Feedback control and synchronization of Mandelbrot sets*

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The movement of a particle could be depicted by the Mandelbrot set from the fractal viewpoint. According to the requirement, the movement of the particle needs to show different behaviors. In this paper, the feedback control method is taken on the classical Mandelbrot set. By amending the feedback item in the controller, the control method is applied to the generalized Mandelbrot set and by taking the reference item to be the trajectory of another system, the synchronization of Mandelbrot sets is achieved.

Keywords: Mandelbrot set, feedback control, synchronization

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1. Introduction

Fractal is one of the branches in nonlinear sciences. Using the complex variables theory and the computer technology, the properties and graphs of fractal are discussed. Mandelbrot and Julia (M–J) sets are two important sets in fractal, which have a complex and fractal structure. With further research of M–J sets, their applications are reported,[1,2] which relate to the zeros of partition functions of certain hierarchical spin systems. Based on the study of the typical Langevin problem and the dynamics of the particle's movement, the physical applications of M–J sets are discussed.[3,4] They show that the fractal structure of the M–J sets may visually reflect the change rule of the particle's velocity, and the classical Mandelbrot set of the complex system

$$z_{n+1} = z_n^2 + c \tag{1}$$

has a concrete physical meaning in the classical particle movement. For all kick strengths c being in the Mandelbrot set, the velocity of the particle remains bounded, whereas for c not being in the Mandelbrot set the velocity asymptotically diverges, i.e., the particle falls off the potential well. Julia sets of (1) also have a physical meaning. For each c, they describe the closure of the union of all possible unstable periodic orbits of the particle.

However, the particle's movement needs to show different behaviors or similar behaviors for different systems according to the practical requirement. Recently, the control and synchronization of Julia sets have been considered,[5–7] which provides a valid method to analyze and control the movement of the particle. In this paper, the feedback control is taken on the Mandelbrot set (1). We show that the Mandelbrot set of the controlled system changes with the changing of values of the control parameter, such as rotating, expanding, shrinking and so on. Subsequently, this control method is extended to control the generalized Mandelbrot set of the complex system $z_{n+1} = z_n^r + c$, where $r \geq 2$ is an arbitrary integer. By taking the reference item to be the trajectory of another system, synchronization of Mandelbrot sets of different systems is achieved.

2. Feedback control of the Mandelbrot set

Chaotic behavior in dynamical systems has been studied extensively in recent years in the context of real maps[8,9] such as the Logistic map[10] and a number of methods have been proposed for feedback control of chaos[11,12] such as the OGY control method.[13]

According to the fractal character of M-set, we introduce the auxiliary reference feedback control[14] into the system (1) in this section. Then we have the following controlled system

$$z_{n+1} = z_n^2 + c + g_n(z_n - z_{\text{ref}}), \tag{2}$$

where z_{ref} is the reference item and g_n is the feedback item, and

$$g_n = kz_{\text{ref}} + \sum_{j=0}^{l} a_j z_{n-j}, \tag{3}$$

where l is a non-negative integer depending on the dimension of the system. The k and a_j depend on the stability of the target. To simplify our illustration, we choose the simplest form, i.e., $l = 0$, then equation (3) becomes

$$g_n = kz_{\text{ref}} + a_0 z_n. \tag{4}$$

Substituting Eq. (4) into Eq. (2), we have

$$z_{n+1} = (1 + a_0)z_n^2 + (k - a_0)z_{\text{ref}} z_n + c - kz_{\text{ref}}^2. \tag{5}$$

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Letting $k = a_0$, we have

$$z_{n+1} = (1+k)z_n^2 + c - kz_{\text{ref}}^2. \tag{6}$$

To discuss the Mandelbrot set of the controlled system (6), we discuss the structure of the Mandelbrot sets of the following systems. Consider

$$z_{n+1} = \frac{1}{d}z_n^2 + dc, \tag{7}$$

$$z_{n+1} = z_n^2 + dc, \tag{8}$$

$$z_{n+1} = z_n^2 + c - d, \tag{9}$$

$$z_{n+1} = z_n^2 + (\cos\theta + i\sin\theta)c. \tag{10}$$

Let $f(z) = z^2 + c$, $f_6(z) = (1+k)z^2 + c - kz_{\text{ref}}^2$, $f_7(z) = z^2/d + dc$, $f_8(z) = z^2 + dc$, $f_9(z) = z^2 + c - d$, and $f_{10}(z) = z^2 + (\cos\theta + \mathrm{i}\sin\theta)c$. When the parameter d is changed in systems (7)–(10), we obtain the information about different Mandelbrot sets, such as the position, size, shape, and so on. Thus we can adjust the values of d according to different requirements. Recall that the Mandelbrot set M is defined to be the parameter set c that enables the Julia set of f to be connected,

$$M = \{c \in C : J(f) \text{ is connected}\}. \tag{11}$$

In addition, the recursive form of f could be used to illustrate another character of the Mandelbrot set,

$$M = \{c \in C : \{f^n(0)\}_{n\geq 1} \text{ is bounded}\}. \tag{12}$$

Denote the Mandelbrot sets of system (1), (6)–(10) by M_1, M_6–M_{10} and the M_1 is illustrated in Fig. 1.

