Extended symmetry transformation of (3+1)-dimensional generalized nonlinear Schrödinger equation with variable coefficients*

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In this paper, the extended symmetry transformation of (3+1)-dimensional (3D) generalized nonlinear Schrödinger (NLS) equations with variable coefficients is investigated by using the extended symmetry approach and symbolic computation. Then based on the extended symmetry, some 3D variable coefficient NLS equations are reduced to other variable coefficient NLS equations or the constant coefficient 3D NLS equation. By using these symmetry transformations, abundant exact solutions of some 3D NLS equations with distributed dispersion, nonlinearity, and gain or loss are obtained from the constant coefficient 3D NLS equation.

Keywords: (3+1)-dimensional nonlinear Schrödinger equation, extended symmetry, exact solution, symbolic computation

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1. Introduction

The nonlinear Schrödinger (NLS) equation appears in many important branches of physics and applied mathematics, such as the propagation of laser beams in nonlinear media, plasma dynamics, the mean field dynamics of Bose–Einstein condensates, condensed matter, nonlinear optics, and quantum electronics.[1–18] The construction of exact solutions for NLS-type equations is one of the most important and essential tasks in the study of NLS equations. In the past several years, in order to obtain exact solutions of NLS equations, many effective methods have been developed, such as the inverse scattering method,[19] the Bäcklund transformation, the Darboux transformation,[20] the Hirota method,[21] the Lie symmetry method,[22,23] the Adomian decomposition method,[24] various tanh methods,[25–28] and the generalized sub-equation expansion method.[29,30] Ever since Bluman and Cole[31] and later Olver and Rosenau[32] generalized the Lie approach to encompass symmetry transformations, lots of research referring to symmetry has been done,[33–39] and the study of symmetries, symmetry groups, symmetry reductions, and group invariant solutions of NLS equations has become one of the most exciting and extremely active areas of research. For example, Gagnon *et al.*[40] studied the symmetries of a (1+1)-dimensional variable-coefficient NLS equation which involved three arbitrary complex functions of space–time and reported that the dimension of the Lie point symmetry group G of the (1+1)-dimensional variable-coefficient NLS equation was $1 < \dim G \leq 5$; Mansfield *et al.*[41] applied a certain nonlinear generalization of Lie's linear method for finding infinitesimal symmetries to a generalization of the (3+1)-dimensional coupled NLS system; Li *et al.* presented finite symmetry transformation groups of a (2+1)-dimensional cubic NLS equation and its corresponding cylindrical NLS equations in Ref. [13]. In Refs. [42]– [45], Ruan *et al.* investigated the exact solutions of some types of (1+1)-dimensional variable-coefficient NLS equations using the extending symmetry approach and applied those results to discuss the propagation in optical fibers and the amplification and compression of pulses in optical fiber amplifiers.

Our main motivation in this paper is to study the group theoretical properties of a generalized (3+1)-dimensional (3D) NLS equation with variable coefficients

$$\mathrm{i}\Psi_T + \Psi_{XX} + \Psi_{YY} + \Psi_{ZZ} + G(X,Y,Z,T,|\Psi|)\Psi = 0, \quad (1)$$

where the potential function G is a function of $\{X,Y,Z,T,\Psi\}$ and is considered as the new dependent variable. Based on the idea of the extended symmetry, we can get an extending symmetry group and the related finite transformation for the original equation (1). Using the extending symmetry group, we can set up a relationship between the solutions of two nonlinear equations with different variable-coefficient equations. Especially, if we assume that the original nonlinear partial differential equation possesses constant coefficients, then we can solve some special types of variable-coefficient nonlinear equations from the corresponding constant-coefficient equations.

The organization of the paper is as follows. In Section 2, we report the extending symmetries of Eq. (1). In Section 3, we derive some different type 3D variable-coefficient NLS equations and their general solutions. In Section 4, some exact solutions of the 3D variable-coefficient NLS equations are given. Finally, the conclusion is given.

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2. Extending symmetries of a generalized (3+1)-dimensional NLS equation with variable coefficients

Usually, the symmetry study on Eq. (1) is restricted by the fact that G is a function of X,Y,Z,T, and $|\Psi|$. Here we assume that $G(X,Y,Z,T,\Psi,\Psi^*)$ is a new dependent variable which is parallel to variable Ψ. When G is considered as a dependent variable, the symmetry of Eq. (1) is called the extending symmetry to distinguish from the usual symmetry. The first-order differential operator of the extending symmetry operator and the corresponding transformation group of Eq. (1) are given by

$$\begin{aligned}
&K=\xi\partial_X+\eta\partial_Y+\zeta\partial_Z+\tau\partial_T+\kappa\partial_\Psi+\kappa^*\partial_{\Psi^*}+\rho\partial_G,\\
&x=X+\varepsilon\xi,\ \ y=Y+\varepsilon\eta,\ \ z=Z+\varepsilon\zeta,\ \ t=T+\varepsilon\tau,\\
&\psi=\Psi+\varepsilon\kappa,\ \ \psi^*=\Psi^*+\varepsilon\kappa^*,\ \ g=G+\varepsilon\rho,
\end{aligned}\tag{2}$$

where ξ, η, ζ, τ, κ, and κ^* are all functions of X, Y, Z, T, Ψ, Ψ^*, $\rho=\rho(X,Y,Z,T,\Psi,\Psi^*,G)$, and ε is the group parameter. Using the invariance condition of Eq. (1) under operator (2) and its prolongation

