Explicit solutions of nonlinear wave equation systems*

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We apply the (G'/G)-expansion method to solve two systems of nonlinear differential equations and construct traveling wave solutions expressed in terms of hyperbolic functions, trigonometric functions, and rational functions with arbitrary parameters. We highlight the power of the (G'/G)-expansion method in providing generalized solitary wave solutions of different physical structures. It is shown that the (G'/G)-expansion method is very effective and provides a powerful mathematical tool to solve nonlinear differential equation systems in mathematical physics.

Keywords: (G'/G)-expansion method, long–short-wave interaction system, coupled integrable dispersionless system

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1. Introduction

The world around us is inherently nonlinear, and nonlinear evolution equations are widely used to describe the complex physical phenomena. The exact solutions of nonlinear partial differential equations (NPDEs) play a significant role in nonlinear science and engineering. In recent years, many researchers who are interested in the nonlinear physical phenomena have investigated exact solutions of NPDEs. They established many powerful and direct methods. For instance, the Lie symmetry method,[1,2] first integral method,[3,4] multiple exp-function method,[5,6] transformed rational function method,[7] Hirota bilinear method,[8,9] sine–cosine method,[10,11] and Jacobi elliptic function method.[12,13]

In this work, we study systems of nonlinear partial differential equations. In order to solve systems of differential equations, there are various techniques that are commonly used. The existing techniques have difficulties related to the size of computational work, especially when the system has several PDEs. In the present work, in order to extend applications of the (G'/G)-expansion method to different problems, this method is described and new families of exact solutions are found.

The main advantage of the proposed expansion method over the basic (G'/G)-expansion method is that it provides more new exact traveling wave solutions along with additional free parameters. All the solutions obtained by the basic (G'/G)-expansion method are thought of as a particular case of the proposed expansion method. In addition, we obtain some new solutions. To clarify the fact, we apply the method in two cases of mathematical physics importance. In both cases, we obtain some additional new exact solutions. The exact solutions have great importance in revealing the internal mechanism of the physical phenomena. Apart from the physical relevance, the close-form solutions of nonlinear evolution equations facilitate the numerical solvers to verify the accuracy of their results, and help them in the stability analysis. In the basic (G'/G)-expansion method, if the order of the reduced ordinary differential equation (ODE) obtained from the PDE by using the traveling wave variable is equal to or less than three, with the help of symbolic computation software, such as Maple 14, it is possible to work out a useful solution of the algebraic equations derived in Step 5 of Section 2. Algebraic manipulation of the proposed expansion method with the help of Maple is much easier than the other methods. Sometimes the method gives solutions in disguised versions of known solutions that may be found by other methods, and if the order of the reduced ordinary differential equation is large enough, then the method is unable to guarantee the existence of solutions of the resultant algebraic equations.

Our aim in this paper is to perform an analytical study on the long–short-wave interaction system and the coupled integrable dispersionless system by means of the (G'/G)-expansion method. Utilizing this method, new families of exact traveling wave solutions are found, which show the reliability and advantages of the method. We describe the method in Section 2, analyze two physically important nonlinear differential equation systems in Section 3, and finally obtain some conclusions in Section 4.

2. Description of the (G'/G)-expansion method for NPDEs

The (G'/G)-expansion method proposed by Wang *et al.*[14] is particularly notable for its power and applicability in solving nonlinear problems, and has been successfully

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applied to solve many kinds of nonlinear partial differential equations.[15–26] All of these applications verified that the (G'/G)-expansion method is a straightforward, efficient, and versatile technique for finding generalized solitary, periodic, and rational solutions for nonlinear evolution equations as well as for revealing intriguing characteristics of various inner-wave interactions.

In this section, we describe the (G'/G)-expansion method to solve nonlinear partial differential equations. Let us consider an NPDE for $u(x,t)$ in the following form:

$$P(u,u_t,u_x,u_{tt},u_{xx},\ldots)=0, \tag{1}$$

where $u(x,t)$ is an unknown function, and P is a polynomial of $u(x,t)$ and its partial derivatives, in which the highest-order derivatives and nonlinear terms are involved. The algorithm of the (G'/G)-expansion method for NPDEs is given as follows:

Step 1 For traveling wave solutions of Eq. (1), we introduce the wave transformation as follows:

$$u(x,t)=U(\xi),\ \xi=x-ct. \tag{2}$$

Based on this wave transformation, equation (1) is reduced to an ordinary differential equation

$$Q(U,-cU',U',c^2U'',-cU'',U'',\ldots)=0, \tag{3}$$

where $U=U(\xi)$, and the prime denotes the derivative with respect to ξ.