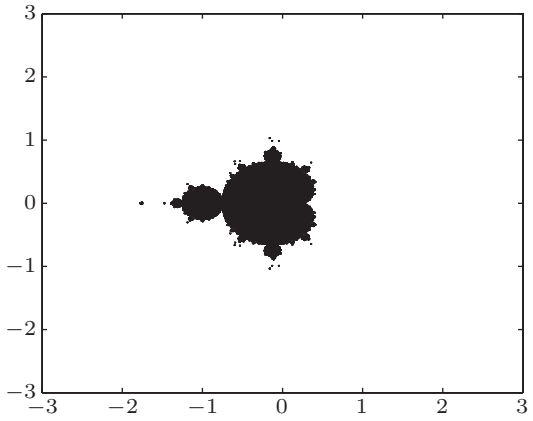


Fig. 1. Mandelbrot set M_1 of system (1).

Let $w_{n+1} = w_n^2/d + dc$ for system (7), then by induction, we can get $w_n = dz_n$. Thus, for any constant d, the iteration of system (7) at 0 is bounded if and only if the iteration of system (1) at 0 is bounded. From the definition of the Mandelbrot set, we have the following conclusion.

Proposition 1 The Mandelbrot sets of system (7) and (1) are identical.

This conclusion shows that the Mandelbrot sets of the controlled system (7) will not be changed with different values of the control parameter d (see Fig. 2).

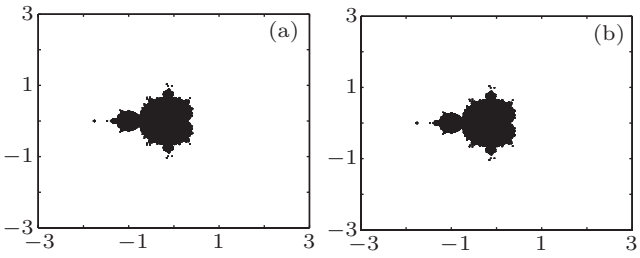


Fig. 2. Control of Mandelbrot sets of system (7). (a) $d = 0.5$; (b) $d = 2$.

Proposition 2 The Mandelbrot set of system (8) $M_8 = \{c/d : c \in M_1\}$.

Proof Let $c' = dc$, then the Mandelbrot set M' of $f_8(z) = z^2 + c'$ is the same as M_1 about the system parameter c'. Since $c = c/d'$, we have $M_8 = \{c/d : c \in M_1\}$. M_8 increases as d decreases, which means that smaller control values can get bigger Mandelbrot sets and bigger control values can get smaller Mandelbrot sets, see Fig. 3.

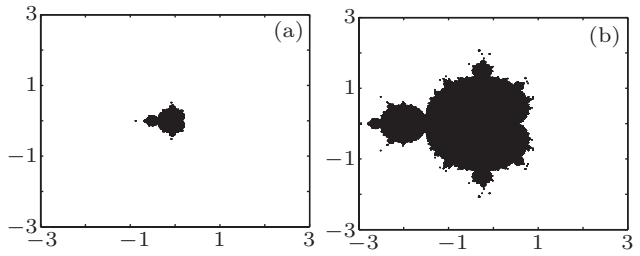


Fig. 3. Control of Mandelbrot sets of system (8). (a) $d = 2$; (b) $d = 0.5$.

Proposition 3 (i) The Mandelbrot set M_9 of system (9) can be obtained by moving M_1 d units.

(ii) The Mandelbrot set M_{10} of system (10) can be obtained by rotating M_1 θ angle (see Fig. 4).

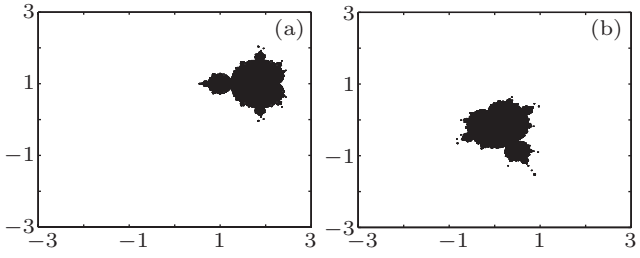


Fig. 4. Control of Mandelbrot sets of system (9)) and (10). (a) $d = 2+i$; (b) $\theta = -2\pi/3$.