$$\mathrm{i}[\kappa_T]+[\kappa_{XX}]+[\kappa_{YY}]+[\kappa_{ZZ}]+G\kappa+\rho\Psi=0,\tag{3}$$

where$[\kappa_T]$, $[\kappa_{XX}]$, $[\kappa_{YY}]$, and $[\kappa_{ZZ}]$ are infinitesimal forms of $\psi_T,\psi_{XX},\psi_{YY}$, and ψ_{ZZ}, respectively, and can be expressed as

$$\begin{aligned}
[\kappa_T] &= (\kappa-\xi\Psi_X-\eta\Psi_Y-\zeta\Psi_Z-\tau\Psi_T)_T\\
&\quad+\xi\Psi_{XT}+\eta\Psi_{YT}+\zeta\Psi_{ZT}+\tau\Psi_{TT},\\
[\kappa_{XX}] &= (\kappa-\xi\Psi_X-\eta\Psi_Y-\zeta\Psi_Z-\tau\Psi_T)_{XX}\\
&\quad+\xi\Psi_{XXX}+\eta\Psi_{YXX}+\zeta\Psi_{ZXX}+\tau\Psi_{TXX},\\
[\kappa_{YY}] &= (\kappa-\xi\Psi_X-\eta\Psi_Y-\zeta\Psi_Z-\tau\Psi_T)_{YY}\\
&\quad+\xi\Psi_{XYY}+\eta\Psi_{YYY}+\zeta\Psi_{ZYY}+\tau\Psi_{TYY},\\
[\kappa_{ZZ}] &= (\kappa-\xi\Psi_X-\eta\Psi_Y-\zeta\Psi_Z-\tau\Psi_T)_{ZZ}\\
&\quad+\xi\Psi_{XZZ}+\eta\Psi_{YZZ}+\zeta\Psi_{ZZZ}+\tau\Psi_{YZZ},
\end{aligned}\tag{4}$$

and considering $G=G(X,Y,Z,T,|\Psi|)$, i.e., $\Psi(\partial G/\partial\Psi)=\Psi^*(\partial G/\partial\Psi^*)$, we can obtain the system of determining equations

$$\begin{aligned}
&\xi_Z=\xi_\Psi=\xi_{\Psi^*}=\eta_Z=\eta_\Psi=\eta_{\Psi^*}=\zeta_X=\zeta_Y\\
&\quad=\zeta_\Psi=\zeta_{\Psi^*}=\tau_X=\tau_Y=\tau_Z=\tau_\Psi=\tau_{\Psi^*}=0,\\
&\kappa_{\Psi^*}=\kappa^*_\Psi=\kappa_{\Psi\Psi}=\kappa^*_{\Psi^*\Psi^*}=0,\\
&\tau_T-2\xi_X=0,\quad \tau_T-2\eta_Y=0,\quad \tau_T-2\zeta_Z=0,\\
&\xi_Y=\eta_X,\quad \mathrm{i}\zeta_T+\zeta_{ZZ}-2\kappa_{Z\Psi}=0,\quad \mathrm{i}\zeta_T-\zeta_{ZZ}+2\kappa^*_{Z\Psi^*}=0,\\
&(\tau_T-\kappa_\Psi)G\Psi+\kappa_{XX}+\kappa_{YY}+\kappa_{ZZ}+\mathrm{i}\kappa_T+\rho\Psi+G\kappa=0,\\
&(\tau_T-\kappa^*_{\Psi^*})G\Psi^*+\kappa^*_{XX}\\
&+\kappa^*_{ZZ}+\kappa^*_{ZZ}-\mathrm{i}\kappa^*_T+\rho\Psi^*+G\kappa^*=0.
\end{aligned}\tag{5}$$

Solving system (5) leads to the following result:

$$\begin{aligned}
\xi &= \frac{1}{2}\alpha'X-f_XY+\beta,\quad \eta=\frac{1}{2}\alpha'Y+f,\\
\zeta &= \frac{1}{2}\alpha'Z+\lambda,\quad \tau=\alpha,\\
\kappa &= \frac{\mathrm{i}}{8}\big[(X^2+Y^2+Z^2)\alpha''+4Y(-\mathrm{i}f_{XX}+f_T)\\
&\quad+4\beta'X+4\lambda'Z+8\mu\big]\Psi,\\
\kappa^* &= -\frac{\mathrm{i}}{8}\big[(X^2+Y^2+Z^2)\alpha''+4Y(\mathrm{i}f_{XX}+f_T)\\
&\quad+4\beta'X+4\lambda'Z+8\mu^*\big]\Psi^*,\\
\rho &= \frac{1}{8}(X^2+Y^2+Z^2)\alpha'''-\frac{3}{4}\mathrm{i}\alpha''+\frac{1}{2}Yf_{XX}\\
&\quad+\frac{1}{2}\beta''X+\frac{1}{2}\lambda''+\mu'-\alpha'G,
\end{aligned}\tag{6}$$

where $\alpha=\alpha(T)$, $\beta=\beta(T)$, $\lambda=\lambda(T)$, $\mu=\mu(T)$, and $\nu=\nu(T)$ are arbitrary smooth functions of T, and $f=f(X,T)$ is an arbitrary function of X,T. In particular, μ and ν are complex functions, while the other arbitrary functions are real. By considering different cases for the above arbitrary functions separately, we can get the general finite transformation of Eq. (1). Let x,y,z,t be new independent variables, and ψ and $g(x,y,z,t,|\psi|)$ be new dependent variables, which satisfy a new evolution equation

$$\mathrm{i}\psi_t+\psi_{xx}+\psi_{yy}+\psi_{zz}+g(x,y,z,t,|\psi|)\psi=0.\tag{7}$$

