Entanglement fidelity of channel adaptive quantum codes*

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We study the performances of quantum channel adaptive [4,1] code transmitting in a joint amplitude damping and dephasing channel, the [6,2] code transmitting in an amplitude damping channel by combining the encoding, noise process, and decoding as one effective channel. We explicitly obtain the entanglement fidelities. The recovery operators of the [6,2] code are given. The performance is nearly optimal compared with that of the optimal method of semidefinite programming.

Keywords: adaptive quantum code, amplitude damping channel, fidelity

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1. Introduction

Quantum error correction codes (QECCs) encode quantum information in large Hilbert spaces so that errors can be detected and corrected without directly measuring the quantum information to be protected.[1,2] The errors come from the inevitable interactions of the system with its environment and the internal operational errors. In quantum communication, the encoding and decoding are assumed to be error free, and all errors occur during the transmission of quantum information through a noisy quantum channel. The most popular and well studied transmission channel is the Pauli channel. Many QECCs have been devised to overcome the Pauli errors. There are X, Z, and Y errors, namely, bit flip error, phase flip error, and simultaneous bit and phase flip error. The usual QECCs are designed to control the noise with equal bit and phase flip probabilities. However, the noise in physical qubits is fundamentally asymmetric, so phase flips appear more frequently than bit errors. Asymmetric QECCs have been proposed for achieving a high encoding efficiency.[3–5] Topological surface codes can be asymmetric and have good performances too.[6,7] The topological codes are constructed based on the fact that the ground state degeneracy of a topologically ordered phase cannot be distinguished by local operations and hence are robust against local perturbations. Applications of QECCs in quantum secret sharing[8] and entanglement assistant codes[9] are also investigated.

Besides the Pauli channel, there are other channels, such as the amplitude damping channel and the dephasing channel, which are practically important. The error types in those channels are different from that in the Pauli channel. It has been found that a different strategy of devising QECCs may be more efficient. The first of which is the [4,1] code devised for the amplitude damping channel;[10] other asymmetric codes or channel adaptive codes have also been developed.[11–13] To analyze the performance of the code, one of the methods of black box nature is to consider the encoding, channel noise, and decoding as one effective channel.[14] For the amplitude damping channel, nonadditive quantum codes of a codeword stabilized type which outperform the best possible additive codes in terms of encoded dimension are studied.[15] Better performing codes up to length 14 are obtained numerically and listed. The asymmetric codes are closely related to the channel adaptive codes, since we can obtain a Pauli channel from a combined amplitude damping and dephasing channel by a technique called twirling.[11] The resultant Pauli channel is most probably asymmetric.

In this paper, we will study the overall performances of [4,1] and [6,2] codes by working out the effective channel parameters using explicit recovery operators. Most quantum codes are based on the stabilizer construction.[16] The stabilizer formalism provides an extremely useful and compact description for quantum error correction codes. For a code encoding k qubits in n qubits, the code space T has 2^k dimensions, and the stabilizer group S has $2^{(n-k)}$ elements, the space $T = \{|\psi\rangle \text{ s.t. } M|\psi\rangle = |\psi\rangle \ \forall M \in S\}$. The so called [4,1] code, which is more efficient in correcting the errors in the amplitude damping channel, can also be written in a stabilizer form, the stabilizer group is $S = \langle M_1, M_2, M_3\rangle$, where $M_1 = XXXX$, $M_2 = ZZII$, and $M_3 = IIZZ$. The summation of all the elements of the stabilizer group S is

$$\begin{aligned}\sum_j S_j &= (I+M_1)(I+M_2)(I+M_3)\\ &= IIII + XXXX + ZZII + IIZZ\\ &\quad - YYXX - XXYY + ZZZZ + YYYY. \end{aligned} \tag{1}$$

So S consists of all Pauli products that stabilize the code. The crucial property of S is that its operators commute with each

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other, i.e. it is an Abelian group.

