Dirac equation for the Hulthén potential within the Yukawa-type tensor interaction

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Using the Nikiforov–Uvarov (NU) method, pseudospin and spin symmetric solutions of the Dirac equation for the scalar and vector Hulthén potentials with the Yukawa-type tensor potential are obtained for an arbitrary spin–orbit coupling quantum number κ. We deduce the energy eigenvalue equations and corresponding upper- and lower-spinor wave functions in both the pseudospin and spin symmetry cases. Numerical results of the energy eigenvalue equations and the upper- and lower-spinor wave functions are presented to show the effects of the external potential and particle mass parameters as well as pseudospin and spin symmetric constants on the bound-state energies and wave functions in the absence and presence of the tensor interaction.

Keywords: Dirac equation, Hulthén potential and Yukawa potential, pseudospin and spin symmetry, tensor interaction

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1. Introduction

About 43 years ago, the concept of pseudospin was introduced in Refs. [1] and [2], independently, into nuclear physics in order to elucidate the experimental observation of the quasi-degeneracy in single-nucleon doublets between normal parity orbitals $(n,l,j=l+1/2)$ and $(n-1,l+2,j=l+3/2)$, where n,l, and j represent the radial, orbital, and total angular momentum quantum numbers, respectively. Since then, this concept has been used to form an effective nuclear shell model,[1–3] and to explain different phenomena in the nuclear structure such as deformation, super-deformation, identical bands, and magnetic moment.[4–10] The relation between the pseudospin and the Dirac equation was established by Ginocchio, who recognized that pseudo-orbital angular momentum $\tilde{l}=l+1$ is the usual orbital angular momentum l of the lower component of the Dirac spinor, and showed that in the presence of an external attractive scalar $S(\boldsymbol{r})$ potential and a repulsive vector $V(\boldsymbol{r})$ potential, pseudospin symmetry occurs in the Dirac equation when $S(\boldsymbol{r})=-V(\boldsymbol{r})$.[11–14] On the other hand, spin symmetry which is relevant to mesons[15] appears in the Dirac equation when substraction of the scalar potential $S(\boldsymbol{r})$ from vector potential $V(\boldsymbol{r})$ is equal to zero.

The Dirac equation has been solved exactly for only a few well-known potentials because of the centrifugal term. Thus, it is generally necessary to use approximations and symmetries in order to obtain approximate analytical solution of the Dirac equation with an external potential. Recently, pseudospin symmetric and spin symmetric solutions of the Dirac equation for vector $V(\boldsymbol{r})$ and scalar $S(\boldsymbol{r})$ potentials have been studied intensively.[16–37] Moreover, solutions of the Dirac equation with a quadratic vector and a scalar potential within the pseudospin and spin symmetry formalism have been obtained in the presence of the tensor interaction in Ref. [38]. The authors investigated the effect of the tensor potential on the pseudospin (and spin) doublets whose energies lie very close, and showed that tensor interactions remove all degeneracies between pseudospin doublets (spin doublets as well). After this pioneering work, the tensor effects on the pseudospin and spin doublets have also been studied in Refs. [39]–[50].

The Hulthén potential is widely used to describe the interactions between the nucleon and heavy nucleus, and is given in the following form:[51]

$$V_H=-\frac{V_0\,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}},\quad \delta=2\alpha, \tag{1}$$

where V_0 and δ are two positive real parameters, and represent the strength and screening range of the potential, respectively. The Hulthén potential acts like a Coulomb potential when r goes to zero. However, the number of the bound-state eigenvalues of this potential is less than the Coulomb potential. On the other hand, one of the other significant short-range potentials is the Yukawa potential which was proposed in Refs. [52] and [53] in order to build the meson theory describing the nucleon–nucleon interactions. The form of the Yukawa potential is given by[52,53]

$$V_{\mathrm{Y}}=-V_1\frac{\mathrm{e}^{-\alpha r}}{r}, \tag{2}$$

where V_1 and α refer to different expressions with respect to the physical system under consideration. For instance, they are called the strength and range of the nucleon force in the meson theory,[54] respectively.

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In the present study, we obtain the approximate analytical solutions of the Dirac equation for the scalar and vector Hulthén potentials together with the Yukawa-type tensor potential within the pseudospin and spin symmetry limit. The effects of the Yukawa-type tensor interaction on the pseudospin (and spin) doublets and on the corresponding wave functions are investigated. We also study how potential parameters and the mass parameter change the bound-state energy levels.

The paper is arranged as follows: in Section 2, we obtain the second-order Schrödinger-type differential equation for the Dirac equation with the scalar and vector Hulthén potentials and the Yukawa-type tensor potential in the presence of the pseudospin and spin symmetry limit; in Section 3, pseudospin and spin symmetric solutions of the second-order differential equations are presented; we discuss the effects of the potential and mass parameters on the bound states in Section 4; a summary is given in the last Section.

2. Dirac equation within tensor interaction

In spherical coordinates, the Dirac equation which describes a nucleon with mass M moving in a repulsive vector potential $V(r)$, an attractive scalar potential $S(r)$, and a tensor potential $U(r)$[38] is given by

$$\begin{aligned}&[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta\,(M+S(r))-\mathrm{i}\beta\boldsymbol{\alpha}\cdot\hat{r}U(r)]\,\Psi(r)\\ &=\,[E-V(r)]\Psi(r),\end{aligned}\tag{3}$$

where E is the relativistic energy of the system, $\boldsymbol{p}$ denotes the three-dimensional momentum operator, $-\mathrm{i}\boldsymbol{\nabla}$, and M is the mass of the fermionic particle. Here, $\boldsymbol{\alpha}$ and β are the 4×4 usual Dirac matrices which are expressed in terms of the three 2×2 Pauli matrices and the 2×2 unit matrix. The Dirac Hamiltonian for a particle in a spherical field commutes with the total angular momentum operator $\boldsymbol{J}$ and the spin–orbit coupling operator $\boldsymbol{K}=-\beta(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, with $\boldsymbol{L}$ being the orbital angular momentum. The eigenvalues of the spin–orbit coupling operator are $\kappa=(j+1/2)>0$ and $\kappa=-(j+1/2)<0$ for the unaligned spin ($\mathrm{p}_{1/2}$, $\mathrm{d}_{3/2}$,...) and the aligned spin ($s_{1/2}, p_{3/2},\ldots$), respectively. The set $(\boldsymbol{H},\boldsymbol{K},\boldsymbol{J}^2,\boldsymbol{J}_z)$ is taken as the complete one of the conservative quantities. Thus, the spinors can be written as

$$\Psi_{n\kappa}(\boldsymbol{r})=\begin{pmatrix}f_{n\kappa}(\boldsymbol{r})\\ g_{n\kappa}(\boldsymbol{r})\end{pmatrix}=\begin{pmatrix}\dfrac{F_{n\kappa}(r)}{r}\mathrm{Y}^{l}_{jm}(\theta,\phi)\\ \mathrm{i}\dfrac{G_{n\kappa}(r)}{r}\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)\end{pmatrix},\tag{4}$$

where $f_{n\kappa}(\boldsymbol{r})$ and $g_{n\kappa}(\boldsymbol{r})$ represent the upper (large) and the lower (small) components of the Dirac spinors, respectively. The $\mathrm{Y}^{l}_{jm}(\theta,\phi)$ and $\mathrm{Y}^{\tilde{l}}_{jm}(\theta,\phi)$ stand for the spin and pseudospin spherical harmonics, respectively, and m is the projection of the angular momentum on the z axis.