Solving the characteristic equation

$$\begin{aligned}
&\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}y}{\eta}=\frac{\mathrm{d}z}{\zeta}=\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}\psi}{\kappa}=\frac{\mathrm{d}\psi^*}{\kappa^*}=\frac{\mathrm{d}g}{\rho}=\mathrm{d}\varepsilon,\\
&x|_{\varepsilon=0}=X,\quad y|_{\varepsilon=0}=Y,\quad z|_{\varepsilon=0}=Z,\quad t|_{\varepsilon=0}=T,\\
&\psi|_{\varepsilon=0}=\Psi,\quad \psi^*|_{\varepsilon=0}=\Psi^*,\quad g|_{\varepsilon=0}=G,
\end{aligned}\tag{8}$$

we can find the general finite transformation x,y,z,t,ψ,ψ^*, and g. For simplification, we only consider some special cases of Eq. (1) which can be determined easily and possess physical significance in the following sections.

2.1. The first type of extending symmetry

If $f(X,T)=AX+BT$, $(A,B\neq0)$, the corresponding finite transformations x,y,z,t,ψ,ψ^*, and g are given by

$$\begin{aligned}
&x=\left(f_1(y,t)-f_1(Y,T)+\frac{X}{\sqrt{\alpha(T)}}\right)\sqrt{\alpha(t)},\\
&f_0(t)=f_0(T)+\varepsilon,\\
&y=\left(f_2(x,t)-f_2(X,T)+\frac{Y}{\sqrt{\alpha(T)}}\right)\sqrt{\alpha(t)},\\
&z=\left(f_3(t)-f_3(T)+\frac{Z}{\sqrt{\alpha(T)}}\right)\sqrt{\alpha(t)},\\
&\psi=\exp(f_4(x,y,z,t)-f_4(X,Y,Z,T))\Psi,\\
&\psi^*=\exp(f_4^*(x,y,z,t)-f_4^*(X,Y,Z,T))\Psi^*,\\
&g=\frac{1}{\alpha(t)}\big(\alpha(T)G+f_5(x,y,z,t)-f_5(X,Y,Z,T)\big),
\end{aligned}\tag{9}$$

where ε is the group parameter, and f_0,f_1,f_2,f_3,f_4,f_5 are defined by

$$\begin{aligned}
&f_0(t)=\int\frac{1}{\alpha(t)}\mathrm{d}t,\quad f_1(y,t)=\int\frac{\beta(t)-Ay}{\alpha(t)^{3/2}}\mathrm{d}t,\\
&f_2(x,t)=\int\frac{Ax+Bt}{\alpha(t)^{3/2}}\mathrm{d}t,\quad f_3(t)=\int\frac{\lambda(t)}{\alpha(t)^{3/2}}\mathrm{d}t,
\end{aligned}$$

$$f_4(x,y,z,t) = \frac{\mathrm{i}}{8}\int \frac{(x^2+y^2+z^2)\alpha''(t)+4By+4\beta'(t)x+4\lambda'(t)z+8\mu(t)}{\alpha(t)}\,\mathrm{d}t,$$
$$f_4^*(x,y,z,t) = -\frac{\mathrm{i}}{8}\int \frac{(x^2+y^2+z^2)\alpha''(t)+4By+4\beta'(t)x+4\lambda'(t)z+8\mu^*(t)}{\alpha(t)}\,\mathrm{d}t,$$
$$f_5(x,y,z,t) = \frac{1}{8}(x^2+y^2+z^2)\alpha''(t)-\frac{3}{4}\mathrm{i}\alpha'(t)+\frac{1}{2}\beta'(t)x+\frac{1}{2}\lambda'(t)z+\mu(t). \tag{10}$$

Under the condition $\alpha = T$, $\mu = \mathrm{i}T$, and $\lambda = \beta = 0$, system (9) can be expressed as

$$\begin{aligned}
t &= T, \quad z = T\exp(\varepsilon/2), \quad g = \mathrm{i}+(G-\mathrm{i})\exp(-\varepsilon),\\
x &= \frac{[-(4A^2+1)Y+2BT]\sin(A\varepsilon)+[(4A^2+1)X+4ABT]\cos(A\varepsilon)-4AB\exp(\varepsilon/2)T}{(4A^2+1)\exp(-\varepsilon/2)},\\
y &= \frac{[(4A^2+1)X+4ABT]\sin(A\varepsilon)+[(4A^2+1)Y-2BT]\cos(A\varepsilon)+2B\exp(\varepsilon/2)T}{(4A^2+1)\exp(-\varepsilon/2)},\\
\psi &= \exp\left[\frac{-B\exp(\varepsilon/2)(X+2AY)\sin(A\varepsilon)+B\exp(\varepsilon/2)(2AX-Y+2BT)\cos(A\varepsilon)}{4A^2+1}\right.\\
&\quad \left.+\frac{\exp(\varepsilon)(4A^2+1-\mathrm{i}B^2)\mathrm{i}T+2ABX+B^2T-BY-\mathrm{i}T-4\mathrm{i}A^2T}{4A^2+1}\mathrm{i}\right]\Psi,\\
\psi^* &= \exp\left[-\left(\frac{-B\exp(\varepsilon/2)(X+2AY)\sin(A\varepsilon)+B\exp(\varepsilon/2)(2AX-Y+2BT)\cos(A\varepsilon)}{4A^2+1}\right.\right.\\
&\quad \left.\left.+\frac{\exp(\varepsilon)(-4A^2-1-\mathrm{i}B^2)\mathrm{i}T+2ABX+B^2T-BY+\mathrm{i}T+4\mathrm{i}A^2T}{4A^2+1}\right)\mathrm{i}\right]\Psi^*.
\end{aligned} \tag{11}$$