Step 2 We assume that the following series expansion is a solution of Eq. (3):

$$U_n(\xi)=\sum_{l=0}^{m}\alpha_l\left(\frac{G'(\xi)}{G(\xi)}\right)^l,\ (\alpha_m\neq 0), \tag{4}$$

where m and α_i are constants to be determined, $G(\xi)$ satisfies a second-order linear ordinary differential equation (LODE) in the form

$$\frac{\mathrm{d}^2G(\xi)}{\mathrm{d}\xi^2}+\lambda\frac{\mathrm{d}G(\xi)}{\mathrm{d}\xi}+\mu G(\xi)=0, \tag{5}$$

where λ and μ are arbitrary constants. Using the general solutions of Eq. (5), we obtain the following cases:

$$\frac{G'(\xi)}{G(\xi)}=\begin{cases}\dfrac{\sqrt{\lambda^2-4\mu}}{2}\left(\dfrac{C_1\sinh(\sqrt{\lambda^2-4\mu}\xi/2)+C_2\cosh(\sqrt{\lambda^2-4\mu}\xi/2)}{C_1\cosh(\sqrt{\lambda^2-4\mu}\xi/2)+C_2\sinh(\sqrt{\lambda^2-4\mu}\xi/2)}\right)-\dfrac{\lambda}{2}, & (\lambda^2-4\mu>0),\\ \dfrac{\sqrt{4\mu-\lambda^2}}{2}\left(\dfrac{-C_1\sin(\sqrt{4\mu-\lambda^2}\xi/2)+C_2\cos(\sqrt{4\mu-\lambda^2}\xi/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}\xi/2)+C_2\sin(\sqrt{4\mu-\lambda^2}\xi/2)}\right)-\dfrac{\lambda}{2}, & (\lambda^2-4\mu<0),\\ \dfrac{C_1}{C_1\xi+C_2}-\dfrac{\lambda}{2}, & (\lambda^2-4\mu=0),\end{cases} \tag{6}$$

where C_1 and C_2 are arbitrary constants.

Step 3 Positive integer m can be determined by following the homogeneous balance procedure. Balancing the highest-order nonlinear term(s) and the highest-order derivatives of $U(\xi)$ in Eq. (3), we can determine the degree m of Eq. (4).

Step 4 Substituting Eqs. (4) and (5) into Eq. (3), and collecting all terms with the same order of $G'(\xi)/G(\xi)$ together, the left-hand side of Eq. (3) is converted into a polynomial in $G'(\xi)/G(\xi)$. Setting the coefficients of this polynomial to be zero, we obtain a system of nonlinear algebraic equations, from which the constants α_l can be explicitly determined by using Maple. Finally, we substitute these values into Eq. (4), and obtain exact solutions of Eq. (1).

3. Applications

In this section, we apply the (G'/G)-expansion method discussed in Section 2 to solve two NPDE systems.

3.1. Long–short-wave interaction system

Let us first apply the above method to the following long–short-wave interaction system:[27]

$$\begin{aligned}&\mathrm{i}\psi_t+\psi_{xx}-\psi\upsilon=0,\\ &\upsilon_t+\upsilon_x+(|\psi|^2)_x=0,\end{aligned} \tag{7}$$

where $\psi(x,t)$ is a complex function, $\upsilon(x,t)$ is a real function, x is the spatial coordinate, and t is the time. This system describes the interaction between one long longitudinal wave and one short transverse wave propagating in a generalized elastic medium. The $\upsilon(x,t)$ characterizes the long longitudinal wave and $\psi(x,t)$ is the slowly varying envelope of the short transverse wave. Wang *et al.*[28] obtained periodic wave solutions of the long–short-wave interaction system by using the F-expansion method. We use the following transformation:

$$\begin{aligned}&\psi(x,t)=\mathrm{e}^{\mathrm{i}\xi}u(x,t),\ \xi=\alpha x+\beta t,\\ &\upsilon(x,t)=V(\xi),\ u(x,t)=U(\xi),\ \xi=x-\omega t,\end{aligned} \tag{8}$$

where α and β are real constants, and $u(x,t)$ is a real function. Substituting Eq. (8) into Eq. (7), we find