Substituting Eq. (4) into Eq. (3), one obtains the following two uncoupled second-order differential equations for the upper and lower radial wave functions (refer to Refs. [38] and [42] for calculation details):

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-U(r)^2-\frac{\mathrm{d}}{\mathrm{d}r}U(r)\right.\\ &\left.+\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\right)\times(M+E_{n\kappa}-\Delta(r))^{-1}\right]\\ &\times F_{n\kappa}(r)+(E_{n\kappa}+M-\Delta(r))\\ &\times(E_{n\kappa}-M-\Sigma(r))\,F_{n\kappa}(r)=0,\end{aligned}\tag{5}$$

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)-U(r)^2+\frac{\mathrm{d}}{\mathrm{d}r}U(r)\right.\\ &\left.+\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\right)\times(M-E_{n\kappa}+\Sigma(r))^{-1}\right]\\ &\times G_{n\kappa}(r)+(E_{n\kappa}+M-\Delta(r))\\ &\times(E_{n\kappa}-M-\Sigma(r))\,G_{n\kappa}(r)=0,\end{aligned}\tag{6}$$

where the spin–orbit quantum number κ is related to the usual orbital angular momentum and pseudo-orbital angular momentum quantum numbers as follows: $\kappa(\kappa+1)=l(l+1)$, and $\kappa(\kappa-1)=\tilde{l}(\tilde{l}+1)$. Here, $\Delta(r)=V(r)-S(r)$, and $\Sigma(r)=V(r)+S(r)$.

2.1. Pseudospin symmetry in the Dirac equation

The pseudospin symmetry occurs in the Dirac equation as $\mathrm{d}\Sigma(r)/\mathrm{d}r=0$ or equivalently $\Sigma(r)=C_{ps}=$ constant.[11] In order to obtain the approximate analytical solution of the Dirac equation under the pseudospin symmetry limit, we consider the following scalar, vector, and tensor potentials:

$$\begin{aligned}&\sum(r)=C_{ps}=\text{constant},\\ &\Delta(r)=-V_0\left(\frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right),\quad U(r)=-V_1\frac{\mathrm{e}^{-\alpha r}}{r}.\end{aligned}\tag{7}$$

Inserting the above potentials into Eq. (6) yields

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}-\left(\frac{2\kappa}{r}+V_1\frac{\mathrm{e}^{-\alpha r}}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)V_1\frac{\mathrm{e}^{-\alpha r}}{r}\right]G^{ps}_{n\kappa}(r)\\ &+\left(E^{ps}_{n\kappa}+M+\frac{V_0\,\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right)(E_{n\kappa}-M-C_{ps})\,G^{ps}_{n\kappa}(r)=0.\end{aligned}\tag{8}$$

Equation (8) cannot be solved exactly due to the centrifugal term $1/r^2$. To get an approximate solution, we use the following approximations:

$$1/r^2\approx\frac{4\alpha^2\,\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2},\tag{9}$$

$$1/r^2\approx\frac{4\alpha^2\,\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}.\tag{10}$$

One can see from Fig. 1 that the above approximations behave similar to each other for the small values of the α parameter. By using approximation (9) for the terms $(\kappa(\kappa-1)r^{-2}$, $V_1^2r^{-2}\,\mathrm{e}^{-2\alpha r})$, and $\left(\alpha V_1r^{-1}\,\mathrm{e}^{-\alpha r}\right)$, and approximation (10) for

the terms $\left(2\kappa V_1 r^{-2}\mathrm{e}^{-\alpha r}\right)$, and $\left(V_1 r^{-2}\mathrm{e}^{-\alpha r}\right)$, and applying the transformation $z=\mathrm{e}^{-2\alpha r}$, Eq. (8) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}G^{\mathrm{ps}}_{n\kappa}(z)+\frac{(1-z)}{z(1-z)}\frac{\mathrm{d}}{\mathrm{d}z}G^{\mathrm{ps}}_{n\kappa}(z)+\frac{1}{z^2(1-z)^2}\left(K_{\mathrm{ps}}z^2+L_{\mathrm{ps}}z-f_{\mathrm{ps}}\right)G^{\mathrm{ps}}_{n\kappa}(z)=0, \quad (11)$$

where

$$K_{\mathrm{ps}} = -V_1\left(V_1+\frac{1}{2}\right)+\frac{1}{4\alpha^2}(M-E^{\mathrm{ps}}_{n\kappa}+C_{\mathrm{ps}})(V_0-E^{\mathrm{ps}}_{n\kappa}-M), \quad (12)$$

$$L_{\mathrm{ps}} = -\kappa(\kappa-1)-\left(2\kappa-\frac{3}{2}\right)V_1+\frac{1}{4\alpha^2}(M-E^{\mathrm{ps}}_{n\kappa}+C_{\mathrm{ps}})(2M+2E^{\mathrm{ps}}_{n\kappa}-V_0), \quad (13)$$

$$f_{\mathrm{ps}} = \frac{1}{4\alpha^2}(M-E^{\mathrm{ps}}_{n\kappa}+C_{\mathrm{ps}})(E^{\mathrm{ps}}_{n\kappa}+M). \quad (14)$$

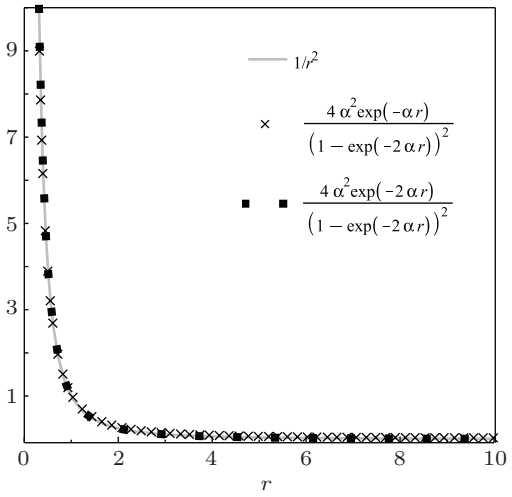


Fig. 1. Curves for $1/r^2$ and its approximations for $\alpha=0.05$ fm.

2.2. Spin symmetry in the Dirac equation

In the spin symmetry case, we use the following scalar, vector, and tensor potentials:

$$\Delta(r)=C_s=\text{constant}, \sum(r)=-V_0\left(\frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right), \quad U(r)=-V_1\frac{\mathrm{e}^{-\alpha r}}{r}, \quad (15)$$

which lead to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}-\left(\frac{2\kappa}{r}+V_1\frac{\mathrm{e}^{-\alpha r}}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)V_1\frac{\mathrm{e}^{-\alpha r}}{r}\right]F^s_{n\kappa}(r)+(E^s_{n\kappa}+M-C_s)\left(E^s_{n\kappa}-M+V_0\left(\frac{\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right)\right)F^s_{n\kappa}(r)=0. \quad (16)$$

By using approximation (9) for the terms $(\kappa(\kappa+1)r^{-2}$, $V_1^2 r^{-2}\mathrm{e}^{-2\alpha r})$, and $\left(\alpha V_1 r^{-1}\mathrm{e}^{-\alpha r}\right)$, and approximation (10) for the terms $\left(2\kappa V_1 r^{-2}\mathrm{e}^{-\alpha r}\right)$ and $\left(V_1 r^{-2}\mathrm{e}^{-\alpha r}\right)$, the above second-order differential equation turns into the following form in view of $y=\mathrm{e}^{-2\alpha r}$ transformation:

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}F^{\mathrm{s}}_{n\kappa}(y)+\frac{(1-y)}{y(1-y)}\frac{\mathrm{d}}{\mathrm{d}y}F^{\mathrm{s}}_{n\kappa}(y)+\frac{1}{y^2(1-y)^2}\left(K_s y^2+L_s y-f_s\right)F^{\mathrm{s}}_{n\kappa}(y)=0, \quad (17)$$

where

$$K_{\mathrm{s}} = -V_1\left(V_1-\frac{1}{2}\right)+\frac{1}{4\alpha^2}(M+E^{\mathrm{s}}_{n\kappa}-C_{\mathrm{s}})(E^{\mathrm{s}}_{n\kappa}-M-V_0), \quad (18)$$

$$L_{\mathrm{s}} = -\kappa(\kappa+1)-\left(2\kappa+\frac{3}{2}\right)V_1+\frac{1}{4\alpha^2}(M+E^{\mathrm{s}}_{n\kappa}-C_{\mathrm{s}})(2M-2E^{\mathrm{s}}_{n\kappa}+V_0), \quad (19)$$

$$f_{\mathrm{s}} = \frac{1}{4\alpha^2}(M+E^{\mathrm{s}}_{n\kappa}-C_{\mathrm{s}})(M-E^{\mathrm{s}}_{n\kappa}). \quad (20)$$

3. Pseudospin and spin symmetric solutions

In this section, we investigate the solutions of Eqs. (11) and (17) by using the parametric generalization of the NU method.[55] In the Appendix section, we give only necessary ingredients of the method.

3.1. Pseudospin symmetric solution

Comparing Eq. (11) with Eq. (A1), we obtain

$$\gamma_1=1,\gamma_2=1,\gamma_3=1, \quad \eta_1=-K_{\mathrm{ps}},\eta_2=L_{\mathrm{ps}},\eta_3=f_{\mathrm{ps}}, \quad (21)$$

and then the other parameters are obtained as follows:

$$\gamma_4=0,\quad \gamma_5=-\frac{1}{2},\quad \gamma_6=\frac{1}{4}+\eta_1,\quad \gamma_7=-\eta_2,$$
$$\gamma_8=\eta_3,\quad \gamma_9=\eta_1-\eta_2+\eta_3+\frac{1}{4},$$
$$\gamma_{10}=1+2\sqrt{\eta_3},$$
$$\gamma_{11}=2+2\left(\sqrt{\eta_1-\eta_2+\eta_3+\frac{1}{4}}+\sqrt{\eta_3}\right),$$
$$\gamma_{12}=\sqrt{\eta_3},$$
$$\gamma_{13}=-\frac{1}{2}-\left(\sqrt{\eta_1-\eta_2+\eta_3+\frac{1}{4}}+\sqrt{\eta_3}\right). \quad (22)$$

Substituting Eqs. (21) and (22) into Eq. (A5) yields

$$\mu+\nu\sqrt{f_{\mathrm{ps}}}+\frac{V_0}{4\alpha^2}(M-E^{\mathrm{ps}}_{n\kappa}+C_{\mathrm{ps}})=0, \quad (23)$$

where

$$\mu = n^2+n+\frac{1}{2}+(2n+1)\sqrt{(\kappa+V_1)(\kappa+V_1-1)+\frac{1}{4}}+\kappa(\kappa-1)+\left(2\kappa-\frac{3}{2}\right)V_1, \quad (24)$$

$$\nu = 2n+1+2\sqrt{(\kappa+V_1)(\kappa+V_1-1)+\frac{1}{4}}. \quad (25)$$

We can obtain the following two roots for the $E_{n\kappa}^{\rm ps}$ with the help of the Mathematica from Eq. (23):

$$E_{n\kappa}^{\rm ps\pm} = \frac{1}{V_0^2+4\nu^2\alpha^2}[(C_{\rm ps}+M)V_0^2+2\alpha^2(\nu^2 C_{\rm ps}+2\mu V_0) \pm 2(\nu^2\alpha^4[\nu^2(C_{\rm ps}+2M)^2 - 4\mu(C_{\rm ps}V_0+2MV_0+4\mu\alpha^2)])^{1/2}]. \quad (26)$$

The energy eigenvalue equation (26) gives two different energy spectra for each set of the quantum numbers (n,κ). In Table 1, we present some numerical results based on the energy eigenvalue equation (26) for various values of n and κ. For every pair of orbitals, we give two energy eigenvalues obtained by using $E_{n\kappa}^{\rm ps+}$ (first line) and $E_{n\kappa}^{\rm ps-}$ (second line). One can observe from Table 1 that pseudospin doublets whose energies lie very close are formed by a pair of orbitals, such as $(ns_{1/2},(n-1)d_{3/2})$, $(np_{3/2},(n-1)f_{5/2})$, and $(nd_{5/2},(n-1)g_{7/2})$. Besides, there are extra degeneracies among the bound states, i.e., $(1d_{5/2},0g_{7/2},2p_{3/2},1f_{5/2},3s_{1/2},2d_{3/2})$, $(1p_{3/2},0f_{5/2},2s_{1/2},1d_{3/2})$, etc. It is readily seen in Table 1 that tensor interaction $(V_1\neq 0)$ removes all degeneracies between the two states in the pseudospin doublet. The reason can be found by considering the energy eigenvalue equation (26). In this equation, μ and ν offer different contributions to each energy level in the pseudospin doublet. However, there are still extra degeneracies among the energy levels, such as ($1d_{5/2}$, $2p_{3/2}$, and $3s_{1/2}$), for $\kappa<0$, and ($0g_{7/2}$, $1f_{5/2}$, and $2d_{3/2}$) for $\kappa>0$ because μ and ν offer the same contributions to the degenerate energy levels which have the same $(n+|\kappa|)$ for $\kappa<0$ and $(n-1+|\kappa|)$ for $\kappa>0$. On the other hand, we investigate the minimum value of the parameter $C_{\rm ps}$ which gives the negative-energy bound state in the pseudospin symmetry limit. If we take the following set of parameters $V_0=4$ fm^{-1} (1 fm $=10^{-15}$ m), $M=5$ fm^{-1}, $\alpha=0.05$ fm for $n=1$ and $\kappa=-1$, the minimum value of the $C_{\rm ps}$ is found to be -4.9411 fm^{-1} ($E_{1,-1}^{\rm ps-}=-0.00000412655$ fm^{-1}) in the absence of the tensor interaction and -4.95731 fm^{-1} ($E_{1,-1}^{\rm ps-}=-0.000000218751$ fm^{-1}) in the presence of the tensor interaction. Furthermore, we obtain numerical solutions of the Dirac equation without any approximation under the condition of the pseudospin symmetry by using the finite difference method[57–59] in order to check the exactness of our energy eigenvalue equation (26). The results are also given in Table 1.

Table 1. Negative-energy bound states (in fm^{-1}) for the pseudospin symmetry with $V_0=4$ fm^{-1}, $M=5$ fm^{-1}, $\alpha=0.05$ fm, and $C_{\rm ps}=-5$ fm^{-1}.