We now discuss the Mandelbrot set of the controlled system (6). From (6) we have

$$z_{n+1} = (1+k)z_n^2 + \frac{1}{1+k}(1+k)(c - kz_{\text{ref}}^2). \tag{13}$$

By Proposition 1, we know that the Mandelbrot sets of system (13) or system (6) are the same as the Mandelbrot sets of the following system

$$\begin{aligned} z_{n+1} &= z_n^2 + (1+k)(c - kz_{\text{ref}}^2), \\ &= z_n^2 + (1+k)c - (1+k)kz_{\text{ref}}^2. \end{aligned} \tag{14}$$

From Propositions 2 and 3, we can get that the Mandelbrot set of system (13) is also that of the controlled system (6). According to our expectation, we could select appropriate reference item z_{ref} in the controlled system (6) to control the Mandelbrot sets at any position on the complex plane, and control the size of the Mandelbrot set by changing k, see Fig. 5.

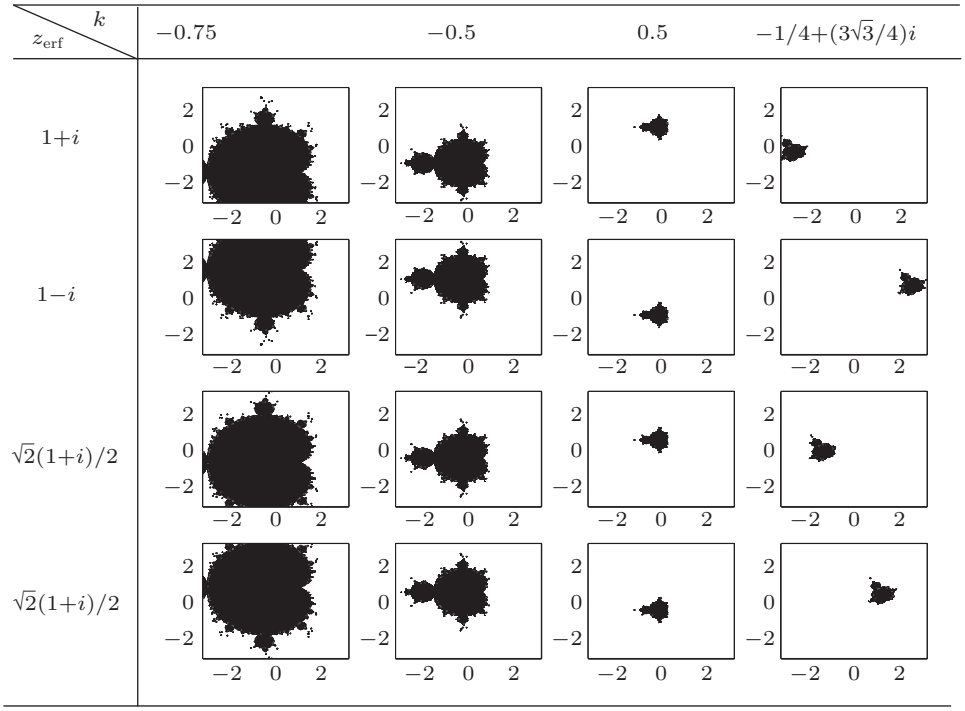


Fig. 5. Control of Mandelbrot sets of system (6) with different parameters.

Figure 5 shows that the Mandelbrot sets rotate θ, or expand, or shrink once the auxiliary reference control item z_{ref} and the feedback coefficient k are changed.

In fact, by amending the feedback item in Eq. (3), the auxiliary reference feedback control method can be applied to the generalized Mandelbrot set of the complex system

$$z_{n+1} = z_n^r + c, \tag{15}$$

where $r > 2$ is an integer. For example, take $g_n = k\sum_{i=1}^{r} z_n^{r-i} z_{\text{ref}}^{i-1}$, then the controlled system of Eq. (15) is

$$z_{n+1} = z_n^r + c + k(z_n^r - z_{\text{ref}}^r). \tag{16}$$

The changing of Mandelbrot sets of (16) can be analyzed in a similar way. Thus we omit the discussion here.

