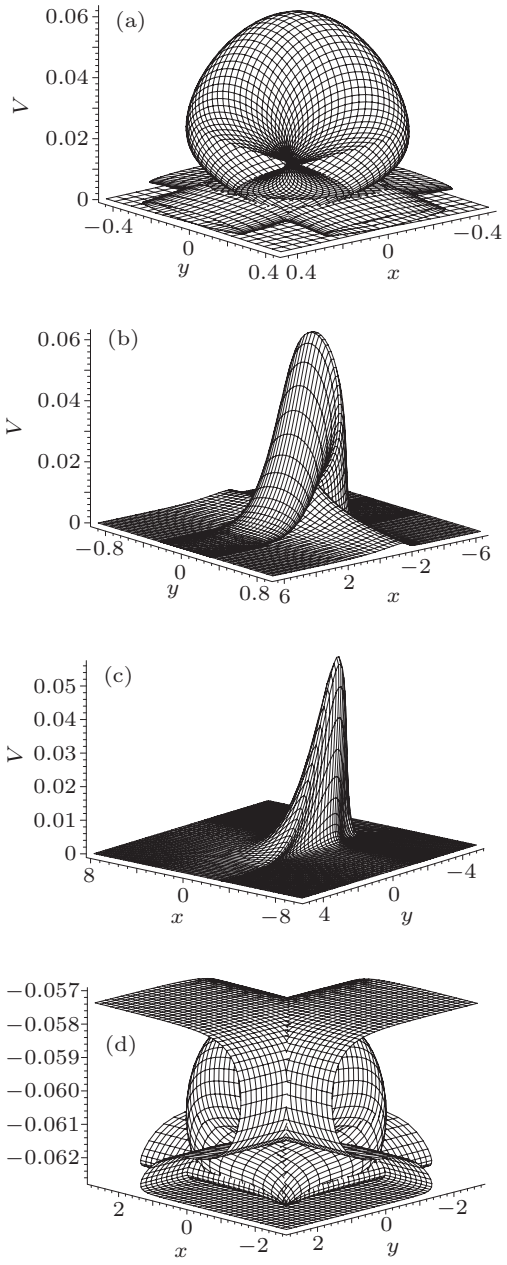


Fig. 1. Four types of folded localized excitations are depicted for the field V determined by Eq. (31) at $\sigma = 1$, $t = 0$ with the choices: (a) $\chi_x = \mathrm{sech}^2(\varepsilon - ct)$, $x = \varepsilon - 1.6\tanh(\varepsilon - ct)$, $\varphi_y = -\mathrm{sech}^2(\zeta)$, $y = \zeta - 1.6\tanh(\zeta)$; (b) same χ_x and φ_y with (a), but $x = \varepsilon - 0.8\,\mathrm{sech}(\varepsilon - ct)$, $y = \zeta - 0.8\tanh(\zeta)$; (c) same χ_x and φ_y with (a), but $x = \varepsilon - 1.2\,\mathrm{sech}(\varepsilon - ct)$, $y = \zeta - 1.2\,\mathrm{sech}(\zeta)$; (d) $\chi_x = \mathrm{sech}^2(\varepsilon - ct) + \tanh^2(\varepsilon - ct)$, $x = \varepsilon - 3.2\tanh(\varepsilon - ct)$, $\varphi_y = -\mathrm{sech}^2(\zeta) + \tanh^2(\zeta)$, $y = \zeta - 1.6\tanh(\zeta)$.

3.2. The evolution of folded waves

Now we focus our attention on the intriguing evolution of two folded solitary waves for the solution V. If we choose χ and φ to be some appropriate multi-valued functions, then we find that the interactions among the folded solitary waves are completely elastic. For instance, if we choose χ and φ as

$$\begin{aligned}\chi_x &= -\mathrm{sech}(\varepsilon - t)^2 - 0.7\mathrm{sech}(\varepsilon + t)^2,\\ x &= \varepsilon - \mathrm{sech}(\varepsilon - t) - 0.7\mathrm{sech}(\varepsilon + t),\\ \varphi_y &= \mathrm{sech}(\zeta)^2, \quad y = \zeta - 0.7\mathrm{sech}(\zeta),\end{aligned} \tag{34}$$

we can derive the time evolution of the folded solitary waves for the physical quantity V, as presented in Fig. 2 with fixed parameter $\sigma = 1$, at different times (a) $t = -13$, (b) $t = -8$, (c) $t = 0$, (d) $t = 6$, (e) $t = 12$.