$$\omega=2\alpha,\quad \beta=-2\alpha^2, \tag{9}$$

and equation (7) turns into the following form:

$$\alpha^2U''(\xi)-U(\xi)V(\xi)+\alpha^2U(\xi)=0, \tag{10}$$

$$(1-2\alpha)V'(\xi)+2\alpha U(\xi)U'(\xi)=0, \tag{11}$$

where

$$U(\xi)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'(\xi)}{G(\xi)}\right)^m,\ (\alpha_m\neq 0), \tag{12}$$

$$V(\xi)=\sum_{i=0}^{n}\beta_i\Big(\frac{G'(\xi)}{G(\xi)}\Big)^n,\ (\beta_n\neq 0). \tag{13}$$

Here, m and n are the balancing terms of $(U(\xi))$ and $(V(\xi))$, respectively. Then, according to the homogeneous balance procedure, balancing the nonlinear term $(U(\xi)V(\xi))$ in Eq. (10) with the highest-order derivative term $(U''(\xi))$, we find

$$m+n=m+2. \tag{14}$$

In Eq. (11), balancing the nonlinear term $(U(\xi)U'(\xi))$ with the highest-order derivative term $(V'(\xi))$, we find

$$m+(m+1)=n+1. \tag{15}$$

Thus, from Eqs. (14) and (15), we obtain the balancing terms as follows:

$$m=1,\ n=2. \tag{16}$$

Substituting these terms into Eqs. (12) and (13), we obtain

$$U(\xi)=\alpha_0+\alpha_1\Big(\frac{G'(\xi)}{G(\xi)}\Big),\ (\alpha_1\neq 0), \tag{17}$$

$$V(\xi)=\beta_0+\beta_1\Big(\frac{G'(\xi)}{G(\xi)}\Big)+\beta_2\Big(\frac{G'(\xi)}{G(\xi)}\Big)^2,\ (\beta_2\neq 0), \tag{18}$$

for the travelling wave solutions of Eqs. (10) and (11). Substituting Eqs. (17) and (18) together with Eq. (5) into Eqs. (10) and (11), and setting the coefficients of $(G'(\xi)/(G(\xi)))^i (0\leq i\leq 3)$ to be zero, we obtain an algebraic system for α_0, α_1, β_0, β_1, and α. Solving this system by use of Maple, we find

$$\begin{aligned}&\alpha_0=\mp\lambda\sqrt{\alpha^2-\frac{\alpha}{2}},\ \alpha_1=\mp 2\sqrt{\alpha^2-\frac{\alpha}{2}},\\ &\beta_0=\alpha^2(2\mu+1),\ \beta_1=2\alpha^2\lambda,\ \beta_2=2\alpha^2,\\ &\alpha=\alpha.\end{aligned} \tag{19}$$

Thus, we obtain solutions for Eqs. (10) and (11) in different cases. Let us consider the following cases.

Case 1 When $\lambda^2-4\mu>0$, we obtain hyperbolic function solution of Eqs. (7) as

$$u(x,t)=\mp\sqrt{\alpha^2-\frac{\alpha}{2}}\sqrt{\lambda^2-4\mu}\left(\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)+C_2\cosh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)}{C_1\cosh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)+C_2\sinh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)}\right), \tag{20}$$

$$\psi(x,t)=\mp e^{i(\alpha x-\alpha^2 t)}\sqrt{(\alpha^2-\frac{\alpha}{2})(\lambda^2-4\mu)}\left(\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)+C_2\cosh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)}{C_1\cosh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)+C_2\sinh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)}\right), \tag{21}$$

and

$$\upsilon(x,t)=\alpha^2+\alpha^2\frac{\lambda^2-4\mu}{2}\left(\left(\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)+C_2\cosh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)}{C_1\cosh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)+C_2\sinh(\sqrt{\lambda^2-4\mu}(x-2\alpha t)/2)}\right)^2-1\right), \tag{22}$$

where C_1 and C_2 are arbitrary constants.