2. Amplitude damping and dephasing channel

The amplitude damping channel is characterized by the Kraus operators

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \tag{2}$$

The parameter γ indicates the probability of decaying from state $|1\rangle$ to state $|0\rangle$, i.e., the probability of losing one photon.

The combined amplitude damping and dephasing channel is characterized by the Kraus operators[11]

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-r-\lambda} \end{pmatrix},$$
$$K_1 = \begin{pmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \tag{3}$$

where $\sqrt{1-\gamma-\lambda} = \mathrm{e}^{-t/T_2}$, $1-\gamma = \mathrm{e}^{-t/T_1}$, T_1 is the relaxation time, and T_2 is the dephasing time. In general, relaxation leads to both bit flip and phase flip errors, whereas dephasing only leads to phase flip errors.

In Ref. [10], the [4,1] code was proposed for combating the amplitude damping errors, the authors proposed a recovery circuit and demonstrated its good performance. The logic codewords are given by

$$|0_{\mathrm{L}}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$
$$|1_{\mathrm{L}}\rangle = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle).$$

The recovery of all the first and some of the second order damping errors are considered.[13] The recovery operators should restore the damped states. For example, $K_1^{(2)}$ acts on the codewords of the [4,1] code as

$$I \otimes K_1 \otimes I \otimes I\,|0_{\mathrm{L}}\rangle = \sqrt{\gamma}|1011\rangle,$$
$$I \otimes K_1 \otimes I \otimes I\,|1_{\mathrm{L}}\rangle = \sqrt{\gamma}|1000\rangle,$$

$K_1^{(2)}$ and $K_1^{(4)}$ act simultaneously on the codewords of the [4, 1] code as

$$K_1^{(2)}K_1^{(4)}\,|0_{\mathrm{L}}\rangle = \gamma|1010\rangle,$$
$$K_1^{(2)}K_1^{(4)}\,|1_{\mathrm{L}}\rangle = 0.$$

The recovery operations are $R_3 - R_6$ and $R_7 - R_{10}$, which can be found in Table 1 of Ref. [13]. Then the recovery from the distortion of the no damping case must be considered. The recovery operation is $R_1 - R_2$.[13]

3. Effective channel

We consider the processes of encoding, noise, and decoding as a whole effective channel. The action of the encoding operation $\mathscr{E} : \rho_0 \to \rho(0)$ is given by $\rho(0) = \mathscr{E}[\rho_0]$. Notice that ρ_0 is the logical qubit and $\rho(0)$ is the initial register state of physical qubits. With the conventions of $\sigma_0 = I_2$ being the 2×2 identity matrix and σ_i $(i = 1,2,3)$ being the Pauli matrices, we can write $\rho_0 = \frac{1}{2}\sum_{i=0}^{3} u_i\sigma_i$. The vector $\boldsymbol{u} = (1, \boldsymbol{r})$, where $\boldsymbol{r}$ is the Bloch vector of ρ_0. Let $E_\sigma = \frac{1}{2}\mathscr{E}[\sigma]$, in the representation of encoded logical state basis $\{|0_{\mathrm{L}}\rangle, |1_{\mathrm{L}}\rangle\}$, the encoding operators E_σ can be expressed as

$$E_I = \frac{1}{2}(|0_{\mathrm{L}}\rangle\langle 0_{\mathrm{L}}| + |1_{\mathrm{L}}\rangle\langle 1_{\mathrm{L}}|),$$
$$E_X = \frac{1}{2}(|0_{\mathrm{L}}\rangle\langle 1_{\mathrm{L}}| + |1_{\mathrm{L}}\rangle\langle 0_{\mathrm{L}}|),$$
$$E_Y = \frac{1}{2}(-\mathrm{i}\,|0_{\mathrm{L}}\rangle\langle 1_{\mathrm{L}}| + \mathrm{i}\,|1_{\mathrm{L}}\rangle\langle 0_{\mathrm{L}}|),$$
$$E_Z = \frac{1}{2}(|0_{\mathrm{L}}\rangle\langle 0_{\mathrm{L}}| - |1_{\mathrm{L}}\rangle\langle 1_{\mathrm{L}}|). \tag{4}$$