$\tilde{l}$	n,κ	State	$E_{n\kappa}^{\rm ps}$		$E_{n\kappa}^{\rm ps}$ a)		$n-1,\kappa$	States	$E_{n\kappa}^{\rm ps}$		$E_{n\kappa}^{\rm ps}$ a)	
			$V_1=0$	$V_1=0.6$	$V_1=0$	$V_1=0.6$			$V_1=0$	$V_1=0.6$	$V_1=0$	$V_1=0.6$
1	$1,-1$	$1s_{1/2}$	-0.00862755	-0.00381316	-0.0084782	-0.0037895	$0,2$	$0d_{3/2}$	-0.00862755	-0.01055249	-0.0084782	-0.010083
			-0.05738712	-0.04202674					-0.05738712	-0.08679350		
2	$1,-2$	$1p_{3/2}$	-0.01538462	-0.00922321	-0.014939	-0.0071448	$0,3$	$0f_{5/2}$	-0.01538462	-0.01841811	-0.014939	-0.016630
			-0.10000000	-0.07822485					-0.10000000	-0.13544265		
3	$1,-3$	$1d_{5/2}$	-0.02413384	-0.01668239	-0.022794	-0.011109	$0,4$	$0g_{7/2}$	-0.02413384	-0.02832645	-0.022794	-0.024595
			-0.15233675	-0.12496202					-0.15233675	-0.19283453		
4	$1,-4$	$1f_{7/2}$	-0.03492388	-0.02617976	-0.031992	-0.015704	$0,5$	$0h_{9/2}$	-0.03492388	-0.04031877	-0.031992	-0.033912
			-0.21278254	-0.18073608					-0.21278254	-0.25727644		
1	$2,-1$	$2s_{1/2}$	-0.01538462	-0.00922321	-0.021582	-0.0098341	$1,2$	$1d_{3/2}$	-0.01538462	-0.01841811	-0.021582	-0.025467
			-0.10000000	-0.07824885					-0.10000000	-0.13544265		
2	$2,-2$	$2p_{3/2}$	-0.02413384	-0.01668239	-0.032684	-0.015919	$1,3$	$1f_{5/2}$	-0.02413384	-0.02832645	-0.032684	-0.036480
			-0.15233675	-0.12496202					-0.15233675	-0.19283453		
3	$2,-3$	$2d_{5/2}$	-0.03492388	-0.02617976	-0.045208	-0.022495	$1,4$	$1g_{7/2}$	-0.03492388	-0.04031877	-0.045208	-0.048998
			-0.21278254	-0.18073608					-0.21278254	-0.25727344		
4	$2,-4$	$2f_{7/2}$	-0.04781723	-0.03775078	-0.059121	-0.029669	$1,5$	$1h_{9/2}$	-0.04781723	-0.05445980	-0.059121	-0.062947
			-0.27957698	-0.24390742					-0.27957698	-0.32696278		

a) The numerical results have been obtained by using the finite difference method[57–59] without any approximations.

Now, we find the corresponding Dirac spinor $G_{n\kappa}^{\rm ps}$. Considering Eqs. (21) and (22) together with Eq. (A6), we obtain

$$G_{n\kappa}^{\rm ps}(r) = {\rm e}^{-2\alpha r\sqrt{f_{\rm ps}}}(1-{\rm e}^{-2\alpha r})^{1/2+\sqrt{(\kappa+V_1)(\kappa+V_1-1)+1/4}} \times P_n^{\left(2\sqrt{f_{\rm ps}},2\sqrt{(\kappa+V_1)(\kappa+V_1-1)+1/4}\right)}(1-{\rm e}^{-2\alpha r}), \quad (27)$$

and find the upper component of the Dirac spinor via

$$F_{n\kappa}^{\rm ps}(r) = \frac{1}{M-E_{n\kappa}^{\rm ps}+C_{\rm ps}}\left(\frac{{\rm d}}{{\rm d}r}-\frac{\kappa}{r}+U(r)\right)G_{n\kappa}^{\rm ps}(r), \quad (28)$$

where $P_n^{(\zeta_1,\zeta_2)}(r)$ is the Jacobi polynomial. The upper and lower components of the Dirac spinor for the Hulthén potential in the presence and absence of the tensor interaction are presented in Fig. 2. From Fig. 2, it is seen that the node structure

of the energy levels remains the same in the presence of the tensor interaction. However, we readily observe from Fig. 2 that tensor interaction changes the shape of the wave functions.

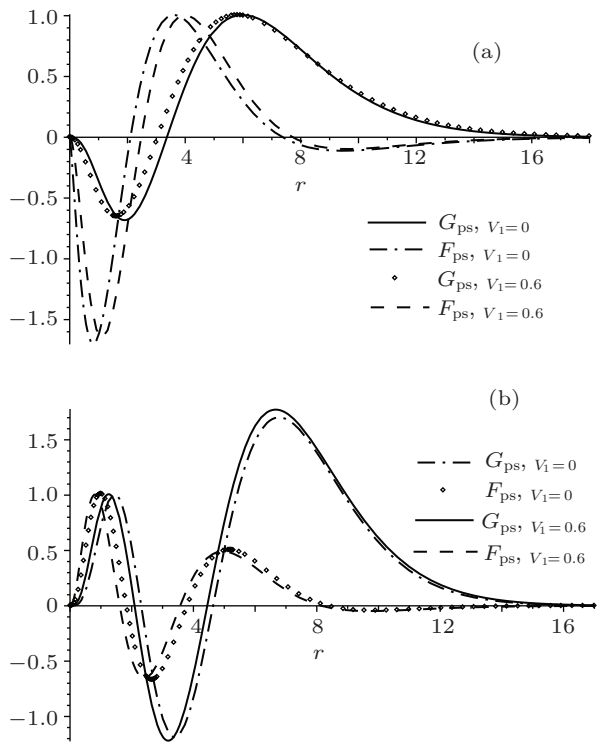


Fig. 2. Wave functions of (a) $1p_{3/2}$ and (b) $2d_{5/2}$ in the pseudospin symmetric case for $V_0 = 4$ fm^{-1}, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$ fm^{-1}, and $\alpha = 0.05$ fm.

3.2. Spin symmetric solution

Equation (17) obtained in the spin symmetry limit has the same form with Eq. (11). Thus, we give only the final results which are obtained by using the same procedure used in the previous section. The energy eigenvalues equation and corresponding wave function of the Dirac particles for the Hulthén potential in the presence of the tensor interaction are obtained as follows:

$$\begin{aligned} E^{\rm s\pm}_{n\kappa} &= \frac{1}{V_0^2+4\tilde{\nu}^2\alpha^2}[(C_{\rm s}-M)V_0^2+2\alpha^2(\tilde{\nu}^2C_{\rm s}+2\tilde{\mu}V_0) \\ &\quad \pm 2(\tilde{\nu}^2\alpha^4[\tilde{\nu}^2(C_{\rm s}-2M)^2 \\ &\quad -4\tilde{\mu}(C_{\rm s}V_0-2MV_0+4\tilde{\mu}\alpha^2)])^{1/2}], \qquad (29) \\ F^{\rm s}_{n\kappa}(r) &= {\rm e}^{-2\alpha r\sqrt{f_{\rm s}}}(1-{\rm e}^{-2\alpha r})^{1/2+\sqrt{(\kappa+V_1)(\kappa+V_1+1)+1/4}} \\ &\quad \times P_n^{\left(2\sqrt{f_{\rm s}},2\sqrt{(\kappa+V_1)(\kappa+V_1+1)+1/4}\right)}(1-{\rm e}^{-2\alpha r}), \qquad (30) \end{aligned}$$

where

$$\begin{aligned} \tilde{\mu} &= n^2+n+\frac{1}{2}+(2n+1)\sqrt{(\kappa+V_1)(\kappa+V_1+1)+\frac{1}{4}} \\ &\quad +\kappa(\kappa+1)+\left(2\kappa+\frac{3}{2}\right)V_1, \qquad (31) \\ \tilde{\nu} &= 2n+1+2\sqrt{(\kappa+V_1)(\kappa+V_1+1)+\frac{1}{4}}. \qquad (32) \end{aligned}$$