As a special example, when $\lambda^2-4\mu=4$ and $C_2=0$, the solution ψ of Eq. (21) becomes

$$\psi(x,t)=\mp 2e^{i(\alpha x-\alpha^2 t)}\sqrt{\alpha^2-\frac{\alpha}{2}}\tanh(x-2\alpha t). \tag{23}$$

When $\lambda^2-4\mu=4$ and $C_1=0$, the solution ψ of Eq. (21) becomes

$$\psi(x,t)=\mp 2e^{i(\alpha x-\alpha^2 t)}\sqrt{\alpha^2-\frac{\alpha}{2}}\coth(x-2\alpha t). \tag{24}$$

When $\lambda^2-4\mu=4$ and $C_2=0$, the solution υ of Eq. (22) becomes

$$\upsilon(x,t)=\alpha^2(1-2\,\mathrm{sech}^2(x-2\alpha t)). \tag{25}$$

When $\lambda^2-4\mu=4$ and $C_1=0$, the solution υ of Eq. (22) becomes

$$\upsilon(x,t)=\alpha^2(2\,\mathrm{cosech}^2(x-2\alpha t)+1). \tag{26}$$

Case 2 When $\lambda^2-4\mu<0$, we obtain trigonometric function solutions of Eq. (7) as

$$u(x,t)=\mp\sqrt{\alpha^2-\frac{\alpha}{2}}\sqrt{4\mu-\lambda^2}\left(\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)+C_2\cos(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)+C_2\sin(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)}\right), \tag{27}$$

$$\psi(x,t)=\mp e^{i(\alpha x-\alpha^2 t)}\sqrt{\left(\alpha^2-\frac{\alpha}{2}\right)(4\mu-\lambda^2)}\left(\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)+C_2\cos(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)+C_2\sin(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)}\right), \tag{28}$$

and

$$\upsilon(x,t)=\alpha^2+\alpha^2\frac{4\mu-\lambda^2}{2}\left(\left(\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)+C_2\cos(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)+C_2\sin(\sqrt{4\mu-\lambda^2}(x-2\alpha t)/2)}\right)^2+1\right), \tag{29}$$

where C_1 and C_2 are arbitrary constants.

As a special example, if we take $4\mu-\lambda^2=4$ and $C_2=0$, the solution ψ of Eq. (28) becomes

$$\psi(x,t)=\pm 2\mathrm{e}^{\mathrm{i}(\alpha x-\alpha^2 t)}\sqrt{\alpha^2-\frac{\alpha}{2}}\tan(x-2\alpha t). \tag{30}$$

When $4\mu-\lambda^2=4$ and $C_1=0$, the solution ψ of Eq. (28) becomes

$$\psi(x,t)=\pm 2\mathrm{e}^{\mathrm{i}(\alpha x-\alpha^2 t)}\sqrt{\alpha^2-\frac{\alpha}{2}}\cot(x-2\alpha t). \tag{31}$$

When $4\mu-\lambda^2=4$ and $C_2=0$, the solution υ of Eq. (29) becomes

$$\upsilon(x,t)=\alpha^2(1+2\sec^2(x-2\alpha t)). \tag{32}$$

When $4\mu-\lambda^2=4$ and $C_1=0$, the solution υ of Eq. (29) becomes

$$\upsilon(x,t)=\alpha^2(1+2\operatorname{cosec}^2(x-2\alpha t)). \tag{33}$$

Case 3 When $\lambda^2-4\mu=0$, we obtain the rational function solution of Eq. (7) as

$$u(x,t)=\mp 2\sqrt{\alpha^2-\frac{\alpha}{2}}\frac{C_1}{C_1(x-2\alpha t)+C_2}, \tag{34}$$

$$\psi(x,t)=\mp 2\mathrm{e}^{\mathrm{i}(\alpha x-\alpha^2 t)}\sqrt{\alpha^2-\frac{\alpha}{2}}\frac{C_1}{C_1(x-2\alpha t)+C_2}, \tag{35}$$

and

$$\upsilon(x,t)=\alpha^2+\frac{2\alpha^2C_1^2}{(C_1(x-2\alpha t)+C_2)^2}, \tag{36}$$

where C_1 and C_2 are arbitrary constants.