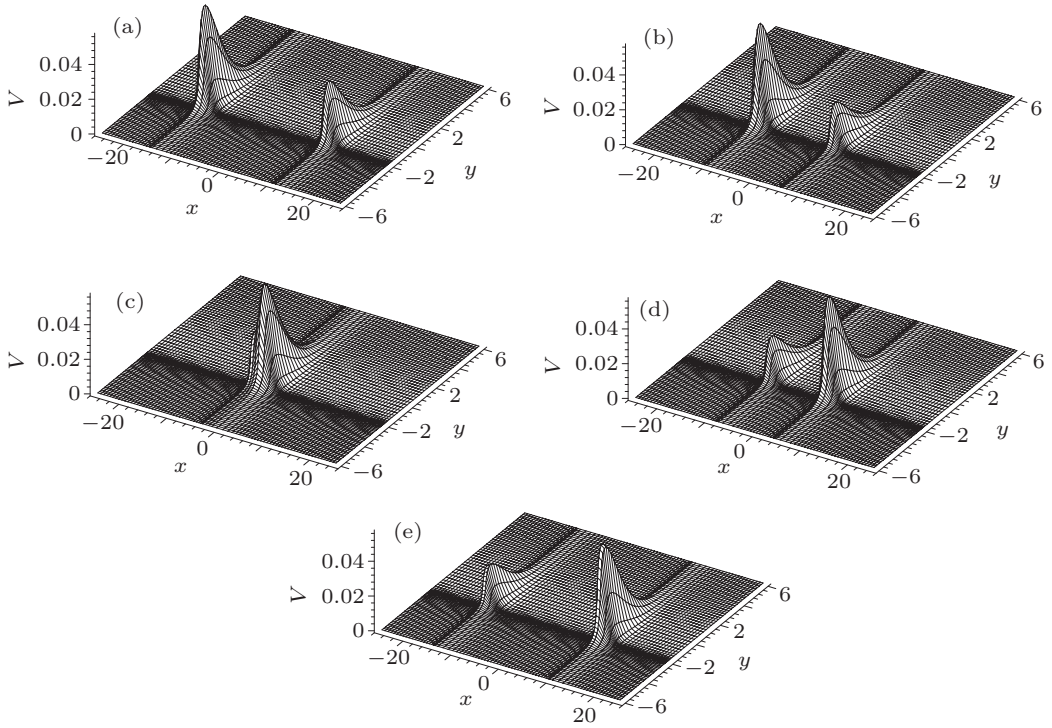


Fig. 2. Evolutional profiles of two folded solitary waves for the solution V given by Eq. (31) with condition (34) at different times (a) $t = -13$, (b) $t = -8$, (c) $t = 0$, (d) $t = 6$, (e) $t = 12$.