The other Dirac spinor can be found via

$$G^{\rm s}_{n\kappa}(r) = \frac{1}{M+E^{\rm s}_{n\kappa}-C_{\rm s}}\left(\frac{{\rm d}}{{\rm d}r}+\frac{\kappa}{r}-U(r)\right)F^{\rm s}_{n\kappa}(r). \qquad (33)$$

In the spin symmetric case, the numerical results for some energy levels are presented in Table 2. For the energy levels, we calculate two energy eigenvalues obtained by using $E^{\rm s+}_{n\kappa}$ (first line) and $E^{\rm s-}_{n\kappa}$ (second line). From Table 2, it is seen that spin doublets which have two bound states with the same energy are formed for a pair of orbitals $(np_{3/2},\ (n-1)p_{1/2})$, $(nd_{5/2},\ (n-1)d_{3/2})$, $(nf_{7/2},\ (n-1)f_{5/2})$, etc. On the other hand, degeneracy between the two states in the spin doublets is removed by tensor interaction. However, one can observe from Table 2

Table 2. Positive-energy bound states (in fm^{-1}) for the spin symmetry with $V_0 = 4$ fm^{-1}, $M = 5$ fm^{-1}, $\alpha = 0.05$ fm, and $C_{\rm s} = 5$ fm^{-1}.

l	n,κ	State	$E^{\rm s}_{n\kappa}$		$E^{\rm s}_{n\kappa}$ [a]		$n-1,\kappa$	States	$E^{\rm s}_{n\kappa}$		$E^{\rm s}_{n\kappa}$ [a]	
			$V_1=0$	$V_1=0.6$	$V_1=0$	$V_1=0.6$			$V_1=0$	$V_1=0.6$	$V_1=0$	$V_1=0.6$
1	$0,-2$	$0p_{3/2}$	0.06784763	0.03315706	0.068074	0.040537	0,1	$0p_{1/2}$	0.06784763	0.11361047	0.068074	0.106040
			0.00145930	0.00067716					0.00145930	0.00242847		
2	$0,-3$	$0d_{5/2}$	0.15074992	0.0970098	0.151470	0.109870	0,2	$0d_{3/2}$	0.15074992	0.21469947	0.151470	0.202790
			0.00328431	0.00206350					0.00328431	0.00469224		
3	$0,-4$	$0f_{7/2}$	0.26338976	0.19214524	0.26484	0.21044	0,3	$0f_{5/2}$	0.26338976	0.34372173	0.26484	0.328000
			0.00584101	0.00418093					0.00584101	0.00768866		
4	$0,-5$	$0g_{9/2}$	0.40263365	0.31584227	0.405060	0.339540	0,4	$0g_{7/2}$	0.40263365	0.49720420	0.405060	0.478240
			0.00913106	0.00703068					0.00913106	0.01141968		
1	$1,-2$	$1p_{3/2}$	0.15074992	0.09700984	0.150770	0.103410	1,1	$1p_{1/2}$	0.15074992	0.21469947	150770	0.206970
			0.00328431	0.00206350					0.00328431	0.00469224		
2	$1,-3$	$1d_{5/2}$	0.26338976	0.19214524	0.263960	0.204580	1,2	$1d_{3/2}$	0.26338976	0.34372173	0.26396	0.33179
			0.00584101	0.00418093					0.00584101	0.00768866		
3	$1,-4$	$1f_{7/2}$	0.40263365	0.31584227	0.4033930	0.333560	1,3	$1f_{5/2}$	0.40263365	0.49720420	0.403930	0.481600
			0.00913106	0.00703068					0.00913106	0.01141968		
4	$1,-5$	$1g_{9/2}$	0.56482502	0.46473259	0.567080	0.487610	1,4	$1g_{7/2}$	0.56482502	0.67127970	0.567080	0.652610
			0.01315663	0.1061460					0.01315663	0.01588776		

[a] The numerical results have been obtained by using the finite difference method[57–59] without any approximations.

that there are still extra degeneracies among the energy levels such as $(0f_{7/2},1d_{5/2},2p_{3/2})$ for $\kappa<0$ and $(0f_{5/2},1d_{3/2},2p_{1/2})$ for $\kappa>0$ due to $\tilde{\mu}$ and $\tilde{\nu}$ giving the same contributions to the degenerate bound states which have the same $(n+|\kappa|)$ for $\kappa<0$ and $(n-1+|\kappa|)$ for $\kappa>0$. Under the spin symmetry limit, the minimum value of the C_s has to be 4.93146 fm^{-1} ($E^{s+}_{0,-2}=0,00000100935$ fm^{-1}) for the set of parameter $V_0=4$ fm^{-1}, $M=5$ fm^{-1}, $\alpha=0.05$ fm, $n=0$, $\kappa=-2$, and $V_1=0$ and 4.966678 fm^{-1} ($E^{s+}_{0,-2}=0,000000908797$ fm^{-1}) for the set of parameter $V_0=4$ fm^{-1}, $M=5$ fm^{-1}, $\alpha=0.05$ fm, $n=0$, $\kappa=-2$, and $V_1=0.6$ to obtain positive-energy bound states. Moreover, numerical solutions of the Dirac equation without any approximation in view of the spin symmetry have been obtained by using the finite difference method[57–59] to check the exactness of our energy eigenvalue equation (29). These results are also given in Table 2.

In Fig. 3, we present the node structures of the wave functions $F^s_{n\kappa}$ and $G^s_{n\kappa}$ for $V_1=0$ and $V_1=0.6$. It is readily seen in Fig. 3 that tensor interaction changes only the shape of the wave functions. The node structures remain unchanged in the presence of the tensor potential.

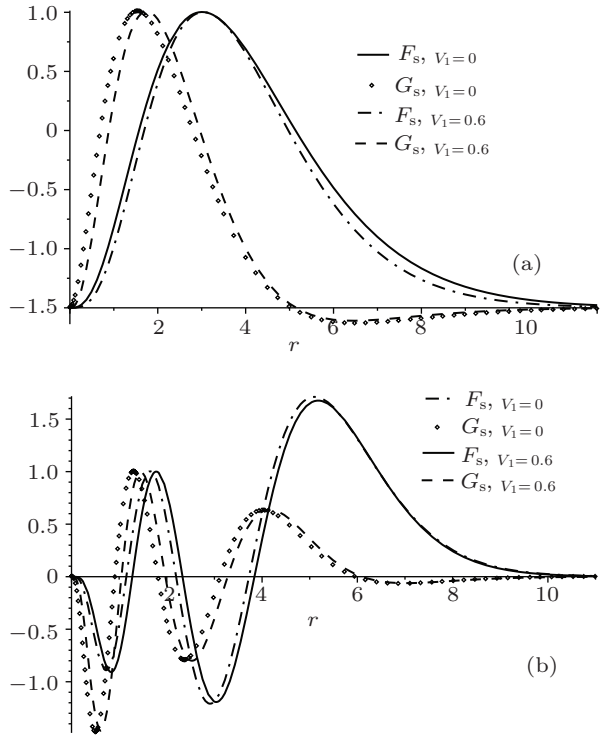


Fig. 3. Wave functions of (a) $0d_{3/2}$ and (b) $3f_{5/2}$ in the spin symmetric case for $V_0=4$ fm^{-1}, $M=5$ fm^{-1}, $C_s=5$ fm^{-1}, and $\alpha=0.05$ fm.