Under the condition $B = 0$, $\alpha = T^2$, $\mu = \frac{3}{2}\mathrm{i}T$, $\lambda = T$, and $\beta = 0$, system (9) can be expressed as

$$\begin{aligned}
&t = \frac{T}{1-\varepsilon T}, \quad x = \frac{X\cos(A\varepsilon)-Y\sin(A\varepsilon)}{1-\varepsilon T},\\
&y = \frac{X\sin(A\varepsilon)+Y\cos(A\varepsilon)}{1-\varepsilon T}, \quad z = \frac{Z+\varepsilon T}{1-\varepsilon T},\\
&\psi = \frac{\Psi}{(1-\varepsilon T)^{3/2}}\exp\left[\frac{\mathrm{i}\varepsilon(X^2+Y^2+Z^2+2Z+\varepsilon T)}{4(1-\varepsilon T)}\right],\\
&\psi^* = \frac{\Psi^*}{(1-\varepsilon T)^{3/2}}\exp\left[-\frac{\mathrm{i}\varepsilon(X^2+Y^2+Z^2+2Z+\varepsilon T)}{4(1-\varepsilon T)}\right],\\
&g = G(1-\varepsilon T)^2.
\end{aligned} \tag{12}$$

2.2. The second type of extending symmetry

When $f(X,T) = BT$, we find that the finite transformations x, y, z, t, ψ, ψ^*, and g are given by

$$\begin{aligned}
&g = \big(h_2(t)-h_2(T)+G\alpha(T)\big)/\alpha(t),\\
&x = \left(\int\frac{\beta(t)}{\alpha(t)^{3/2}}\mathrm{d}t-\int\frac{\beta(T)}{\alpha(T)^{3/2}}\mathrm{d}T+\frac{X}{\sqrt{\alpha(T)}}\right)\sqrt{\alpha(t)},\\
&\int\frac{1}{\alpha(t)}\mathrm{d}t = \varepsilon+\int\frac{1}{\alpha(T)}\mathrm{d}T,\\
&y = \left(\int\frac{Bt}{\alpha(t)^{3/2}}\mathrm{d}t-\int\frac{BT}{\alpha(T)^{3/2}}\mathrm{d}T+\frac{Y}{\sqrt{\alpha(T)}}\right)\sqrt{\alpha(t)},\\
&\psi = \exp(h_1(t)-h_1(T)\Psi,\\
&z = \left(\int\frac{\lambda(t)}{\alpha(t)^{3/2}}\mathrm{d}t-\int\frac{\lambda(T)}{\alpha(T)^{3/2}}\mathrm{d}T+\frac{Z}{\sqrt{\alpha(T)}}\right)\sqrt{\alpha(t)},\\
&\psi^* = \exp(h_1^*(t)-h_1^*(T))\Psi^*,
\end{aligned} \tag{13}$$

where ε is the group parameter, and h_1, h_2, are defined as

$$\begin{aligned}
h_1(t) &= \frac{\mathrm{i}}{8}\int[(x^2+y^2+z^2)\alpha''(t)+4\beta'(t)x\\
&\quad +4By+4\lambda'(t)z+8\mu(t)]/\alpha(t)\,\mathrm{d}t,\\
h_1^*(t) &= -\frac{\mathrm{i}}{8}\int[(x^2+y^2+z^2)\alpha''(t)+4\beta'(t)x\\
&\quad +4By+4\lambda'(t)z+8\mu^*(t)]/\alpha(t)\,\mathrm{d}t,\\
h_2(t) &= \frac{1}{8}(x^2+y^2+z^2)\alpha''(t)-\frac{3}{4}\mathrm{i}\alpha'(t)\\
&\quad +\frac{1}{2}\beta'(t)x+\frac{1}{2}\lambda'(t)z+\mu(t),
\end{aligned} \tag{14}$$

with x, y, and z being determined by Eq. (13).

If $B = 0$, $\alpha = T^2$, $\beta = \lambda = 0$, and $\mu = \frac{3}{2}\mathrm{i}T$, system (13) can be expressed as

$$\begin{aligned}
&t = \frac{T}{1-\varepsilon T}, \quad x = \frac{X}{1-\varepsilon T}, \quad y = \frac{Y}{1-\varepsilon T},\\
&z = \frac{Z}{1-\varepsilon T}, \quad g = G(1-\varepsilon T)^2,\\
&\psi = (1-\varepsilon T)^{3/2}\exp\left[\frac{\varepsilon\mathrm{i}(X^2+Y^2+Z^2)}{4(1-\varepsilon T)}\right]\Psi,\\
&\psi^* = (1-\varepsilon T)^{3/2}\exp\left[\frac{-\varepsilon\mathrm{i}(X^2+Y^2+Z^2)}{4(1-\varepsilon T)}\right]\Psi^*.
\end{aligned} \tag{15}$$