3.2. Coupled integrable dispersionless system

We next consider a coupled integrable dispersionless system of the following form:[29]

$$u_{xt}+(\upsilon\omega)_x=0,\ \upsilon_{xt}-2\upsilon u_x=0,\ \omega_{xt}-2\omega u_x=0. \tag{37}$$

Coupled integrable dispersionless equations are presented and solved based on the inverse scattering method proposed in Ref. [30]. Using the traveling wave transformation

$$\begin{aligned} &u(x,t)=U(\xi),\ \upsilon(x,t)=V(\xi),\\ &\omega(x,t)=W(\xi),\ \xi=x-ct, \end{aligned} \tag{38}$$

where c is the wave speed and $c\neq 0$, equations (37) become

$$cU''(\xi)-(V(\xi)W(\xi))'=0 \tag{39}$$

$$cV''(\xi)+2V(\xi)U'(\xi)=0 \tag{40}$$

$$cW''(\xi)+2W(\xi)U'(\xi)=0. \tag{41}$$

Here, the prime denotes the derivation with respect to ξ. Integrating Eq. (39), we obtain

$$U'(\xi)=\frac{1}{c}V(\xi)W(\xi)+A, \tag{42}$$

where A is the integration constant. Substituting Eq. (42) into Eqs. (40) and (41) yields

$$V''(\xi)+\frac{2}{c^2}V^2(\xi)W(\xi)+BV(\xi)=0, \tag{43}$$

$$W''(\xi)+\frac{2}{c^2}V(\xi)W^2(\xi)+BW(\xi)=0, \tag{44}$$

where $B=2A/c$. Multiplying Eqs. (43) and (44) by $V(\xi)$ and $W(\xi)$, and subtracting the latter from the former, we obtain

$$V(\xi)W''(\xi)-W(\xi)V''(\xi)=0. \tag{45}$$

From Eq. (45), we obtain

$$(V(\xi)W'(\xi)-W(\xi)V'(\xi))'=0, \tag{46}$$

or

$$\left(\frac{W(\xi)}{V(\xi)}\right)'=0, \tag{47}$$

and equation (47) can be rewritten as follows:

$$W(\xi)=DV(\xi), \tag{48}$$

where D is the integration constant. Substituting Eq. (48) into Eqs. (45) and (46), we obtain the second-order nonlinear ordinary differential equation as

$$V''(\xi)+\frac{2A}{c}V(\xi)+\frac{2D}{c^2}V^3(\xi)=0. \tag{49}$$

Now we can solve this equation using the (G'/G)-expansion method. Let

$$V(\xi)=\sum_{i=0}^{m}\alpha_i\left(\frac{G'(\xi)}{G(\xi)}\right)^m,\ (\alpha_m\neq 0), \tag{50}$$

where m is the balancing term of $V(\xi)$. Then, according to the homogeneous balance procedure, balancing the nonlinear term $V^3(\xi)$ in Eq. (49) with the highest-order derivative term $V''(\xi)$, we find

$$3m=m+2, \tag{51}$$

$$m=1. \tag{52}$$

Substituting this balancing term into Eq. (50), we obtain

$$V(\xi)=\alpha_0+\alpha_1\left(\frac{G'(\xi)}{G(\xi)}\right),\ \alpha_1\neq 0, \tag{53}$$

for the traveling wave solution of Eq. (49). Substituting Eqs. (53) and (5) into Eq. (49), and setting the coefficients of $(G'(\xi)/(G(\xi)))^{\mathrm{i}}$ $(0\leq i\leq 3)$ to be zero, we obtain an algebraic system for α_0, α_1, c, and A. Solving this system by use of Maple, we find

$$\begin{aligned} &\alpha_0=\mp\frac{\lambda c}{2}\sqrt{-\frac{1}{D}},\ \alpha_1=\mp c\sqrt{-\frac{1}{D}},\\ &A=\frac{c(\lambda^2-4\mu)}{4},\ \ c=c. \end{aligned} \tag{54}$$

Thus, we obtain solutions of Eq. (49) in different cases. Let us consider the following cases.

Case 1 When $\lambda^2-4\mu>0$, we obtain the hyperbolic function solution of Eq. (49) as

$$V(\xi)=\mp c\sqrt{-\frac{1}{D}}\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}\xi/2)+C_2\cosh(\sqrt{\lambda^2-4\mu}\xi/2)}{C_1\cosh(\sqrt{\lambda^2-4\mu}\xi/2)+C_2\sinh(\sqrt{\lambda^2-4\mu}\xi/2)}\right), \tag{55}$$

$$W(\xi)=\mp cD\sqrt{-\frac{1}{D}}\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}\xi/2)+C_2\cosh(\sqrt{\lambda^2-4\mu}\xi/2)}{C_1\cosh(\sqrt{\lambda^2-4\mu}\xi/2)+C_2\sinh(\sqrt{\lambda^2-4\mu}\xi/2)}\right). \tag{56}$$

