Bound states of the Dirac equation with position-dependent mass for the Eckart potential*

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(Received 15 March 2012; revised manuscript received 11 April 2012)

Studying with the asymptotic iteration method, we present approximate solutions of the Dirac equation for the Eckart potential in the case of position-dependent mass. The centrifugal term is approximated by an exponential form, and the relativistic energy spectrum and the normalized eigenfunctions are obtained explicitly.

Keywords: Dirac equation, position-dependent mass, Eckart potential, asymptotic iteration method

PACS: 03.65.–w, 03.65.Ge, 03.65.Pm **DOI:** 10.1088/1674-1056/22/1/010301

1. Introduction

In recent years, quantum mechanical systems with a position-dependent mass have been investigated more efficiently in the literature.[1] Systems with a position-dependent mass have been found to be very useful for studying physical properties of different microstructures,[2] such as quantum wells, quantum dots,[3,4] quantum liquids,[5] and semiconductor heterostructures.[6] The analytical solutions of the nonrelativistic Schrödinger equation with a position-dependent mass for solvable potentials have been discussed by using different methods.[7–18] On the other hand, the investigation of relativistic effects is important in quantum mechanical systems such as heavy atoms and heavy ion doping. Therefore, the investigation of the relativistic Dirac equation with a position-dependent mass is very important for these circumstances. In the recent literature, considerable studies have been carried out to investigate the exact or quasi-exact solutions of the Dirac equation with a position-dependent mass and the properties of these solutions for different potentials and mass distributions.[19–31]

In this paper, our aim is to solve the Dirac equation for the Eckart potential in the case of a position-dependent mass with an approximation to the centrifugal term for arbitrary spin orbit quantum number κ by using an asymptotic iteration method (AIM). This method for solving second-order homogeneous linear differential equations has been proposed by Ciftci *et al.* recently.[32–34]

This paper is organized as follows. In Section 2, approximate solutions of the Dirac equation with a position-dependent mass for the Eckart potential are given. The relativistic energy eigenvalues and the corresponding eigenfunctions are studied in detail by using the AIM. The paper ends with a brief conclusion in Section 3.

2. Approximate solutions of the Dirac equation with position-dependent mass for the Eckart potential

The Dirac equation[35–37] for a single particle position-dependent mass formalism $\mu(r)$ in vector potential $V(r)$ and scalar potential $S(r)$ is given by (in atomic units $\hbar = c = 1$)

$$[\boldsymbol{\alpha}\cdot p + \beta(\mu(r)+S(r)) + V(r)]\psi(r) = E\psi(r), \quad (1)$$

$$p = -\mathrm{i}\nabla,\ \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},\ \beta = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix}, \quad (2)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{I}$ are the vector Pauli spin matrix and the identity matrix, respectively, p is the momentum operator, $S(r)$ and $V(r)$ are the scalar and the vector potentials, respectively, and E is the total relativistic energy of the system. The $\boldsymbol{\sigma}_{i=x,y,z}$ are 2×2 Pauli matrices

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

The Dirac eigenfunction can be written with $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ as

$$\psi(r) = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \dfrac{F_{n\kappa}(r)}{r}\mathrm{Y}^{\ell}_{jm}(\theta,\phi) \\ \dfrac{\mathrm{i}G_{n\kappa}(r)}{r}\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\phi) \end{pmatrix}, \quad (4)$$

where $\mathrm{Y}^{\ell}_{jm}(\theta,\varphi)$ and $\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi)$ are the spin and the pseudo spin spherical harmonics, respectively, $F_{n\kappa}(r)/r$ and $\mathrm{i}G_{n\kappa}(r)/r$ are the radial functions for the upper and the lower components, respectively, n is the radial quantum number, and m is the projection of the angular momentum on the z axis. The orbital angular momentum quantum numbers ℓ and $\tilde{\ell}$ refer to the spin and the pseudo spin quantum numbers, respectively. For a given spin–orbit coupling term $\kappa = 0, \pm 1, \pm 2, \ldots$, the total angular momentum, the orbital angular momentum,