If $B = 0$, $\alpha = 0$, $\mu(T) = T$, and $\beta = \lambda = 0$, the corresponding finite transformations x, y, z, t, ψ, ψ^*, and g are given by

$$\begin{aligned}
&t = T\exp(\varepsilon), \quad x = X\exp(\varepsilon/2),\\
&y = Y\exp(\varepsilon/2), \quad z = Z\exp(\varepsilon/2),
\end{aligned}$$

$$\psi = \Psi\exp(T(1-\exp(\varepsilon)), \quad \psi^* = \Psi^*\exp(T(1-\exp(\varepsilon)),$$
$$g = \mathrm{i} + (G-\mathrm{i})\exp(-\varepsilon). \tag{16}$$

If $B=0$, $\alpha = T$, $\mu(T) = -\mathrm{i}T\cos(T)/(\sin(T)+C)$, and $\beta=\lambda=0$, the corresponding finite transformations x,y,z,t,ψ,ψ^*, and g are given by

$$t = T\exp(\varepsilon), \quad x = X\exp(\varepsilon/2),$$
$$y = Y\exp(\varepsilon/2), \quad z = Z\exp(\varepsilon/2),$$
$$\psi = \frac{\sin(T\exp(\varepsilon))+C}{\sin(T)+C}\Psi, \quad \psi^* = \frac{\sin(T\exp(\varepsilon))+C}{\sin(T)+C}\Psi^*,$$
$$g = \exp(-\varepsilon)\left(G + \mathrm{i}\frac{\cos(T)(-\sin(T)+C)}{C^2+\cos^2(T)-1}\right)$$
$$-\mathrm{i}\left(\frac{\cos(T\exp(\varepsilon))(-\sin(T\exp(\varepsilon))+C)}{C^2+\cos^2(T\exp(\varepsilon))-1}\right). \tag{17}$$

If $B=0$, $\alpha = 1/T$, and $\mu(T)=\beta=\lambda=0$, the corresponding finite transformations x,y,z,t,ψ,ψ^*, and g are given by

$$t = \sqrt{T^2+2\varepsilon}, \quad x = \frac{X\sqrt{T}}{\sqrt[4]{T^2+2\varepsilon}},$$
$$y = \frac{Y\sqrt{T}}{\sqrt[4]{T^2+2\varepsilon}}, \quad z = \frac{Z\sqrt{T}}{\sqrt[4]{T^2+2\varepsilon}},$$
$$\psi = \Psi\exp\left(\frac{\mathrm{i}(X^2+Y^2+Z^2)\varepsilon}{4(T^2+2\varepsilon)T}\right),$$
$$\psi^* = \Psi^*\exp\left(\frac{-\mathrm{i}(X^2+Y^2+Z^2)\varepsilon}{4(T^2+2\varepsilon)T}\right),$$
$$g = \frac{G\sqrt{T^2+2\varepsilon}}{T} - \frac{3\varepsilon(\varepsilon+T^2)}{4T^3(T^2+2\varepsilon)^{3/2}}$$
$$\times (X^2+Y^2+Z^2) - \frac{3\varepsilon\mathrm{i}}{2T^2\sqrt{T^2+2\varepsilon}}. \tag{18}$$

3. Several variable-coefficient NLS equations and their solutions

Using the extending symmetry operator (2) and the related finite transformations (9) and (13), we can establish the relations among some different 3D variable-coefficient NLS equations. Especially, if the original equation is a constant-coefficient equation, some types of variable-coefficient equations may be solved via the solutions of the constant-coefficient NLS equation. Here, we only consider the above mentioned five concrete finite transformations to obtain some important equations in physics application.

Case 1 With the aid of the finite transformation (11), we can get the following variable-coefficient equation:

$$\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz} + \mathrm{i}\psi(1-\gamma^2)$$
$$+\gamma^2\psi G(X,Y,Z,T,|\Psi|) = 0, \tag{19}$$

where

$$X = \frac{(4ABt+4A^2x+x)\cos(A\varepsilon)+(4A^2y+y-2Bt)\sin(A\varepsilon)-4AB\gamma t}{4A^2+1},$$
$$Y = \frac{\gamma(4A^2y+y-2Bt)\cos(A\varepsilon)-\gamma(4ABt+4A^2x+x)\sin(A\varepsilon)+2B\gamma^2 t}{4A^2+1},$$
$$Z = \gamma z, \quad T = \gamma^2 t, \quad |\Psi| = |\psi|\exp(1-\gamma^2)t, \gamma = \exp(-\varepsilon/2). \tag{20}$$

The solution of Eq. (19) can be written as

$$\psi = \Psi(X,Y,Z,T)\exp\left[\frac{\mathrm{i}B\gamma\left[(2Ax-y+2Bt)\cos(A\varepsilon)+(x+2Ay)\sin(A\varepsilon)\right]}{4A^2+1}\right.$$
$$\left.+\frac{\left[(\gamma^2-1)(4A^2+1)-\mathrm{i}B^2(\gamma^2+1)\right]t-\mathrm{i}B(2Ax-y)}{4A^2+1}\right], \tag{21}$$

where $\Psi(X,Y,Z,T)$ is a solution of the original equation (1).