Substituting Eqs. (55) and (56) into Eq. (42), we obtain

$$U'(\xi)=\frac{c(\lambda^2-4\mu)}{4}\left(1-\left(\frac{C_1\tanh(\sqrt{\lambda^2-4\mu}\xi/2)+C_2}{C_1+C_2\tanh(\sqrt{\lambda^2-4\mu}\xi/2)}\right)^2\right). \tag{57}$$

From Eq. (57), we can find $U(\xi)$ in the following form:

$$U(\xi)=\frac{c\sqrt{\lambda^2-4\mu}}{2}\left(\frac{(-C_1^2/C_2^2+C_2)}{C_1+C_2(\tanh(\sqrt{\lambda^2-4\mu}\xi/2))}+\frac{\sqrt{\lambda^2-4\mu}\xi}{2}+\frac{1}{2}\ln\left(\frac{\tanh(\sqrt{\lambda^2-4\mu}\xi/2)-1}{\tanh(\sqrt{\lambda^2-4\mu}\xi/2)+1}\right)\right). \tag{58}$$

Thus, when $\lambda^2-4\mu>0$, the traveling wave solutions for Eqs. (37) is obtained

$$\upsilon(x,t)=\mp ic\sqrt{\frac{1}{D}}\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}(x-ct)/2)+C_2\cosh(\sqrt{\lambda^2-4\mu}(x-ct)/2)}{C_1\cosh(\sqrt{\lambda^2-4\mu}(x-ct)/2)+C_2\sinh(\sqrt{\lambda^2-4\mu}(x-ct)/2)}\right), \tag{59}$$

$$\omega(x,t)=\mp ic\sqrt{D}\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\sqrt{\lambda^2-4\mu}(x-ct)/2)+C_2\cosh(\sqrt{\lambda^2-4\mu}(x-ct)/2)}{C_1\cosh(\sqrt{\lambda^2-4\mu}(x-ct)/2)+C_2\sinh(\sqrt{\lambda^2-4\mu}(x-ct)/2)}\right), \tag{60}$$

$$u(x,t)=\frac{c\sqrt{\lambda^2-4\mu}}{2}\left[\frac{(-C_1^2/C_2^2+C_2)}{C_1+C_2(\tanh(\sqrt{\lambda^2-4\mu}(x-ct)/2))}+\frac{\sqrt{\lambda^2-4\mu}(x-ct)}{2}\right.$$
$$\left.+\frac{1}{2}\ln\left(\frac{\tanh(\sqrt{\lambda^2-4\mu}(x-ct)/2)-1}{\tanh(\sqrt{\lambda^2-4\mu}(x-ct)/2)+1}\right)\right]+C_3, \tag{61}$$

where C_3 is an integration constant, and C_1 and C_2 are arbitrary constants.

As a special example, when $\lambda^2-4\mu=4$ and $C_2=0$, the solution υ of Eq. (59) becomes

$$\upsilon(x,t)=\mp ic\sqrt{\frac{1}{D}}\tanh(x-ct). \tag{62}$$

When $\lambda^2-4\mu=4$ and $C_1=0$, the solution υ of Eq. (59) becomes

$$\upsilon(x,t)=\mp ic\sqrt{\frac{1}{D}}\coth(x-ct). \tag{63}$$

When $\lambda^2-4\mu=4$ and $C_2=0$, the solution ω of Eq. (60) becomes

$$\omega(x,t)=\mp ic\sqrt{D}\tanh(x-ct). \tag{64}$$

When $\lambda^2-4\mu=4$ and $C_1=0$, the solution ω of Eq. (60) becomes

$$\omega_+(x,t)=ic\sqrt{D}\coth(x-ct). \tag{65}$$

When $\lambda^2-4\mu=4$ and $C_1=0$, the solution u of Eq. (61) becomes

$$u(x,t)=c\left(\frac{C_2}{C_2\tanh(x-ct)}+(x-ct)+\frac{1}{2}\ln\left(\frac{\tanh(x-ct)-1}{\tanh(x-ct)+1}\right)\right)+C_3. \tag{66}$$