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and the pseudo orbital angular momentum are given by $j = |\kappa| - 1/2, \ell = |\kappa + 1/2| - 1/2$, and $\tilde{\ell} = |\kappa - 1/2| - 1/2$, respectively. By substituting Eqs. (2) and (4) into Eq. (1), the Dirac equation reduces to the following coupled differential equations:

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right) F_{n\kappa}(r) = (\mu(r) + E - V(r) + S(r)) G_{n\kappa}(r), \quad (5)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right) G_{n\kappa}(r) = (\mu(r) - E + V(r) + S(r)) F_{n\kappa}(r). \quad (6)$$

Using function $G_{n\kappa}(r)$ in Eq. (5) and substituting it into Eq. (6), we obtain the following second-order differential equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} - (\mu(r) + E - V_-(r))(\mu(r) - E + V_+(r)) - \frac{\left(\frac{\mathrm{d}\mu(r)}{\mathrm{d}r} - \frac{\mathrm{d}V_-(r)}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r}\right)}{\mu(r) + E - V_-(r)}\right] F_{n\kappa}(r) = 0. \quad (7)$$

In a similar way, we have

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} - (\mu(r) + E - V_-(r))(\mu(r) - E + V_+(r)) - \frac{\left(\frac{\mathrm{d}\mu(r)}{\mathrm{d}r} + \frac{\mathrm{d}V_+(r)}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r}\right)}{\mu(r) - E + V_+(r)}\right] G_{n\kappa}(r) = 0, \quad (8)$$

where $V_-(r) = V(r) - S(r)$ and $V_+(r) = V(r) + S(r)$.

2.1. Energy eigenvalues of the equation

To solve the Dirac equation for a single particle with position-dependent mass in the Eckart potential and to obtain an analytical approximate solution for the Eckart potential with the spin–orbit coupling term, we have to use an approximation in the following form:[38–40]

$$\frac{1}{r^2} \sim 4\alpha^2 \frac{\mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2}, \quad (9)$$

where α is the screening parameter determining the range of the Eckart potential, which is valid for $\alpha r \ll 1$. The Eckart potential[41] is defined as

$$V(r) = V_1 \mathrm{cosech}^2(\alpha r) - V_2 \coth(\alpha r). \quad (10)$$

In this study, if the scalar potential $S(r)$ is taken as zero, the Eckart potential can be written in the exponential form

$$V(r) = 4V_1 \frac{\mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2} - V_2 \frac{1 + \mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}}. \quad (11)$$

To solve Eq. (7), first, we eliminate the last term in Eq. (7) by using equality $\mathrm{d}\mu(r)/\mathrm{d}r - \mathrm{d}V_-(r)/\mathrm{d}r = 0$, which gives a position-dependent mass function

$$\mu(r) = m_0 + 4V_1 \frac{\mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2} - V_2 \frac{1 + \mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}}, \quad (12)$$

where m_0 is the integration constant. Substituting Eqs. (9), (11), and (12) into Eq. (7), we obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \kappa(\kappa+1)\frac{4\alpha^2 \mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2} - (m_0 + E)\left(m_0 - E + 8V_1 \frac{\mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2} - 2V_2 \frac{1 + \mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}}\right)\right] F_{n\kappa}(r) = 0. \quad (13)$$

We rewrite Eq. (13) by using a new variable $s = \mathrm{e}^{-2\alpha r}$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s} - \left(\kappa(\kappa+1) + \frac{2V_1(m_0+E)}{\alpha^2}\right)\frac{1}{s(1-s)^2} - \frac{m_0^2 - E^2}{4\alpha^2 s^2} + \frac{(m_0+E)V_2}{2\alpha^2}\frac{1+s}{s^2(1-s)}\right] F_{n\kappa}(s) = 0. \quad (14)$$