Case 2 According to the finite transformation (12), we may obtain the equation

$$\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz}$$
$$+\frac{1}{(1+\varepsilon t)^2}G(X,Y,Z,T,|\Psi|)\psi = 0, \tag{22}$$

where

$$X = \frac{y\sin(A\varepsilon)+x\cos(A\varepsilon)}{1+\varepsilon t}, \quad Y = \frac{y\cos(A\varepsilon)-x\sin(A\varepsilon)}{1+\varepsilon t},$$
$$Z = \frac{z-\varepsilon t}{1+\varepsilon t}, \quad T = \frac{t}{1+\varepsilon t}, \quad |\Psi| = \frac{|\psi|}{(1+\varepsilon t)^{3/2}}. \tag{23}$$

The solution of Eq. (22) can be written as

$$\psi = \Psi(X,Y,Z,T)$$
$$\times\exp\left(\frac{-\mathrm{i}\varepsilon(-x^2-y^2-z^2-2z+\varepsilon t)}{1+\varepsilon t}\right)(1+\varepsilon t)^{3/2}, \tag{24}$$

where $\Psi(X,Y,Z,T)$ is also a solution of Eq. (1).

Case 3 Now we will give some significant equations related to the optical fiber from the finite transformations (15)–(18) and give some new exact solutions in the following discussion. Here we only consider the original equation (1) as the

3D standard constant-coefficient NLS equation

$$\mathrm{i}\Psi_T + \Psi_{XX} + \Psi_{YY} + \Psi_{ZZ} + |\Psi|^2\Psi = 0. \tag{25}$$

(I) From the finite transformation (15), we can obtain the variable-coefficient equation

$$\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz} + (1+\varepsilon t)|\psi|^2\psi = 0, \tag{26}$$

with the solution

$$\psi = \frac{1}{(1+\varepsilon t)^{3/2}}\Psi(X,Y,Z,T)\exp\left(\frac{\mathrm{i}\varepsilon(x^2+y^2+z^2)}{4(1+\varepsilon t)}\right), \tag{27}$$
$$X = \frac{x}{1+\varepsilon x},\quad Y = \frac{y}{1+\varepsilon y},\quad Z = \frac{z}{1+\varepsilon z},\quad T = \frac{t}{1+\varepsilon t},$$

where $\Psi(X,Y,Z,T)$ is a solution of Eq. (25).

With the help of the transformation $t = -(1+\exp(-\varepsilon t_1))/\varepsilon$ in Eq. (26), then equation (26) becomes the defocusing NLS equation of dispersion slow-changing optical fiber

$$\mathrm{i}\psi_{t_1} + \exp(-\varepsilon t_1)(\psi_{xx} + \psi_{yy} + \psi_{zz}) - |\psi|^2\psi = 0. \tag{28}$$

The solution of Eq. (28) can be easily obtained by the solution of Eq. (26), namely, Eq. (27).

(II) From the finite transformation (16), we can obtain the variable-coefficient equation which describes the soliton propagation in an optical fibre with exponentially changing nonlinearity and constant gain or loss,

$$\begin{aligned}&\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz} - \mathrm{i}\gamma_1\psi\\ &+(1+\gamma_1)\exp(-2\gamma_1 t)|\psi|^2\psi = 0,\end{aligned} \tag{29}$$

where we have taken $\gamma_1 = \exp(-\varepsilon) - 1$. The solution of Eq. (29) reads

$$\begin{aligned}\psi = \Psi\big(&X = x\sqrt{\gamma_1+1},\ Y = y\sqrt{\gamma_1+1},\\ &Z = z\sqrt{\gamma_1+1},\ T = (\gamma_1+1)t\big)\exp(\gamma_1 t),\end{aligned} \tag{30}$$

where $\Psi(X,Y,Z,T)$ is a solution of Eq. (25).

(III) From the finite transformation (17), we can obtain the variable-coefficient equation whose dispersion, nonlinearity, gain or absorption become periodical functions of distance

$$\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz} + \theta(t)|\psi|^2\psi + \mathrm{i}\Theta(t)\psi = 0, \tag{31}$$

where θ and Θ are periodic functions of t

$$\begin{aligned}\theta(t) &= \frac{\gamma_2(\sin(\gamma_2 t)+C)^2}{(\sin(t)+C)^2},\\ \Theta(t) &= \frac{\gamma_2\cos(\gamma_2 t)\big(C-\sin(\gamma_2 t)\big)}{\cos^2(\gamma_2 t)+C^2-1} - \frac{\cos(t)\big(C-\sin(t)\big)}{\cos^2(t)+C^2-1},\end{aligned} \tag{32}$$

and $\gamma_2 = \exp(-\varepsilon)$.

The solution of Eq. (32) can be expressed as

$$\begin{aligned}\psi = \Psi\Bigg(&X = \sqrt{\gamma_2}x,\quad Y = \sqrt{\gamma_2}y,\\ &Z = \sqrt{\gamma_2}z,\quad T = \gamma_2 t\frac{\sin(t)+C}{\sin(\gamma_2 t)+C}\Bigg),\end{aligned} \tag{33}$$

where $\Psi(X,Y,Z,T)$ is a solution of Eq. (25).

From the above discussion, we can see that if we know any solution to constant-coefficient NLS equation (25), we can obtain the corresponding exact solutions of variable-coefficient NLS equations (26), (29), and (31).