4. Effects of the potential and mass parameters on the bound states

In this work, we investigate the effects of the potential parameters as well as the mass parameter on the bound states under the pseudospin and spin symmetry limits.

4.1. The parameter α

In order to investigate how the bound states are sensitive to the parameter α in the absence of the tensor interaction, we keep all other parameters fixed and change the α. The results are given in Fig. 4. From Fig. 4(a), we can conclude that energy levels become less bounded with increasing parameter α in the pseudospin symmetry limit. However, energy levels become more bounded with increasing parameter α for the spin symmetry Fig. (4b).

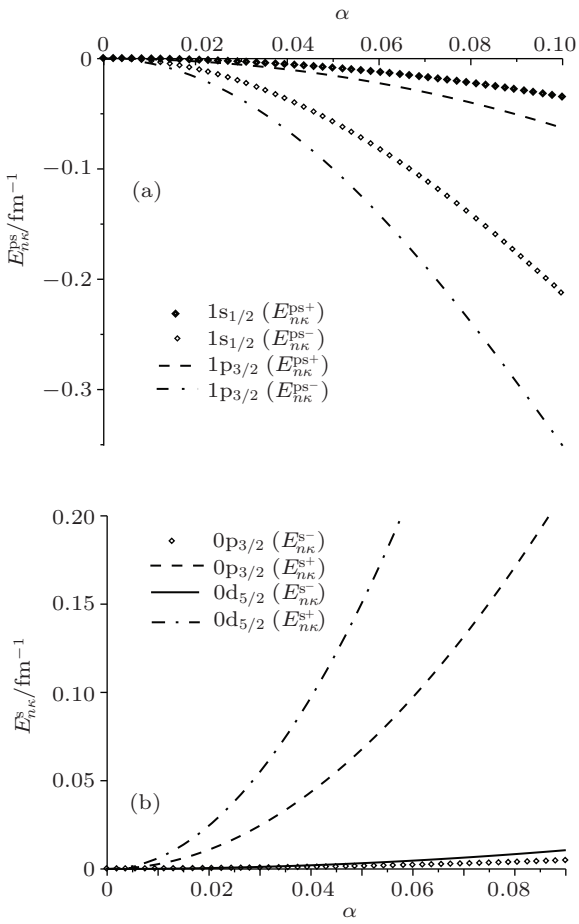


Fig. 4. (a) Energy versus α in the pseudospin symmetric case for $V_0=4$ fm^{-1}, $M=5$ fm^{-1}, $C_{ps}=-5$ fm^{-1}, and $V_1=0$. (b) Energy versus α in the spin symmetric case for $V_0=4$ fm^{-1}, $M=5$ fm^{-1}, $C_s=5$ fm^{-1}, and $V_1=0$.

4.2. The parameter V_0

In Fig. 5, we investigate the effect of the parameter V_0 on the pseudospin doublet $(1s_{1/2}, 0d_{3/2})$ and spin doublet $(0p_{3/2}, 0p_{1/2})$ in the presence and absence of the tensor interaction. One can see in Fig. 5 that energy levels become more (less) bounded in the pseudospin (spin) symmetry with increasing parameter V_0 for both $V_1=0$ and $V_1\neq 0$.

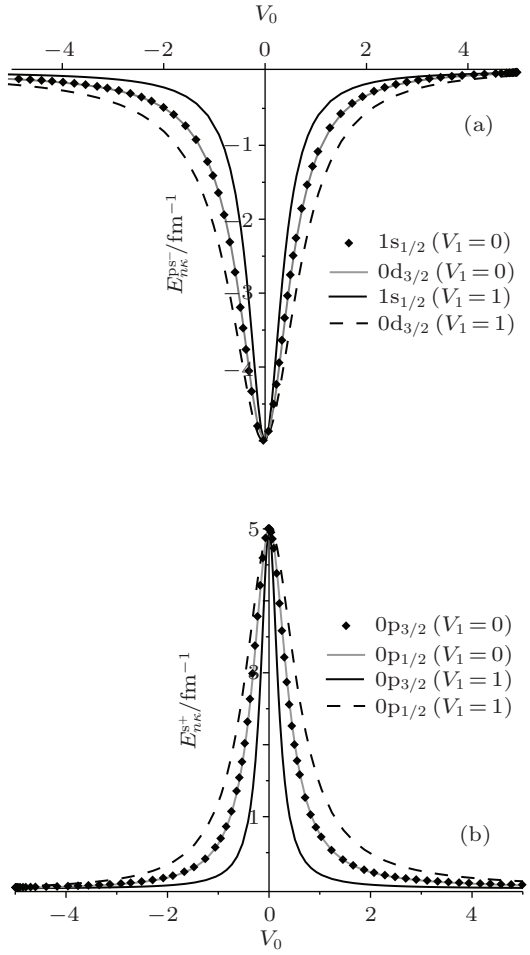


Fig. 5. (a) Sensitivity of the negative-energy bound states to the parameter V_0 with $M = 5$ fm^{-1}, $C_{\mathrm{ps}} = -5$ fm^{-1}, and $\alpha = 0.05$ fm. (b) Sensitivity of the positive-energy bound states to the parameter V_0 with $M = 5$ fm^{-1}, $C_{\mathrm{s}} = 5$ fm^{-1}, and $\alpha = 0.05$ fm.

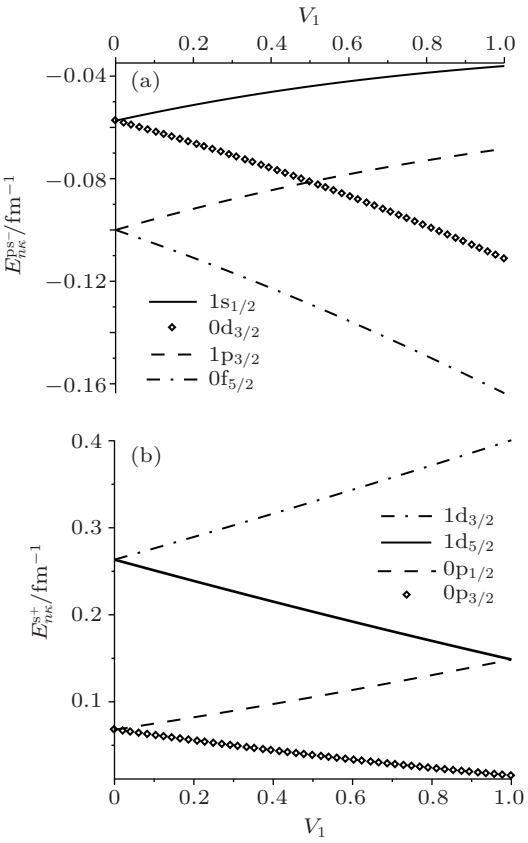


Fig. 6. (a) Effects of the tensor interaction on the pseudospin doublets for $V_0 = 4$ fm^{-1}, $M = 5$ fm^{-1}, $C_{\mathrm{ps}} = -5$ fm^{-1}, and $\alpha = 0.05$ fm. (b) Effects of the tensor interaction on the spin doublets for $V_0 = 4$ fm^{-1}, $M = 5$ fm^{-1}, $C_{\mathrm{s}} = 5$ fm^{-1}, and $\alpha = 0.05$ fm.

4.3. The parameter V_1

Figure 6 represents the effects of the tensor interaction on the pseudospin and spin doublets. From Fig. 6, it is seen that degeneracies are removed by tensor interaction. Furthermore, magnitude of the energy difference between two states in the doublets increases with increasing V_1. It is explained by considering the energy eigenvalue equations. In these equations, the term $2\kappa V_1$ is positive for $\kappa > 0$ and negative for $\kappa < 0$, and its contribution to the energy levels increases with increasing V_1.