In terms of the new parameters, we have

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s} - \frac{\xi_0}{s^2} - \frac{\xi_1}{s(1-s)^2} + \xi_2 \frac{1+s}{s^2(1-s)}\right) F_{n\kappa}(s) = 0, \quad (15)$$

where

$$\xi_0 = \frac{m_0^2 - E^2}{4\alpha^2}, \quad \xi_1 = \kappa(\kappa+1) + \frac{2V_1(m_0+E)}{\alpha^2}, \quad \xi_2 = \frac{(m_0+E)V_2}{2\alpha^2}. \quad (16)$$

Now, we will solve Eq. (15) by using the AIM. We briefly outline the AIM here; the details can be found in Refs. [32]– [34]. This method was proposed for solving second-order differential equations of the form

$$y'' = \lambda_0(x) y' + s_0(x) y, \quad (17)$$

where $\lambda_0(x) \neq 0$ and $s_0(x), \lambda_0(x)$ are in $\mathrm{C}_\infty(a,b)$. The variables $s_0(x)$ and $\lambda_0(x)$ are sufficiently differentiable. The differential Eq. (17) has a general solution

$$y(x) = \exp\left(-\int^x \alpha \,\mathrm{d}x'\right)\left[C_2 + C_1 \int^x \exp\left(\int^{x'} [\lambda_0(x'') + 2\alpha(x'')]\,\mathrm{d}x''\right)\mathrm{d}x'\right], \quad (18)$$

if for sufficiently large n,

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} = \alpha, \quad (19)$$

where

$$\lambda_n(x) = \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x), \quad s_n(x) = s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x), \quad n = 1, 2, 3, \ldots. \quad (20)$$

The termination condition of the method together with Eq. (20) can also be written as follows:

$$\delta(x) = \lambda_{n-1}(x)s_n(x) - \lambda_n(x)s_{n-1}(x) = 0. \tag{21}$$

For a given potential, the idea is to convert the relativistic Dirac equation to the form of Eq. (17). Then, s_0 and λ_0 are determined, and parameters s_n and λ_n are calculated. The energy eigenvalues are obtained by using the termination condition given in Eq. (21). However, the exact eigenfunctions can be derived from the following wave function generator:

$$y_n(x) = C_2 \exp\left(-\int^x \alpha_k \mathrm{d}x'\right), \tag{22}$$

where $n = 0, 1, 2, \ldots$, and k is the iteration step number and usually greater than n.

In order to solve Eq. (15) with the aid of the AIM, we should transform this equation to the form of Eq. (17). The wave function should satisfy the boundary conditions, i.e., $F_{n\kappa}(0) = 0$ at $s = 0$ for $r \to \infty$ and $F_{n\kappa}(1) = 0$ at $s = 1$ for $r \to 0$. Therefore, the reasonable physical wave function is proposed as follows:

$$F_{nk}(s) = s^{\sqrt{\xi_0 - \xi_2}}(1-s)^{\frac{1}{2}(1+\sqrt{1+4\xi_1})} f(s). \tag{23}$$

If we insert Eq. (23) into Eq. (15), we have a second-order homogeneous linear differential equation in the following form:

$$\begin{aligned} & f''(s) \\ &= -\frac{-1+s(2+\sqrt{1+4\xi_1}+2\sqrt{\xi_0-\xi_2})-2\sqrt{\xi_0-\xi_2}}{s(-1+s)} f'(s) \\ &\quad - \frac{(2\xi_1+(1+\sqrt{1+4\xi_1})(1+2\sqrt{\xi_0-\xi_2})-4\xi_2)}{2s(-1+s)} f(s), \end{aligned} \tag{24}$$

which is now convenient for obtaining an AIM solution. By comparing the above equation with Eq. (17), we can obtain $\lambda_0(s)$ and $s_0(s)$, and with the aid of Eq. (20) we may calculate $\lambda_n(s)$ and $s_n(s)$. This gives