(IV) From the finite transformation (18), we can obtain the following equation:

$$\begin{aligned}&\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + \psi_{zz} - \frac{3\varepsilon(t^2-\varepsilon)}{4t^2(t^2-2\varepsilon)}(x^2+y^2+z^2)\psi\\ &+\frac{t}{\sqrt{t^2-2\varepsilon}}|\psi|^2\psi - \frac{3\varepsilon\mathrm{i}}{2t(t^2-2\varepsilon)}\psi = 0.\end{aligned} \tag{34}$$

The solution of Eq. (34) can be expressed as

$$\begin{aligned}\psi = \Psi\Bigg(&X = \frac{x\sqrt{t}}{\sqrt[4]{t^2-2\varepsilon}}, Y = \frac{y\sqrt{t}}{\sqrt[4]{t^2-2\varepsilon}},\\ &Z = \frac{z\sqrt{t}}{\sqrt[4]{t^2-2\varepsilon}}, T = \sqrt{t^2-2\varepsilon}\Bigg)\\ &\times\exp\left(\frac{\mathrm{i}(x^2+y^2+z^2)\varepsilon}{4(t^2-2\varepsilon)t}\right),\end{aligned} \tag{35}$$

where $\Psi(X,Y,Z,T)$ is a solution of Eq. (25). For Eq. (34), by a scale conversion ($x_1 = \frac{\sqrt{2}}{2}x, y_1 = \frac{\sqrt{2}}{2}y, z_1 = \frac{\sqrt{2}}{2}z, t_1 = t$), we can obtain the variable-coefficient Gross–Pitaevskii equation

$$\begin{aligned}&\mathrm{i}\psi_{1t_1} + \frac{1}{2}(\psi_{1x_1x_1} + \psi_{1y_1y_1} + \psi_{1z_1z_1})\\ &-\frac{3\varepsilon(t_1^2-\varepsilon)}{2t_1^2(t_1^2-2\varepsilon)}(x_1^2+y_1^2+z_1^2)\psi_1\\ &+\frac{t_1}{\sqrt{t_1^2-2\varepsilon}}|\psi_1|^2\psi_1 - \frac{3\varepsilon\mathrm{i}}{2t_1(t_1^2-2\varepsilon)}\psi_1 = 0.\end{aligned} \tag{36}$$

4. Discussion of the exact solutions of the NLS equations with dispersion, nonlinearity, and gain or loss

In this section, we discuss the exact solutions of the above variable-coefficient NLS equations (26), (29), and (31) via Eq. (25). For example, the single soliton solution of Eq. (25) is given by

$$\begin{aligned}\Psi = &\Psi_0\mathrm{sech}(p_1X + p_2Y + p_3Z + p_4T + p_5)\\ &\times\exp((k_1X + k_2Y + k_3Z + k_4T + k_5)\mathrm{i}),\end{aligned} \tag{37}$$

where

$$\Psi_0 = \sqrt{2(p_1^2+p_2^2+p_3^2)},\quad p_4 = -2(k_1p_1 + k_2p_2 + k_3p_3),$$
$$k_4 = \frac{\Psi_0^2}{2} - k_1^2 - k_2^2 - k_3^2,$$

and $k_1,k_2,k_3,k_5,p_1,p_2,p_3,p_5$ are arbitrary constants. Substituting Eq. (37) into Eq. (27), we can obtain a solution of Eq. (26)

$$\psi = \frac{\Psi_0}{(1+\varepsilon t)^{3/2}}\operatorname{sech}\left(\frac{p_1x+p_2y+p_3z+p_4t}{1+\varepsilon t}+p_5\right) \times \exp\left[\frac{\mathrm{i}}{4}\left(\frac{\varepsilon(x^2+y^2+z^2)+4(k_1x+k_2y+k_3z+k_4t)}{1+\varepsilon t} +4k_5\right)\right]. \tag{38}$$

Applying this to Eq. (30), we can obtain an exact solution of Eq. (29)

$$\psi = \Psi_0\exp(\gamma_1 t)\operatorname{sech}(p_1\sqrt{\gamma_1+1}x+p_2\sqrt{\gamma_1+1}y +p_3\sqrt{\gamma_1+1}z+p_4t(\gamma_1+1)+p_5) \times \exp\left[\mathrm{i}\left(k_1\sqrt{\gamma_1+1}x+k_2\sqrt{\gamma_1+1}y +k_3\sqrt{\gamma_1+1}z+k_4t(\gamma_1+1)+k_5\right)\right]. \tag{39}$$

From Eq. (39), we find that the amplitude of the soliton solution increases exponentially but the width of the pulse remains unchanged. According to Eq. (33), we can write down the exact solution of Eq. (31)

$$\psi = \Psi_0\frac{\sin(t)+C}{\sin(t\gamma_2)+C} \times \operatorname{sech}(p_1\sqrt{\gamma_2}x+p_2\sqrt{\gamma_2}y+p_3\sqrt{\gamma_2}z+p_4\gamma_2t+p_5) \times \exp(k_1\sqrt{\gamma_2}x+k_2\sqrt{\gamma_2}y+k_3\sqrt{\gamma_2}z+k_4\gamma_2t+k_5). \tag{40}$$

In addition, a simple solution of Eq. (25) can be obtained by using the methods in Ref. [10] and it can be expressed as

$$\psi = \left(1+\frac{2(1-\delta^2)\cos\left(\frac{2t}{\delta^2}\sqrt{1-\delta^2}\right)+2\mathrm{i}\sqrt{1-\delta^2}\sin\left(\frac{2t}{\delta^2}\sqrt{1-\delta^2}\right)}{\delta\cosh\left(-c_2x-c_1y+C_1z+2ct(c_1+c_2-C_1)\right)+\delta^2\cos\left(\frac{2t}{\delta^2}\sqrt{1-\delta^2}\right)}\right)\exp(c\mathrm{i}(x+y+z+t-3ct)), \tag{41}$$

where $C_1=\sqrt{\frac{2}{\delta^2}-c_1^2-c_2^2-2}$. Based on Eqs. (27), (30), and (33), some solutions of Eqs. (26), (29) and (31) can be given. For simplicity, we only give the solution of Eq. (31)