4.4. The parameter M

The sensitivity of the pseudospin doublet $(1s_{1/2}, 0d_{3/2})$ and spin doublet $(0p_{3/2}, 0p_{1/2})$ to the mass of the Dirac particle is given in Fig. 7, which shows that bound states obtained both for pseudospin and spin symmetric cases become unbounded when the parameter M increases in the presence and absence of the tensor interaction.

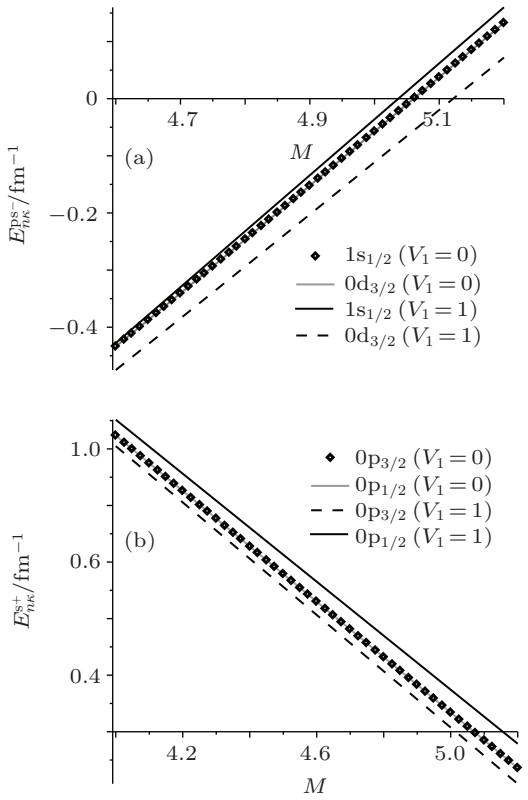


Fig. 7. (a) Energy versus M in the pseudospin symmetric case for $V_0 = 4$ fm^{-1}, $C_{\mathrm{ps}} = -5$ fm^{-1}, and $\alpha = 0.05$ fm. (b) Energy versus M in the spin symmetric case for $V_0 = 4$ fm^{-1}, $C_{\mathrm{s}} = 5$ fm^{-1}, and $\alpha = 0.05$ fm.

4.5. The parameters C_{ps} and C_s

Eventually, we investigate the dependence of the bound-state energy levels to the parameters C_{ps} and C_s for both $V_1 = 0$ and $V_1 \neq 0$. The results are given in Fig. 8. It is seen in Fig. 8 that the parameters C_{ps} and C_s determine whether the Dirac particles constitute bound state or not. From Fig. (8a), we can see that energy level $1s_{1/2}$ constitutes bound state with negative energy on condition that $C_{ps} < -4.9411$ fm^{-1} for $V_1 = 0$ and $C_{ps} < -4.95731$ fm^{-1} for $V_1 \neq 0$ in the pseudospin symmetric case. On the other hand, in the spin symmetric case given in Fig. 8(b), it is observed that positive energy bound-state for $0p_{3/2}$ is formed on condition that $C_s > 4.93146$ fm^{-1} in the absence of tensor interaction and $C_s > 4.966678$ fm^{-1} in the presence of the tensor interaction.

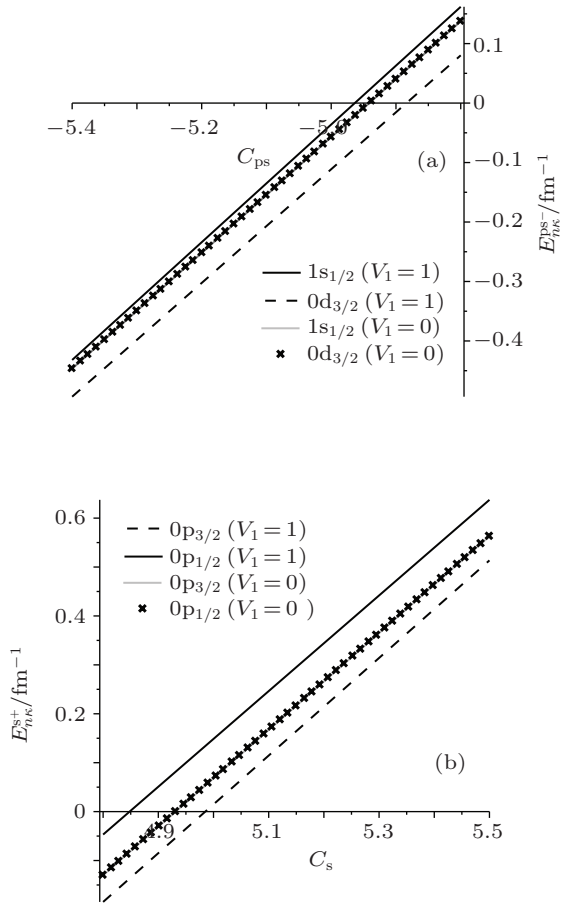


Fig. 8. (a) Energy versus C_{ps} in the pseudospin symmetric case for $V_0 = 4$ fm^{-1}, $M = 5$ fm^{-1}, $\alpha = 0.05$ fm. (b) Energy versus C_s in the spin symmetric case for $V_0 = 4$ fm^{-1}, $M = 5$ fm^{-1}, and $\alpha = 0.05$ fm.

5. Summary

In the present study, we have investigated the approximate analytical pseudospin and spin symmetric solutions of the Dirac equation with the Hulthén potential for the Yukawa-type tensor potential. The energy eigenvalue equations have been obtained in the closed form and the corresponding wave functions have been found in terms of the Jacobi polynomial. In the calculations, we use the generalized NU method. In order to expose the positive- and negative-energy bound states and wave functions, some numerical results have been presented. The effects of the tensor interaction on the pseudospin and spin doublets have been investigated. It has been shown that tensor interaction removes the degeneracy between two states in the pseudospin (spin) doublet. The sensitivity of the positive- and negative-energy bound states to the potential parameters as well as the mass parameter has been also investigated.

Appendix A

In the generalized NU method, we consider the following second-order differential equation in the form of a general Schrödinger-type equation

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Psi_n(s) + \frac{\gamma_1 - \gamma_2 s}{s(1-\gamma_3 s)}\frac{\mathrm{d}}{\mathrm{d}s}\Psi_n(s) + \frac{-\eta_1 s^2 + \eta_2 s - \eta_3}{s^2(1-\alpha_3 s)^2}\Psi_n(s) = 0. \tag{A1}$$

In order to obtain the solution of the above equation by using the generalized NU method, one can use the following parameters and relations:

$$\gamma_4 = \frac{1}{2}(1-\gamma_1), \quad \gamma_5 = \frac{1}{2}(\gamma_2 - 2\gamma_3),$$
$$\gamma_6 = \gamma_5^2 + \eta_1, \quad \gamma_7 = 2\gamma_4\gamma_5 - \eta_2, \tag{A2}$$
$$\gamma_8 = \gamma_4^2 + \eta_3, \quad \gamma_9 = \gamma_3\gamma_7 + \gamma_3^2\gamma_8 + \gamma_6,$$
$$\gamma_{10} = \gamma_1 + 2\gamma_4 + 2\sqrt{\gamma_8}, \tag{A3}$$
$$\gamma_{11} = \gamma_2 - 2\gamma_5 + 2\left(\sqrt{\gamma_9} + \gamma_3\sqrt{\gamma_8}\right),$$
$$\gamma_{12} = \gamma_4 + \sqrt{\gamma_8}, \quad \gamma_{13} = \gamma_5 - \left(\sqrt{\gamma_9} + \gamma_3\sqrt{\gamma_8}\right). \tag{A4}$$

The energy eigenvalue equation reads

$$\gamma_2 n - (2n+1)\gamma_5 + (2n+1)\left(\sqrt{\gamma_9} + \gamma_3\sqrt{\gamma_8}\right) + n(n-1)\gamma_3 + \gamma_7 + 2\gamma_3\gamma_8 + 2\sqrt{\gamma_8\gamma_9} = 0. \tag{A5}$$

The general solution is given by

$$\Psi_n(s) = s^{\gamma_{12}}(1-\gamma_3 s)^{-\gamma_{12}-(\gamma_{13}/\gamma_3)} \times P_n^{(\gamma_{10}-1,(\gamma_{11}/\gamma_3)-\gamma_{10}-1)}(1-2\gamma_3 s). \tag{A6}$$

For more information about the parametric generalization of the NU method, one can refer to Ref. [56].