The error model is the one with identical, uncorrelated noise acting on each physical qubit. For such a model, we may write

$$\mathscr{N} = N \otimes N \otimes \cdots \otimes N = N^{\otimes n}. \tag{5}$$

The decoding operation $\mathscr{D}$ is characterized by operators D_σ[14]

$$D_\sigma = 2\sum_j R_j^\dagger E_\sigma R_j. \tag{6}$$

We can derive the coding map $\mathscr{G}$ between physical and effective channels from a given coding and recovery procedure as follows:

$$\mathscr{G} = \mathscr{D} \circ \mathscr{N} \circ \mathscr{E}. \tag{7}$$

To characterize the effective channel $\mathscr{G}$, we calculate the elements of the corresponding matrix $\boldsymbol{G}$, which are given by[14]

$$G_{\sigma\sigma'} = \mathrm{Tr}(D_\sigma \mathscr{N}[E_{\sigma'}]). \tag{8}$$

After decoding, the final state $\rho_{\mathrm{f}} = \frac{1}{2}\sum_{i=0}^{3} v_i\sigma_i$ is related with the input state $\rho_0 = \frac{1}{2}\sum_{i=0}^{3} u_i\sigma_i$ by

$$\boldsymbol{v} = \boldsymbol{G}\boldsymbol{u}.$$

Notice that any transformation taking the quantum state ρ_0 to ρ_{f} can be written in the following form:

$$\rho_{\mathrm{f}} = \mathscr{G}(\rho_0) = \sum_\mu A_\mu \rho_0 A_\mu^\dagger = \frac{1}{2}\sum_{i,j,k=0}^{3} Q_{jk} v_i \sigma_j \sigma_i \sigma_k,$$

where $A_\mu = \sum_j A_{\mu j}\sigma_j$ are the Kraus operators of the effective channel, and $Q_{jk} = \sum_\mu A_{\mu j} A_{\mu k}^*$. Hence

$$G_{lm} = \frac{1}{2}\sum_{j,k=0}^{3} \mathrm{Tr}(\sigma_l\sigma_j\sigma_m\sigma_k) Q_{jk}. \tag{9}$$

So we get the relationship between matrices $\boldsymbol{G}$ and $\boldsymbol{Q}$ as follows:

$$[Q_{00}, Q_{11}, Q_{22}, Q_{33},]^{\mathrm{T}} = \frac{1}{4}\boldsymbol{H}_0\,[G_{00}, G_{11}, G_{22}, G_{33}]^{\mathrm{T}},$$

$$[Q_{01},Q_{10},Q_{23},Q_{32}]^{\mathrm{T}} = \frac{1}{4}\boldsymbol{H}_1\,[G_{01},G_{10},G_{23},G_{32}]^{\mathrm{T}},$$
$$[Q_{02},Q_{20},Q_{31},Q_{13}]^{\mathrm{T}} = \frac{1}{4}\boldsymbol{H}_1\,[G_{02},G_{20},G_{31}G_{13}]^{\mathrm{T}},$$
$$[Q_{03},Q_{30},Q_{12},Q_{21}]^{\mathrm{T}} = \frac{1}{4}\boldsymbol{H}_1\,[G_{03},G_{30},G_{12},G_{21}]^{\mathrm{T}}, \quad (10)$$

with

$$\boldsymbol{H}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \boldsymbol{H}_1 = \begin{bmatrix} 1 & 1 & -\mathrm{i} & \mathrm{i} \\ 1 & 1 & \mathrm{i} & -\mathrm{i} \\ \mathrm{i} & -\mathrm{i} & 1 & 1 \\ -\mathrm{i} & \mathrm{i} & 1 & 1 \end{bmatrix}.$$