$$\begin{aligned} \lambda_0(s) &= \frac{1-s\left(2+\sqrt{1+4\xi_1}+2\sqrt{\xi_0-\xi_2}\right)+2\sqrt{\xi_0-\xi_2}}{s(-1+s)}, \quad s_0(s) = -\frac{2\xi_1+\left(1+\sqrt{1+4\xi_1}\right)\left(1+2\sqrt{\xi_0-\xi_2}\right)-4\xi_2}{2s(-1+s)}, \\ \lambda_1(s) &= \frac{4+8\xi_0-s\left(11+16\xi_0-2\xi_1+3\sqrt{1+4\xi_1}+30\sqrt{\xi_0-\xi_2}+6\sqrt{1+4\xi_1}\sqrt{\xi_0-\xi_2}-12\xi_2\right)}{2s^2(-1+s)^2} \\ &\quad + \frac{s^2\left(13+8\xi_0+6\xi_1+9\sqrt{1+4\xi_1}+18\sqrt{\xi_0-\xi_2}+6\sqrt{1+4\xi_1}\sqrt{\xi_0-\xi_2}-4\xi_2\right)+12\sqrt{\xi_0-\xi_2}-8\xi_2}{2s^2(-1+s)^2}, \\ s_1(s) &= \frac{\left(-2\left(1+\sqrt{\xi_0-\xi_2}\right)+s\left(4+\sqrt{1+4\xi_1}+2\sqrt{\xi_0-\xi_2}\right)\right)\left(2\xi_1+\left(1+\sqrt{1+4\xi_1}\right)\left(1+2\sqrt{\xi_0-\xi_2}\right)-4\xi_2\right)}{2s^2(-1+s)^2}, \\ &\vdots \end{aligned} \tag{25}$$

Combining these results obtained by the AIM with the quantization condition given by Eq. (21) yields

$$\begin{aligned} & s_0\lambda_1 - s_1\lambda_1 = 0 \Rightarrow \xi_{01} = \xi_2 + \frac{1}{4}\left(-1 - \frac{2(\xi_1 - 2\xi_2)}{1+\sqrt{1+4\xi_1}}\right)^2, \quad \text{for } n = 0, \\ & s_1\lambda_2 - s_2\lambda_1 = 0 \Rightarrow \xi_{02} = \xi_2 + \frac{1}{4}\left(-1 - \frac{2(1+\xi_1+\sqrt{1+4\xi_1}-2\xi_2)}{3+\sqrt{1+4\xi_1}}\right)^2, \quad \text{for } n = 1, \\ & s_2\lambda_3 - s_3\lambda_2 = 0 \Rightarrow \xi_{03} = \xi_2 + \frac{1}{4}\left(-1 - \frac{2(2+\xi_1+2(1+\sqrt{1+4\xi_1})-2\xi_2}{5+\sqrt{1+4\xi_1}}\right)^2, \quad \text{for } n = 2, \\ & \vdots \end{aligned} \tag{26}$$

When the above expressions are generalized, the eigenvalues turn out to be

$$\xi_{0n} = \xi_2 + \frac{1}{4}\left(-1 - \frac{2[(-1+n)n+\xi_1+n(1+\sqrt{1+4\xi_1})-2\xi_2]}{1+2n+\sqrt{1+4\xi_1}}\right)^2, \quad n = 0, 1, 2, \ldots. \tag{27}$$

By using Eq. (27), the energy of state of the Dirac equation for this potential can be obtained by the following expression:

$$E_n^2 = m_0^2 - 4\alpha^2\left[\frac{1}{4}\left(-1 - \frac{2[(-1+n)n+\xi_1+n(1+\sqrt{1+4\xi_1})-2\xi_2]}{1+2n+\sqrt{1+4\xi_1}}\right)^2 + \xi_2\right]. \tag{28}$$

Table 1. Energies for $V_1 = V_2 = 0.8$ ($\hbar = c = m_0 = 1$).