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References

[1] Hecht K T and Adler A 1969 *Nucl. Phys.* A **137** 129
[2] Arima A, Harvey M and Shimizu K 1969 *Phys. Lett.* B **30** 517
[3] Troltenier D, Bahri C and Draayer J P 1995 *Nucl. Phys.* A **586** 53
[4] Bohr A, Hamamoto I and Mottelson B R 1982 *Phys. Scr.* **26** 267
[5] Dudek J, Nazarewicz W, Szymanski Z and Leander G A 1987 *Phys. Rev. Lett.* **59** 1405
[6] Troltenier D, Nazarewicz W, Szymanski Z and Draayer J P 1994 *Nucl. Phys.* A **567** 591
[7] Stuchbery A E 1999 *J. Phys. G.* **25** 611
[8] Stuchbery A E 2002 *Nucl. Phys.* A **700** 83
[9] Nazarewicz W, Twin P J, Fallon P and Garrett J D 1990 *Phys. Rev. Lett.* **64** 1654
[10] Stephens F S, Deleplanque M A, Draper J E, Diamond R M, Macchiavelli A O, Beausang C W, Korten W, Kelly W H and Azaiez F 1990 *Phys. Rev. Lett.* **65** 301
[11] Ginocchio J N 1997 *Phys. Rev. Lett.* **78** 436
[12] Ginocchio J N and Leviatan A 1998 *Phys. Lett.* B **245** 1
[13] Ginocchio J N 2004 *Phys. Rev.* C **69** 034318
[14] Ginocchio J N 2005 *Phys. Rep.* **414** 165
[15] Page P R, Goldman T and Ginocchio J N 2001 *Phys. Rev. Lett.* **86** 204
[16] Xu Y, He S and Jia C S 2008 *J. Phys.* A: *Math. Theor.* **41** 255302
[17] Berkdemir C 2006 *Nucl. Phys.* A **770** 32
[18] Zhang L H, Li X P and Jia C S 2008 *Phys. Lett.* A **372** 2201
[19] Soylu A, Bayrak O and Boztosun I 2007 *J. Math. Phys.* **48** 082302
[20] Berkdemir C and Sever R 2008 *J. Phys.* A: *Math. Theor.* **41** 045302
[21] Qiang W C, Zhou R S and Gao Y 2007 *J. Phys.* A: *Math. Theor.* **40** 1677
[22] Alhaidari A D, Bahlouli H and Al-Hasan A 2006 *Phys. Lett.* A **349** 87
[23] Guo J Y, Han J C and Wang R D 2006 *Phys. Lett.* A **353** 378
[24] Jia C S, Guo P and Peng X L 2006 *J. Phys.* A: *Math. Gen.* **39** 7737
[25] Wei G F and Dong S H 2008 *Phys. Lett.* A **373** 49
[26] ALberto P, Fiolhais M, Malheiro M, Delfino A and Chiapparini M 2002 *Phys. Rev.* C **65** 034307
[27] Chen T S, Lü H F, Meng J, Zhang S Q and Zhou S G 2003 *Chin. Phys. Lett.* **20** 358
[28] Xu Q and Zhu S J 2006 *Nucl. Phys.* A **768** 161
[29] Zhou S G, Meng J and Ring P 2003 *Phys. Rev. Lett.* **91** 262501
[30] He X T, Zhou S G, Meng J, Zhao E G and Scheid W 2006 *Eur. Phys. J.* A **28** 265
[31] Taskin F and Kocak G 2011 *Chin. Phys.* B **20** 070302
[32] Aydogdu O and Sever R 2010 *Ann. Phys.* **325** 373
[33] Zarrinkamar S, Rajabi A A and Hassanabadi H 2010 *Ann. Phys.* **325** 2522
[34] Zarrinkamar S, Rajabi A A and Hassanabadi H 2010 *Ann. Phys.* **325** 1720
[35] Zarrinkamar S, Rajabi A A and Hassanabadi H 2011 *Phys. Scr.* **83** 015009
[36] Zhou Y and Guo J Y 2008 *Chin. Phys.* B **17** 380
[37] Chen W L and Wei G F 2011 *Chin. Phys.* B **20** 062101
[38] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais M 2004 *Phys. Rev.* C **69** 024319
[39] Alberto P, Lisboa R, Malheiro M and de Castro A S 2005 *Phys. Rev.* C **71** 034313
[40] Akcay H 2009 *Phys. Lett.* A **373** 616
[41] Akcay H and Tezcan C 2009 *Int. J. Mod. Phys.* C **20** 931
[42] Aydogdu O and Sever R 2011 *Phys. Lett.* B **703** 379
[43] Aydogdu O and Sever R 2010 *Eur. Phys. J.* A **43** 73
[44] Aydogdu O and Sever R 2010 *Few-Body Syst.* **47** 193
[45] Hassanabadi H, Maghsoodi E, Zarrinkamar S and Rahimov H 2012 *J. Math. Phys.* **53** 022104
[46] Hassanabadi H, Maghsoodi E, Zarrinkamar S and Rahimov H 2011 *Mod. Phys. Lett.* A **26** 2703
[47] Hassanabadi H, Maghsoodi E and Zarrinkamar S 2012 *Eur. Phys. J. Plus.* **127** 31
[48] Maghsoodi E, Hassanabadi E and Zarrinkamar S 2012 *Few-Body Syst.* **53** 525
[49] Eshghi M and Mehraban H 2011 *J. Sci. Res.* **3** 239
[50] Maghsoodi E, Hassanabadi E, Zarrinkamar S and Rahimov H 2012 *Phys. Scr.* **85** 055007
[51] Hulthén L 1942 *Ark. Mat. Astron. Fys.* **28A** 5
[52] Yukawa H 1935 *Proc. Phys. Math. Soc. Jpn.* **17** 48
[53] Yukawa H and Sakata S 1937 *Proc. Phys. Math. Soc. Jpn.* **19** 1084
[54] Luo X Q, Li Y Y and Kröger H 2006 *Sci. China Ser.* G **49** 60
[55] Nikiforov A F and Uvarov V B 1988 *Special Functions of Mathematical Physics* (Basel: Birkhauser)
[56] Ikhdair S M 2010 *J. Math. Phys.* **51** 023525
[57] Chari M V K and Salon S J 1999 *Numerical Methods in Electromagnetism* (Academic Press)
[58] available at www.netlib.org
[59] available at ftp://ftp.caam.rice.edu/pub/software/ARPACK