The entanglement fidelity can be written in the following form:[17]

$$F_{\mathrm{e}}(\rho_0,\boldsymbol{A}) = \sum_{\mu}\left|\mathrm{Tr}\left(\rho_0 A_\mu\right)\right|^2. \quad (11)$$

We therefore have

$$F_{\mathrm{e}}(\rho_0,\boldsymbol{A}) = \sum_{jk} u_j Q_{jk} u_k = \boldsymbol{u}\boldsymbol{Q}\boldsymbol{u}^{\mathrm{T}}. \quad (12)$$

Since the input state is usually chosen to be $\rho_0 = I_2/2$ for the cause of simplicity, we have $\boldsymbol{\nu} = (1,0,0,0)$, the entanglement fidelity for the effective channel with input state $I_2/2$ is

$$F_{\mathrm{e}}\left(\frac{I_2}{2},\boldsymbol{A}\right) = Q_{00} = \frac{1}{4}\mathrm{Tr}(\boldsymbol{G}). \quad (13)$$

For an effective channel encoding two logical qubits with the input state $\rho_0 = I_4/4$, where I_4 is the 4×4 identity matrix, a similar derivation leads to

$$F_{\mathrm{e}}\left(\frac{I_4}{4},\boldsymbol{A}\right) = \frac{1}{16}\mathrm{Tr}(\boldsymbol{G}), \quad (14)$$

where $\boldsymbol{G}$ is a 16×16 matrix characterizing the effective channel $\mathscr{G}$.

4. Performance of [4,1] code

We consider quantum error-correction [4,1] code for the amplitude damping channel. The stabilizer group is $S = \langle XXXX, ZZII, IIZZ\rangle$. We choose the logical Pauli operators $\bar{X} = XXII$ and $\bar{Z} = ZIZI$. As $XY = \mathrm{i}Z$, we have $\bar{Y} = YXZI$. The [4,1] code is a stabilizer code, hence[14]

$$E_\sigma = \frac{1}{2|S|}\sum_k S_k \overline{\sigma}.$$

Using Eq. (1), we have

$$\begin{aligned} E_I &= \frac{1}{16}\left[(II+ZZ)^{\otimes 2} + (XX-YY)^{\otimes 2}\right], \\ E_X &= \frac{1}{16}[(XX-YY)(II+ZZ) \\ &\quad + (II+ZZ)(XX-YY)], \\ E_Y &= \frac{1}{16}[(YX+XY)(ZI+IZ) \\ &\quad + (ZI+IZ)(YX+XY)], \\ E_Z &= \frac{1}{16}\left[(ZI+IZ)^{\otimes 2} - (YX+XY)^{\otimes 2}\right], \end{aligned} \quad (15)$$

where the products between the parentheses are tensor products. For each physical qubit in the amplitude damping channel, we have $N(I) = I+\gamma Z$, $N(X) = \sqrt{1-\gamma}X$, $N(Y) = \sqrt{1-\gamma}Y$, and $N(Z) = (1-\gamma)Z$. Therefore

$$\begin{aligned} \mathscr{N}(E_I) &= \frac{1}{16}\left[(1-\gamma)^2(XX-YY)^{\otimes 2}\right] \\ &\quad + \frac{1}{16}\left[(I+\gamma Z)^{\otimes 2} + (1-\gamma)^2 ZZ\right]^{\otimes 2}, \\ \mathscr{N}(E_X) &= \frac{1-\gamma}{16}(XX-YY)\left[(I+\gamma Z)^{\otimes 2} + (1-\gamma)^2 Z^{\otimes 2}\right] \\ &\quad + \frac{1-\gamma}{16}\left[(I+\gamma Z)^{\otimes 2} + (1-\gamma)^2 Z^{\otimes 2}\right](XX-YY), \\ \mathscr{N}(E_Y) &= \frac{(1-\gamma)^2}{16}(YX+XY)[Z(I+\gamma Z) + (I+\gamma Z)Z] \\ &\quad + \frac{(1-\gamma)^2}{16}[Z(I+\gamma Z) + (I+\gamma Z)Z](YX+XY), \\ \mathscr{N}(E_Z) &= \frac{(1-\gamma)^2}{16}[Z(I+\gamma Z) + (I+\gamma Z)Z]^{\otimes 2} \\ &\quad - \frac{(1-\gamma)^2}{16}(YX+XY)^{\otimes 2}. \end{aligned} \quad (16)$$