α	$\|n,l,\kappa\rangle$	Energy
0.05	$\|0,0,0\rangle$	–0.999307
	$\|1,0,0\rangle$	–0.986568
	$\|2,1,1\rangle$	–0.967129
	$\|3,2,2\rangle$	–0.940083
0.1	$\|0,0,0\rangle$	–0.997211
	$\|1,0,0\rangle$	–0.944109
	$\|2,1,1\rangle$	–0.849050
	$\|3,2,2\rangle$	–
0.3	$\|0,0,0\rangle$	–0.974060
	$\|1,0,0\rangle$	–
	$\|2,1,1\rangle$	–
	$\|3,2,2\rangle$	–

Table 2. Energies for $V_1 = V_2 = 1$ ($\hbar = c = m_0 = 1$).

α	$\|n,l,\kappa\rangle$	Energy
0.001	$\|0,0,0\rangle$	–1.
	$\|1,0,0\rangle$	–0.999994
	$\|2,1,1\rangle$	–0.999984
	$\|3,2,2\rangle$	–0.999970
	$\|4,3,3\rangle$	–0.999952
	$\|5,4,4\rangle$	–0.999930
	$\|6,5,5\rangle$	–0.999904
	$\|7,6,6\rangle$	–0.999874
	$\|8,7,7\rangle$	–0.999840
	$\|9,8,8\rangle$	–0.999802
	$\|10,9,9\rangle$	–0.999761
	$\|11,10,10\rangle$	–0.999715
	$\|12,11,11\rangle$	–0.999666

However, dependences of E_n on ξ_1 and ξ_2 should not be forgotten. For different α values and quantum states, the energy eigenvalues are calculated from Eq. (28). The results are shown in Table 1. As shown in Tables 1 and 2, due to the increase of screening parameter α, eigenvalues E_n also increase. According to Fig. 1(b), the three bound states occur in potential well for relatively small values of screening parameter ($0 < \alpha < 0.2$), while for larger values of screening parameter ($0.2 < \alpha$), only one bound state appears, namely the ground state. The reason for this case can be understood by observing Fig. 1(b). The increase in the screening parameter leads to changes of the depth and width of the potential well, which causes less bound states in the Eckart potential well.

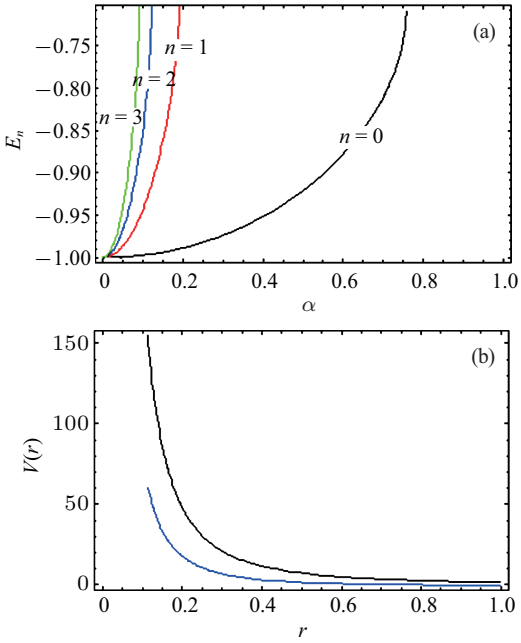


Fig. 1. (color online) (a) The variation of eigenvalues with screening parameter α in atomic units ($\hbar = c = m_0 = 1$) with $V_1 = V_2 = 0.8$. (b) The variation of the Eckart potential with r ($V_1 = V_2 = 0.8, \alpha = 0.2$ black line, $\alpha = 0.7$ blue line).