3. Synchronization of Mandelbrot sets

Chaos synchronization has been studied extensively and there are many methods to realize chaos synchronization. In this section, we will consider the synchronization of generalized Mandelbrot sets (15) and the following complex system

$$w_{n+1} = w_n^s + c', \tag{17}$$

where s is not equal to r in Eq. (15). Therefore the generalized Mandelbrot sets (15) and (17) are different. We consider controlling the generalized Mandelbrot set (15) changing it to be the generalized Mandelbrot set (17). Since z_{ref} in the auxiliary reference feedback control item usually is chosen to be the objective, we take $z_{\text{ref}} = (w_n)^{s/r}$. Then

$$z_{n+1} = z_n^r + c + k(z_n^r - w_n^s), \tag{18}$$

and

$$\begin{aligned} & z_{n+1} - w_{n+1} \\ &= z_n^r + c + k(z_n^r - w_n^r) - w_n^s - c' \\ &= (1+k)(z_n^r - w_n^s) + c - c' \\ &= (1+k)(z_n - w_n)(z_n^{r-1} + z_n^{r-2}w_n + \cdots \\ &\quad + z_n w_n^{r-2} + w_n^{r-1}) + (1+k)(w_n^r - w_n^s) + c - c'. \end{aligned}$$

We consider only the iterations which are bounded since c is not in the generalized Mandelbrot set if for some n_0, $f^{n_0}(0;c)$ is not bounded from the definition of the Mandelbrot set. That is to say, there exists a positive real number T and we consider only the iterations included in the set $\{z : |z| \leq T\}$. Let $M = \max\{T^r, T^s\}$. From the definition of the Mandelbrot set, we take the initial values $z_1 = w_1 = 0$ and $c = c'$ in the computing process. Then

$$|z_3 - w_3| = |(1+k)(c^r - c^s)| \leq 2M|1+k|,$$

and thus

$$\begin{aligned}
&|z_{n+1} - w_{n+1}| \leq rM|1+k||z_n - w_n| + 2M|1+k| \\
&\leq rM|1+k|(rM|1+k||z_{n-1} - w_{n-1}| \\
&\quad + 2M|1+k|) + 2M|1+k| \\
&= (rM|1+k|)^2|z_{n-1} - w_{n-1}| \\
&\quad + (rM|1+k|+1)(2M|1+k|) \\
&\leq \cdots \\
&\leq (rM|1+k|)^{n-3}|z_3 - w_3| + [(rM|1+k|)^{n-4} + \cdots \\
&\quad + rM|1+k|+1](2M|1+k|) \\
&\leq [(rM|1+k|)^{n-3} + (rM|1+k|)^{n-4} + \cdots \\
&\quad + rM|1+k|+1](2M|1+k|) \\
&= 2M|1+k|\frac{1-(rM|1+k|)^{n-2}}{1-rM|1+k|}.
\end{aligned}$$

Take $rM|1+k| < 1$, then the right side of the above equation goes to $2M|1+k|/(1-rM|1+k|)$ as $n \to \infty$. If $|1+k| \to 0$ in this case, that is $k \to -1$, we have $|z_{n+1} - w_{n+1}| \to 0$. Thus the synchronization of trajectories of systems (17) and (18) is achieved, that is to say that, the Mandelbrot set of (18) will change to be the Mandelbrot set of (17) with $k \to -1$.

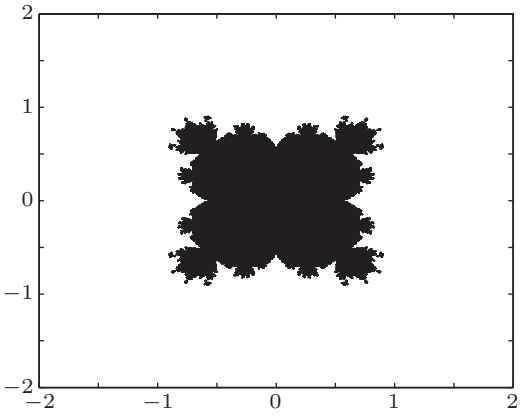


Fig. 6. Mandelbrot set of system (15) when $r = 5$.

Take $s = 2$ in (17) and $r = 5$ in (15), then their Mandelbrot sets are illustrated in Figs. 1 and 6 respectively. From Fig. 7, it is easy to see that Mandelbrot sets? synchronization has not been achieved when the values of k is small. However, the Mandelbrot set (18) is changing to be the Mandelbrot set (15) with the values of k going to -1.