Now we construct the D_σ operators for decoding. According to Eq. (6), using the expressions for E_σ given in Eq. (4) and R_j in Table 1 of Ref. [13], we get the expression for the D_σ operators. Then we can find out the matrix elements of the effective channel using Eq. (8). Since $F_{\mathrm{e}} = \mathrm{Tr}(\boldsymbol{G})/4$ for the input state $\rho_0 = I_2/2$, we need to calculate the diagonal elements of $\boldsymbol{G}$. The calculation yields

$$\begin{aligned} G_{II} &= 1, \\ G_{XX} &= (1-\gamma)\frac{1}{\sqrt{2}}(\alpha_R+\beta_R)\left(1-\gamma+\gamma^2\right) \\ &\quad + (1-\gamma)\frac{\gamma}{\sqrt{2}}(\alpha_R-\beta_R) + 2\gamma(1-\gamma), \\ G_{YY} &= (1-\gamma)^2\frac{1}{\sqrt{2}}(\alpha_R+\beta_R) \\ &\quad + (1-\gamma)^2\frac{\gamma}{\sqrt{2}}(\alpha_R-\beta_R) + 2\gamma(1-\gamma)^2, \\ G_{ZZ} &= (1-\gamma)^2(1+\gamma)^2, \end{aligned} \quad (17)$$

where $\alpha_R = (\alpha+\alpha^*)/2$, $\beta_R = (\beta+\beta^*)/2$, α and β are the adaptive parameters appearing in the recovery operators R_1 and R_2, and $|\alpha|^2 + |\beta|^2 = 1$. So we derive the entanglement fidelity as

$$\begin{aligned} F_{\mathrm{e}} &= \frac{1}{4}(G_{II}+G_{XX}+G_{YY}+G_{ZZ}) \\ &= \frac{1}{4}\left[1+\sqrt{2}\alpha_R t + 8t^2 + \left(\sqrt{2}\beta_R - 8\right)t^3 + t^4\right], \end{aligned} \quad (18)$$

where $t = 1-\gamma$. The maximization over adaptive parameters leads to

$$F_{\mathrm{e}} = \frac{1}{4}\left[1 + t\sqrt{2\left(1+t^4\right)} + t^2\left(8-8t+t^2\right)\right]. \quad (19)$$

The optimal entanglement fidelity F_e of the [4,1] code for the amplitude damping channel is shown in Fig. 1. Notice that this maximum is obtained under the condition of given input and recovery operators with one free complex variable.

The solid line in Fig. 1 displays the maximal entanglement fidelity with respect to the channel damping parameter γ. A careful comparison of our Fig. 1 with Fig. 3 of Ref. [18] shows that our result is better than that of Ref. [10], but a little bit worse than that obtained using the semidefinite programming.[18] This is reasonable because our recovery operators are almost fixed. As γ goes to zero, our optimal entanglement fidelity will become

$$F_e = 1 - 1.5\gamma^2 + O(\gamma^3). \tag{20}$$

In contrast, the entanglement fidelity of Leung *et al.*'s[10] recovery is $F_e = 1 - 2.75\gamma^2 + O(\gamma^3)$, while the optimal entanglement fidelity using the semidefinite programming is numerically determined to be $F_e = 1 - 1.25\gamma^2 + O(\gamma^3)$.[18]

An extension to the joint amplitude damping and dephasing channel (3) is straightforward, which leads to

$$\begin{aligned} F_e = & \frac{1}{4} + \frac{(t-\lambda)}{4}\sqrt{2(1+t^4)} - \lambda t \\ & + (2+\lambda)t^2 - 2t^3 + \frac{t^4}{4}. \end{aligned} \tag{21}$$

The performances of the [4,1] code for the joint amplitude and phase damping channels are also shown in Fig. 1.