2.2. Eigenfunctions of the equation

The exact eigenfunctions can be derived from the following generator:

$$f_n(s) = C_2 \exp\left(-\int^s \alpha_k \, \mathrm{d}s'\right). \tag{29}$$

Using Eqs. (19) and (29), we obtain the following eigenfunctions:

$$\begin{aligned}
f_0(s) &= C_2, \\
f_1(s) &= -C_2(1+2\sqrt{\xi_0-\xi_2})\left(1-\frac{2+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}}{1+2\sqrt{\xi_0-\xi_2}}s\right), \\
f_2(s) &= C_2\left(1+2\sqrt{\xi_0-\xi_2}\right)\left(2+2\sqrt{\xi_0-\xi_2}\right)\left(1-\frac{2\left(3+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)}{1+2\sqrt{\xi_0-\xi_2}}s\right. \\
&\quad \left.+\frac{\left(3+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)\left(4+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)}{\left(1+2\sqrt{\xi_0-\xi_2}\right)\left(2+2\sqrt{\xi_0-\xi_2}\right)}s^2\right), \\
f_3(s) &= -C_2\left(1+2\sqrt{\xi_0-\xi_2}\right)\left(2+2\sqrt{\xi_0-\xi_2}\right)\left(3+2\sqrt{\xi_0-\xi_2}\right)x\left(1-\frac{3\left(4+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)}{1+2\sqrt{\xi_0-\xi_2}}s\right. \\
&\quad +\frac{3\left(4+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)\left(5+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)}{\left(1+2\sqrt{\xi_0-\xi_2}\right)\left(2+2\sqrt{\xi_0-\xi_2}\right)}s^2 \\
&\quad \left.-\frac{\left(4+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)\left(5+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)\left(6+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}\right)}{\left(1+2\sqrt{\xi_0-\xi_2}\right)\left(2+2\sqrt{\xi_0-\xi_2}\right)\left(3+2\sqrt{\xi_0-\xi_2}\right)}s^3\right), \\
&\cdots .
\end{aligned} \tag{30}$$

Finally, the following general formula for the exact solutions $f_n(s)$ is acquired:

$$f_n(s) = (-1)^n C_2 \frac{\Gamma\left(1+2\sqrt{\xi_0-\xi_2}+n\right)}{\Gamma\left(1+2\sqrt{\xi_0-\xi_2}\right)} \times {}_2\mathrm{F}_1\left(-n, 1+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1} + n; 1+2\sqrt{\xi_0-\xi_2}; s\right). \quad (31)$$

Hence, we can write the total radial wave function as

$$F_{n\kappa}(s) = N s^{\sqrt{\xi_0-\xi_2}}(1-s)^{(1+\sqrt{1+4\xi_1})/2} \times {}_2\mathrm{F}_1\left(-n, 1+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1} + n; 1+2\sqrt{\xi_0-\xi_2}; s\right), \quad (32)$$

where N is the normalization constant.

In order to normalize the wave functions, we should determine $G_{n\kappa}(s)$. Inserting Eq. (11) and $\mu(r) = m_0 - 4V_1\,\mathrm{e}^{-2\alpha r}/(1-\mathrm{e}^{-2\alpha r})^2 + V_2(1+\mathrm{e}^{-2\alpha r})/1-\mathrm{e}^{-2\alpha r}$ into Eq. (8), we obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} - (\mu_0 - E_{n\kappa})\left(\mu_0 + E_{n\kappa} - 8V_1\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} + 2V_2\frac{1+\mathrm{e}^{-2\alpha r}}{1-\mathrm{e}^{-2\alpha r}}\right)\right] G_{n\kappa}(r) = 0. \quad (33)$$