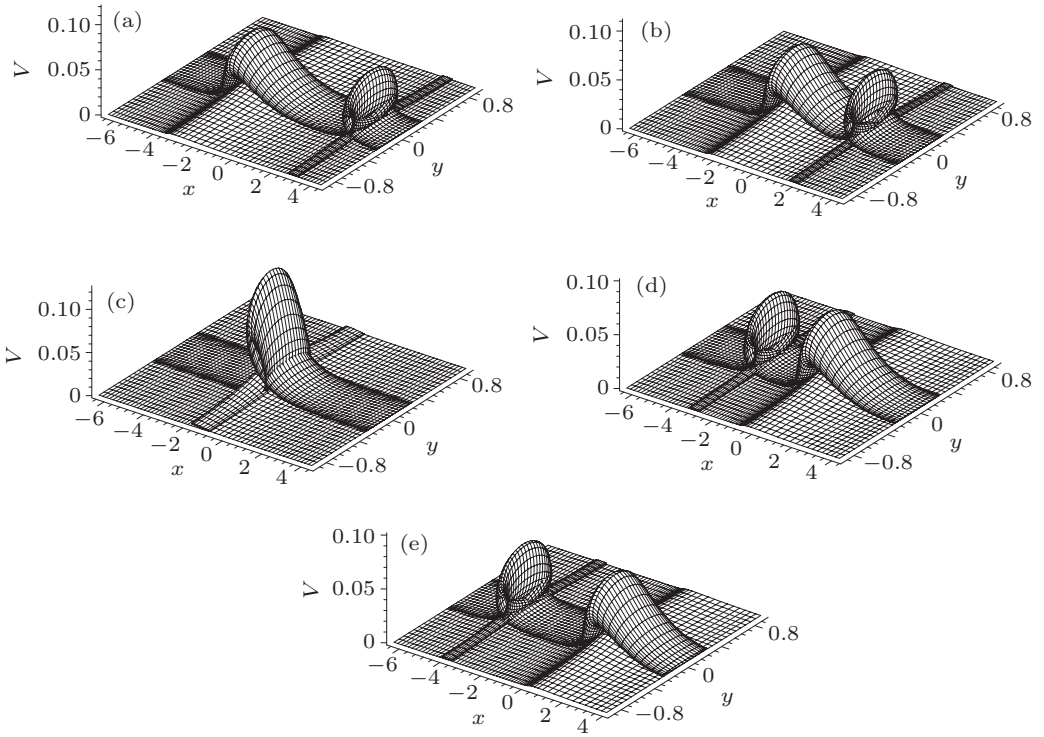


Fig. 3. Evolutional profiles of two folded solitary waves for the solution V given by (31) with condition (35) at different times (a) $t = -3$, (b) $t = -2$, (c) $t = 0$, (d) $t = 3$, (e) $t = 4$.

Furthermore, if we choose χ and φ as

$$\begin{aligned}\chi_x &= -\mathrm{sech}(\varepsilon - t)^2 - 0.6\mathrm{sech}(\varepsilon + t)^2,\\ x &= \varepsilon - \mathrm{sech}(\varepsilon - t) - 0.6\tanh(\varepsilon + t),\\ \varphi_y &= \mathrm{sech}(\zeta)^2, \quad y = \zeta - 0.6\tanh(\zeta),\end{aligned} \tag{35}$$

we can derive another time evolution of folded solitary waves for the physical quantity V, as presented in Fig. 3 with fixed parameter $\sigma = 1$, at different times (a) $t = -3$, (b) $t = -2$, (c) $t = 0$, (d) $t = 3$, (e) $t = 4$. From Figs. 2 and 3, we find that the interactions of the two folded solitary waves are completely elastic since their amplitudes, velocities, and wave shapes do not undergo any change after their collision.

4. Summary and discussion

The mapping method is a classic, efficient, and well-developed method to solve nonlinear evolution equations, the remarkable characteristic of which is that we can have infinitely different ansatzs and thus end up with an abundance of solutions. In summary, via the mapping approach and a linear variable separation approach, we find some new exact solutions of the (2+1)-dimensional modified dispersive water-wave system. Abundant localized coherent soliton structures of the solution V given by Eq. (31), like dromions, peakons, breathers, instantons, etc., can be easily constructed by choosing suitable functions. Except for the single-valued localized excitation, we find a new type of multi-valued localized excitation, i.e., folded solitary waves and/or foldon excitations for the (2+1)-dimensional modified dispersive water-wave system. To our knowledge, the folded solitary waves and/or foldon excitations for the (2+1)-dimensional modified dispersive water-wave system have not been reported in the previous literature. The time evolutions of the two folded solitary waves displayed in Figs. 2 and 3 are different from those presented in the previous work.

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