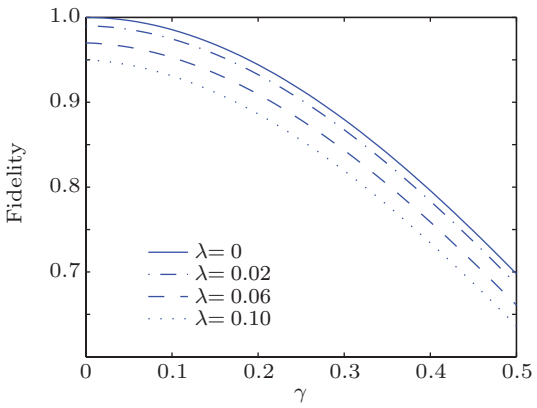


Fig. 1. (color online) Comparison of the performances of the amplitude damping [4,1] code in the amplitude damping channel and the joint amplitude damping and dephasing channel.

5. Performance of [6,2] code in the amplitude damping channel

The performance analysis method used for the [4,1] channel adaptive code can be generalized to higher rate codes. In this section, we consider the [6,2] code and its performance with recovery operations adapting to the amplitude damping channel. Let us write down the logical codewords of the [6,2] code

$$\begin{aligned} |00_L\rangle &= \frac{1}{\sqrt{2}}(|000000\rangle + |111111\rangle), \\ |01_L\rangle &= \frac{1}{\sqrt{2}}(|001001\rangle + |110110\rangle), \\ |10_L\rangle &= \frac{1}{\sqrt{2}}(|000110\rangle + |111001\rangle), \\ |11_L\rangle &= \frac{1}{\sqrt{2}}(|110000\rangle + |001111\rangle). \end{aligned} \tag{22}$$

A proper extension of the encoding operators for two logical qubits is

$$E_{\sigma\sigma'} = E_\sigma \otimes E_{\sigma'}. \tag{23}$$

The stabilizer group of the code is $S = \langle M_1, M_2, M_3, M_4\rangle$ with $M_1 = XXXXXX$, $M_2 = ZZIIII$, $M_3 = IIZIIZ$, and $M_4 = IIIZZI$. The summation of the elements of the stabilizer group S is

$$\begin{aligned} \sum_j S_j = & (III + ZZI)^{\otimes 2} + (XXX - YYX)^{\otimes 2} \\ & + (IIZ + ZZZ)^{\otimes 2} - (XXY - YYY)^{\otimes 2}. \end{aligned} \tag{24}$$

The logical Pauli operators are $\overline{X_1} = IIIXXI$, $\overline{X_2} = IIXIIX$, $\overline{Z_1} = ZIIIZI$, and $\overline{Z_2} = ZIIIIZ$ for the codewords in Eq. (22). The recovery operators are given in Appendix A, where two pairs of adaptive parameters are introduced. The decoding operators

$$D_{\sigma\sigma'} = 4\sum_j R_j^\dagger E_{\sigma\sigma'} R_j \tag{25}$$

can be expressed in the physical qubit basis by making use of the orthonormal property of the logical qubit basis. The 16 diagonal elements of the effective channel matrix $\boldsymbol{G}$ can be evaluated with

$$G_{\sigma\sigma',\tau\tau'} = \mathrm{Tr}(D_{\sigma\sigma'}\mathcal{N}[E_{\tau\tau'}]) \tag{26}$$

in the physical qubit basis. After a lengthy algebra, for logical input $\rho_0 = I_4/4$, we get the formula of entanglement fidelity