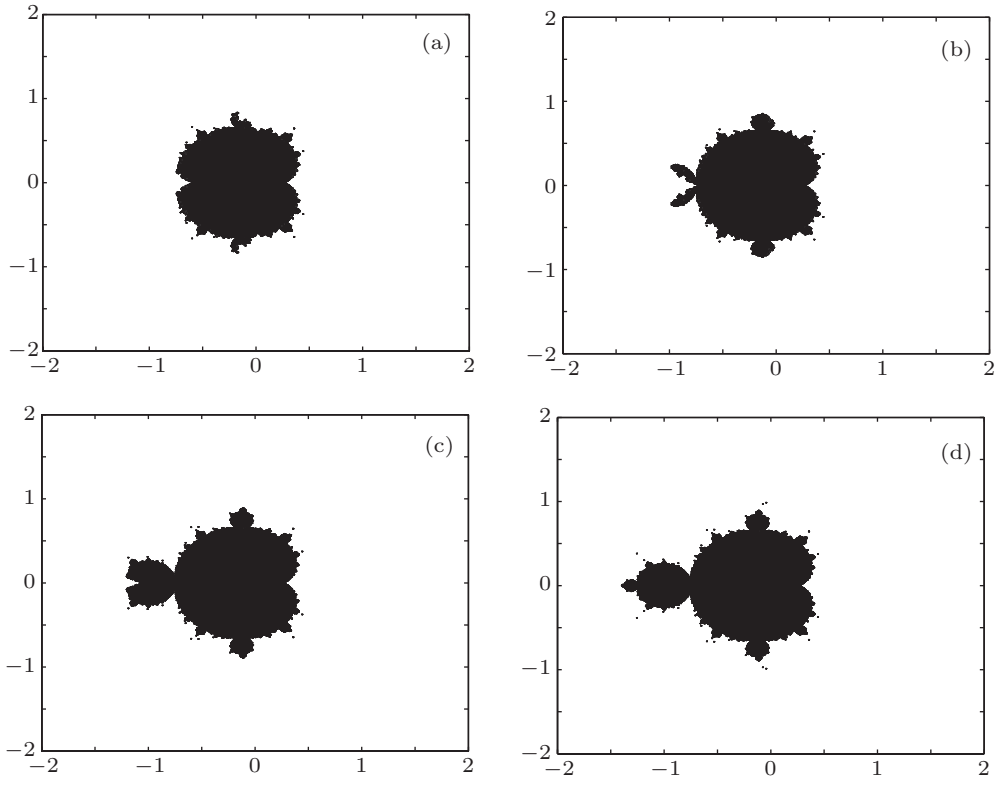


Fig. 7. Mandelbrot set of system (15) when $r = 5$. (a) $k = -0.1$; (b) $k = -0.4$; (c) $k = -0.7$; (d) $k = -0.85$.

4. Conclusions

A new fractal dynamical system is emerged by introducing an auxiliary reference feedback control item into the system. We can obtain various Mandelbrot sets of the controlled system by adjusting the feedback control parameter and the auxiliary reference item. By taking the reference item to be the trajectory of another system, synchronization of Mandelbrot sets of two different systems is achieved. Since M-set is related to the dynamics and the stability of systems, we also provide a method to analyze the dynamical properties of systems from the fractal point of view.

References

[1] Osbaldestin A H 1995 *J. Phys.* A *Math. Gen.* **28** 5951
[2] Hu B and Lin B 1989 *Phys. Rev.* A **39** 4789
[3] Christian Beck 1999 *Physica* D **125** 171
[4] Wang X Y and Meng Q Y 2004 *Acta Phys. Sin.* **53** 388 (in Chinese)
[5] Liu S T and Zhang Y P 2008 *Acta Phys. Sin.* **57** 737 (in Chinese)
[6] Zhang Y P and Liu S T 2008 *Chin. Phys.* B **17** 543
[7] Zhang Y P, Sun W H and Liu C A 2010 *Chin. Phys.* B **19** 050512
[8] Yuen C H and Wong K W 2012 *Chin. Phys.* B **21** 010502
[9] Wand X Q, Ma J, Zhang X H and Wang X G 2011 *Chin. Phys.* B **20** 050510
[10] Duan D H, Xu W, Su J and Zhou B C 2011 *Chin. Phys.* B **20** 030501
[11] Gao T F, Liu F S and Chen J C 2012 *Chin. Phys.* B **21** 020502
[12] Ye Z Y, Yang G and Deng C B 2011 *Chin. Phys.* B **20** 010207
[13] Ott E, Grebogi C and Yorke Y A 1990 *Phys. Rev. Lett.* **64** 1196
[14] Michael E Brandt and Chen G 1997 *IEEE Trans. Cir. Sys.* I **44** 1031