When $1/r^2$ exponential approximation and the defined $s = e^{-2\alpha r}$ transformation are used, equation (33) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s} - \frac{\xi_3}{s(1-s)^2} - \frac{\xi_0}{s^2} - \xi_4\frac{1+s}{s^2(1-s)}\right] G_{n\kappa}(s) = 0, \quad (34)$$

where $\xi_0 = (m_0^2 - E^2)/4\alpha^2$, $\xi_3 = \kappa(\kappa-1) - 2V_1(m_0-E)/\alpha^2$, and $\xi_4 = (m_0-E)V_2/2\alpha^2$. If the processes to find $F_{n\kappa}(s)$ are repeated, we obtain

$$G_{n\kappa}(s) = N s^{\sqrt{\xi_0+\xi_4}}(1-s)^{(1+\sqrt{1+4\xi_3})/2} \times {}_2\mathrm{F}_1\left(-n, 1+2\sqrt{\xi_0+\xi_1}+\sqrt{1+4\xi_3} + n; 1+2\sqrt{\xi_0+\xi_1}; s\right). \quad (35)$$

The total radial wave functions are obtained in terms of confluent hypergeometric functions. In this part of our work, we calculate the normalization constant N in Eqs. (32) and (35). The calculation of the normalization constant is very necessary, which has not been explicitly worked out in most of the studies. The method to compute the normalization constant is given in Ref. [42]. To compute the normalization constant N, first, we start with the normalization condition $\int_0^\infty (f^2+g^2)r^2\,\mathrm{d}r = 1$. According to the transformation of $s = \mathrm{e}^{-2\alpha r}$, $s \to 1$ for $r \to 0$ and $s \to 0$ for $r \to \infty$. Therefore,

$$\int_0^1 \frac{F(s)^2}{2s\alpha}\,\mathrm{d}s + \int_0^1 \frac{G(s)^2}{2s\alpha}\,\mathrm{d}s = 1.$$

So, by means of the above normalization condition, we have

$$N^2\Big[\int_0^1 s^{2\sqrt{\xi_0-\xi_2}-1}(1-s)^{1+\sqrt{1+4\xi_1}}\Big({}_2\mathrm{F}_1(-n, 1 + 2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}+n; 1+2\sqrt{\xi_0-\xi_2}; s)\Big)^2\mathrm{d}s + \int_0^1 s^{2\sqrt{\xi_0+\xi_4}-1}(1-s)^{1+\sqrt{1+4\xi_3}}\Big({}_2\mathrm{F}_1(-n, 1 + 2\sqrt{\xi_0+\xi_1}+\sqrt{1+4\xi_3}+n; 1+2\sqrt{\xi_0+\xi_1}; s)\Big)^2\mathrm{d}s\Big] = 2\alpha. \quad (36)$$

The series representation of the confluent hypergeometric function ${}_2\mathrm{F}_1$ is

$${}_p\mathrm{F}_q(a_1, \ldots a_p; c_1, ..c_q; s) = \sum_{i=0}^{\infty}\frac{(a_1)_i\cdots(a_p)_i}{(c_1)_i\cdots(c_q)_i}\frac{s^i}{i!}. \quad (37)$$

Being Pochhammer symbols $(a_1)_i$, $(c_1)_i$ and a polynomial of degree n in s, we obtain the normalization constant in the following form:

$$N^2\Bigg[\sum_{i=0}^{n}\frac{(-n)_i(1+2\sqrt{\xi_0-\xi_2}+\sqrt{1+4\xi_1}+n)_i(2\sqrt{\xi_0-\xi_2})_i}{(1+2\sqrt{\xi_0-\xi_2})_i i!(2+\sqrt{1+4\xi_1}+2\sqrt{\xi_0-\xi_2})_i} \times B(2\sqrt{\xi_0-\xi_2}, 2+\sqrt{1+4\xi_1})_3\mathrm{F}_2(-n, 1+2\sqrt{\xi_0-\xi_2} + \sqrt{1+4\xi_1}+n, 2\sqrt{\xi_0-\xi_2}+i; 1+2\sqrt{\xi_0-\xi_2}, 2 + \sqrt{1+4\xi_1}+2\sqrt{\xi_0-\xi_2}+i; 1) + \sum_{i=0}^{n}\frac{(-n)_i(1+2\sqrt{\xi_0+\xi_1}+\sqrt{1+4\xi_3}+n)_i(2\sqrt{\xi_0+\xi_4})_i}{(1+2\sqrt{\xi_0+\xi_1})_i i!(2+\sqrt{1+4\xi_3}+2\sqrt{\xi_0+\xi_4})_i} \times B(2+\sqrt{1+4\xi_3}, 2\sqrt{\xi_0+\xi_4})_3\mathrm{F}_2(-n, 1+2\sqrt{\xi_0+\xi_4} + \sqrt{1+4\xi_3}+n, 2\sqrt{\xi_0+\xi_4}+i; 1+2\sqrt{\xi_0+\xi_1}, 2 + \sqrt{1+4\xi_3}+2\sqrt{\xi_0+\xi_4}+i; 1)\Bigg] = 2\alpha. \quad (38)$$