$$F_e = Ak + Bm + C, \tag{27}$$

where

$$\begin{aligned} k &= \frac{3}{8}\left(1 - 3\gamma + 5\gamma^2 - 5\gamma^3 + \frac{5}{2}\gamma^4 - \frac{1}{2}\gamma^5\right), \\ m &= \frac{3}{8}\left(\gamma - \frac{5}{2}\gamma^2 + 2\gamma^3 - \frac{1}{2}\gamma^4\right), \\ C &= \frac{5}{8} + \frac{3}{16}\left(6\gamma - \frac{55}{2}\gamma^2 + \frac{51}{2}\gamma^3 - \frac{15}{2}\gamma^4 + \frac{1}{2}\gamma^5\right), \end{aligned}$$

and A, B rely on the adaptive coefficients $\alpha, \beta, \eta, \delta$

$$\begin{aligned} A &= \frac{1}{2}(\alpha\eta^* + \alpha^*\eta + {}^* + \beta^*\delta), \\ B &= \frac{1}{2}(\alpha\delta^* + \alpha^*\delta - \beta\eta^* + \beta^*\eta), \end{aligned}$$

with $|\alpha|^2 + |\beta|^2 = 1$ and $|\eta|^2 + |\delta|^2 = 1$. Denote $\boldsymbol{a} = (\alpha_R, \beta_R, \alpha_I, \beta_I)$, $\boldsymbol{b} = (\eta_R, \delta_R, \eta_I, \delta_I)$, and $\boldsymbol{c} = (\delta_R, -\eta_R, \delta_I, -\eta_I)$, then $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}| = 1$ and $\boldsymbol{b} \cdot \boldsymbol{c} = 0$. Hence $F_e = \boldsymbol{a} \cdot (k\boldsymbol{b} + m\boldsymbol{c}) + C \leq |k\boldsymbol{b} + m\boldsymbol{c}| + C$. The optimum entanglement fidelity is

$$F_e = \sqrt{k^2 + m^2} + C. \tag{28}$$

The performance is shown in Fig. 2. By a careful comparison between our Fig. 2 with Fig. 5(A) of Ref. [13], we find that our result is almost as good as that obtained using the semidefinite programming. As γ goes to zero, our optimal entanglement fidelity will become

$$F_e = 1 - \frac{99}{32}\gamma^2 + O(\gamma^3). \tag{29}$$

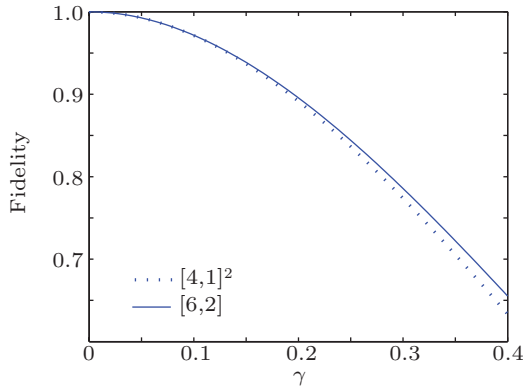


Fig. 2. (color online) Performance of [6,2] code compared with that of $[4,1]^2$ code.

6. Conclusion

We have derived the exact formula of entanglement fidelity for the transmission of [4,1] adaptive quantum code over a joint amplitude damping and dephasing channel. The input logical state is the most random state $\rho_0 = I_2/2$ and the recovery operators are those used in Ref. [13]. The performance of the effective channel including encoding, noise, and decoding is better than Leung *et al.*'s recovery,[10] and comparable with the one using the semidefinite programming. The latter is numerical and complicated, while for the [6,2] code encoding the input state $\rho_0 = I_4/4$ transmitting over the amplitude damping channel, with the given recovery operators, the exact optimal formula of entanglement fidelity is worked out.