$$\psi = \left(1+\frac{2(1-\delta^2)\cos\left(\frac{2\gamma_2t}{\delta^2}\sqrt{1-\delta^2}\right)+2\mathrm{i}\sqrt{1-\delta^2}\sin\left(\frac{2\gamma_2t}{\delta^2}\sqrt{1-\delta^2}\right)}{\delta\cosh\Omega+\delta^2\cos\left(\frac{2\gamma_2t}{\delta^2}\sqrt{1-\delta^2}\right)}\right) \times\exp(c\mathrm{i}(\sqrt{\gamma_2}x+\sqrt{\gamma_2}y+\sqrt{\gamma_2}z+\gamma_2t-3c\gamma_2t))\times\frac{\sin(t\gamma_2)+C}{\sin(t\gamma_2)+C}, \tag{42}$$

where $\Omega=-c_2\sqrt{\gamma_2}x-c_1\sqrt{\gamma_2}y+C_1\sqrt{\gamma_2}z+2c\gamma_2t(c_1+c_2-C_1)$.

In order to understand the significance of solutions (37)–(42), the main features of them are investigated by using direct computer simulations. For simplicity, we only consider some examples of these solutions under some special parameters.

Figures 1(a) and 1(b) show the evolutions of solutions (37) and (40) under different parameters. Figures (2a) and 2(b) show the evolutions of solutions (41) and (42) with varying nonlinear coefficients, from which we can see that the amplitudes decrease with the evolution in time. According to these features, we can adjust various parameters included in the solutions to study optical fibers with different losses, gains, and dispersions or nonlinearity variations.

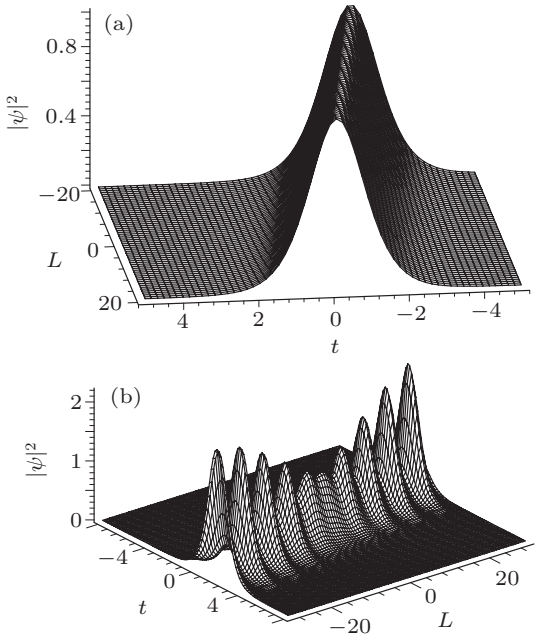


Fig. 1. Density $|\psi|^2$ given by solutions (37) and (40) as a function of propagation distance $L\equiv p_1x+p_2y+p_3z$ and t. In panel (a), $\Psi_0=1$, $p_4=-0.04$, $p_5=0.9$; in panel (b), $\Psi_0=1$, $\gamma_2=0.9$, $p_4=0.05$, $p_5=-0.4$, $C=5$.

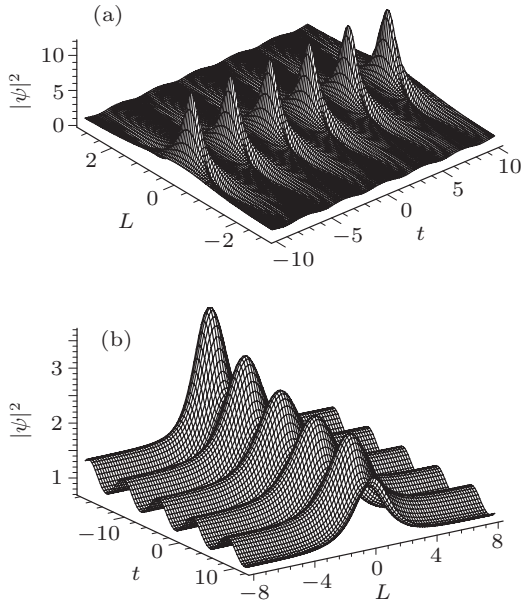


Fig. 2. Density $|\psi|^2$ given by solutions (41) and (42) as a function of propagation distance $L \equiv -c_2x - c_1y + \sqrt{\frac{2}{\delta^2} - c_1^2 - c_2^2 - 2z}$ and t. In panel (a), $\delta = 0.81$, $c_1 = -1$, $c_2 = 0.001$, $c = 0.000005$; in panel (b), $\delta = 0.81$, $\gamma_2 = 0.04$, $c_1 = 1$, $c_2 = -0.01$, $c = -0.01$, $C = 10$.

5. Discussion and summary

In summary, with the help of the extending symmetry, we investigate (3+1)-dimensional variable-coefficient generalized nonlinear Schrödinger equations. We have identified two types of extending symmetries and obtained five kinds of finite transformations. Using the finite transformations, we have found relations among the solutions of different variable-coefficient equations. Because five arbitrary functions of T are included in the extending symmetry groups for the 3D variable-coefficient NLS equation, we can find another variable-coefficient equation by selecting suitable arbitrary functions. If we know one arbitrary solution of all these resulting equations, the corresponding solutions of the other equations can be obtained easily. In particular, when the original model is a constant-coefficient equation, the arbitrary solutions of the constant-coefficient equation can be used to express the solutions of the corresponding variable-coefficient equations.

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