3. Conclusion

To summarize, we have considered analytical approximate bound state solutions of the mass-dependent Dirac equation with the vector Eckart potential and mass function using the AIM by applying an approximation to the centrifugal-like term, which is a different approach. For arbitrary spin–orbit quantum number κ state and small α, we have obtained the energy eigenvalues and the corresponding radial wave functions for the case of position-dependent mass and approximation on the spin–orbit coupling term in a systematic way.

Appendix A

To normalize the wave functions, some of the special procedures for the beta function are given in the following form:[42,43]

(i)

$$B_q(x,y) = \int_0^q s^{x-1}(1-s)^{y-1}\,\mathrm{d}s = 1,$$

$$B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y),$$
$$B_q(x+1,y)=\frac{x}{x+y}B_q(x,y)-\frac{q^x(1-q)^y}{x+y},$$

(ii)

$$\begin{aligned}
&I_q(x,y)\\
&=I_q(x-1,y)-\frac{q^{x-y}(1-q)^y}{x+y}\\
&=I_q(x-2,y)-\frac{q^{x-2}(1-q)^y}{(x-2)B(x-2,y)}\\
&\quad-\frac{q^{x-1}(1-q)^y}{(x-1)B(x-1,y)}\\
&=I_q(x-3,y)-\frac{q^{x-3}(1-q)^y}{(x-3)B(x-3,y)}\\
&\quad-\frac{q^{x-2}(1-q)^y}{(x-2)B(x-2,y)}-\frac{q^{x-1}(1-q)^y}{(x-1)B(x-1,y)}\\
&=\cdots\\
&=I_q(x-m,y)-q^x(1-q)^y\sum_{k=1}^{m}\frac{q^{-k}}{(x-k)B(x-k,y)},\\
&m=1,2,\ldots
\end{aligned}$$

(iii)

$$\begin{aligned}
B_q(x,y)&=\frac{q^x(1-q)^{y-1}}{x}\sum_{k=0}^{\infty}\frac{(1-y)_k}{(1+x)_k}\left(\frac{q}{q-1}\right)^k\\
&=\frac{q^x(1-q)^{y-1}}{x}{}_2\mathrm{F}_1\left(1,1-y;1+x;\frac{q}{q-1}\right)\\
&\quad\text{for}\ \ q\in(-\infty,0)\cup\left(0,\frac{1}{2}\right),\\
B_q(x,y)&=B(x,y)-\frac{q^{x-1}(1-q)^y}{y}\sum_{k=0}^{\infty}\frac{(1-x)_k}{(1+y)_k}\left(\frac{q-1}{q}\right)^k\\
&=B(x,y)-\frac{q^{x-1}(1-q)^y}{y}{}_2\mathrm{F}_1\left(1,1-x;1+y;\frac{q-1}{q}\right)\\
&\quad\text{and}\ B_1(x,y)=B(x,y)\ \ \text{for}\ \ q=1,
\end{aligned}$$

(iv) $(a)_{i+j}=(a)_i(a+i)_j$.

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