Appendix A: The recovery operators of the [6,2] code

They are

$$\begin{aligned}
R_1 &= |00_L\rangle(\alpha\langle 000000| + \beta\langle 111111|) \\
&\quad + |01_L\rangle(\eta\langle 001001| + \delta\langle 110110|) \\
&\quad + |10_L\rangle(\eta\langle 000110| + \delta\langle 111001|) \\
&\quad + |11_L\rangle(\eta\langle 110000| + \delta\langle 001111|), \\
R_2 &= |00_L\rangle(\beta^*\langle 000000| - \alpha^*\langle 111111|) \\
&\quad + |01_L\rangle(\delta^*\langle 001001| - \eta^*\langle 110110|) \\
&\quad + |10_L\rangle(\delta^*\langle 000110| - \eta^*\langle 111001|) \\
&\quad + |11_L\rangle(\delta^*\langle 110000| - \eta^*\langle 001111|), \\
R_3 &= |00_L\rangle\langle 011111| + |01_L\rangle\langle 010110| \\
&\quad + |10_L\rangle\langle 011001| + |11_L\rangle\langle 010000|, \\
R_4 &= |00_L\rangle\langle 101111| + |01_L\rangle\langle 100110| \\
&\quad + |10_L\rangle\langle 101001| + |11_L\rangle\langle 100000|, \\
R_5 &= |00_L\rangle\langle 110111| + |01_L\rangle\langle 000001| \\
&\quad + |10_L\rangle\langle 110001| + |11_L\rangle\langle 000111|, \\
R_6 &= |00_L\rangle\langle 111011| + |01_L\rangle\langle 110010| \\
&\quad + |10_L\rangle\langle 000010| + |11_L\rangle\langle 001011|, \\
R_7 &= |00_L\rangle\langle 111101| + |01_L\rangle\langle 110100| \\
&\quad + |10_L\rangle\langle 000100| + |11_L\rangle\langle 001101|, \\
R_8 &= |00_L\rangle\langle 111110| + |01_L\rangle\langle 001000| \\
&\quad + |10_L\rangle\langle 111000| + |11_L\rangle\langle 001110|, \\
R_9 &= |00_L\rangle\langle 010111| + |10_L\rangle\langle 010001|, \\
R_{10} &= |00_L\rangle\langle 011110| + |10_L\rangle\langle 011000|, \\
R_{11} &= |00_L\rangle\langle 011011| + |01_L\rangle\langle 010010|, \\
R_{12} &= |00_L\rangle\langle 011101| + |01_L\rangle\langle 010100|, \\
R_{13} &= |00_L\rangle\langle 100111| + |10_L\rangle\langle 100001|, \\
R_{14} &= |00_L\rangle\langle 101110| + |10_L\rangle\langle 101000|, \\
R_{15} &= |00_L\rangle\langle 101011| + |01_L\rangle\langle 100010|, \\
R_{16} &= |00_L\rangle\langle 101101| + |01_L\rangle\langle 100100|, \\
R_{17} &= |00_L\rangle\langle 110011| + |11_L\rangle\langle 000011|, \\
R_{18} &= |00_L\rangle\langle 110101| + |11_L\rangle\langle 000101|, \\
R_{19} &= |00_L\rangle\langle 111010| + |11_L\rangle\langle 001010|, \\
R_{20} &= |00_L\rangle\langle 110011| + |11_L\rangle\langle 000011|, \\
R_{21} &= |00_L\rangle\langle 010011|, \\
R_{22} &= |00_L\rangle\langle 010101|, \\
R_{23} &= |00_L\rangle\langle 011010|, \\
R_{24} &= |00_L\rangle\langle 011100|, \\
R_{25} &= |00_L\rangle\langle 100011|, \\
R_{26} &= |00_L\rangle\langle 100101|, \\
R_{27} &= |00_L\rangle\langle 101010|, \\
R_{28} &= |00_L\rangle\langle 101100|.
\end{aligned}$$

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