Case 2 When $\lambda^2-4\mu<0$, we obtain the trigonometric function solution of Eq. (49) as

$$V(\xi)=\mp c\sqrt{-\frac{1}{D}}\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}\xi/2)+C_2\cos(\sqrt{4\mu-\lambda^2}\xi/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}\xi/2)+C_2\sin(\sqrt{4\mu-\lambda^2}\xi/2)}\right), \tag{67}$$

$$W(\xi)=\mp cD\sqrt{-\frac{1}{D}}\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}\xi/2)+C_2\cos(\sqrt{4\mu-\lambda^2}\xi/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}\xi/2)+C_2\sin(\sqrt{4\mu-\lambda^2}\xi/2)}\right). \tag{68}$$

Substituting Eqs. (67) and (68) into Eq. (42), we obtain

$$U'(\xi)=\frac{c(\lambda^2-4\mu)}{4}\left(\left(\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}\xi/2)+C_2\cos(\sqrt{4\mu-\lambda^2}\xi/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}\xi/2)+C_2\sin(\sqrt{4\mu-\lambda^2}\xi/2)}\right)^2+1\right). \tag{69}$$

From Eq. (69), we can derive $U(\xi)$ in the following form:

$$U(\xi) = \frac{c\sqrt{4\mu-\lambda^2}(C_2^2+C_1^2)}{2C_2(C_1+C_2\tan(\sqrt{4\mu-\lambda^2}\xi/2))} + C_3, \tag{70}$$

where C_3 is an integration constant, and C_1 and C_2 are arbitrary constants.

Thus, when $\lambda^2-4\mu<0$, the traveling wave solution for Eqs. (37) is found as follows:

$$\upsilon(x,t) = \mp ic\sqrt{\frac{1}{D}}\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}(x-ct)/2)+C_2\cos(\sqrt{4\mu-\lambda^2}(x-ct)/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}(x-ct)/2)+C_2\sin(\sqrt{4\mu-\lambda^2}(x-ct)/2)}\right), \tag{71}$$

$$\omega(x,t) = \mp ic\sqrt{D}\frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\sqrt{4\mu-\lambda^2}(x-ct)/2)+C_2\cos(\sqrt{4\mu-\lambda^2}(x-ct)/2)}{C_1\cos(\sqrt{4\mu-\lambda^2}(x-ct)/2)+C_2\sin(\sqrt{4\mu-\lambda^2}(x-ct)/2)}\right), \tag{72}$$

$$u(x,t) = \frac{c\sqrt{4\mu-\lambda^2}(C_2^2+C_1^2)}{2C_2(C_1+C_2\tan(\sqrt{4\mu-\lambda^2}(x-ct)/2))} + C_3. \tag{73}$$

As a special example, when $4\mu-\lambda^2=4$ and $C_2=0$, the solution υ of Eq. (71) becomes

$$\upsilon(x,t) = \pm ic\sqrt{\frac{1}{D}}\tan(x-ct). \tag{74}$$

When $4\mu-\lambda^2=4$ and $C_1=0$, the solution υ of Eq. (71) becomes

$$\upsilon(x,t) = \mp ic\sqrt{\frac{1}{D}}\cot(x-ct). \tag{75}$$

When $4\mu-\lambda^2=4$ and $C_2=0$, the solution ω of Eq. (72) becomes

$$\omega(x,t) = \pm ic\sqrt{D}\tan(x-ct). \tag{76}$$

When $4\mu-\lambda^2=4$ and $C_1=0$, the solution ω of Eq. (72) becomes

$$\omega(x,t) = \mp ic\sqrt{D}\cot(x-ct). \tag{77}$$

When $4\mu-\lambda^2=4$ and $C_1=0$, the solution ω of Eq. (73) becomes

$$u(x,t) = c\cot(x-ct)+C_3. \tag{78}$$

Case 3 When $\lambda^2-4\mu=0$, we obtain the rational function solution of Eq. (49) as

$$V(\xi) = \pm ic\sqrt{\frac{1}{D}}\frac{C_1}{C_1\xi+C_2}, \tag{79}$$

$$W(\xi) = \pm ic\sqrt{D}\frac{C_1}{C_1\xi+C_2}. \tag{80}$$

Substituting Eqs. (79) and (80) into Eq. (42), we obtain

$$U'(\xi) = -c\frac{C_1^2}{(C_1\xi+C_2)^2}. \tag{81}$$

From Eq. (81), we find

$$U(\xi) = c\frac{C_1}{C_1\xi+C_2}+C_3, \tag{82}$$

where C_3 is an integration constant, and C_1 and C_2 are arbitrary constants. Thus, the rational function solution of Eq. (37) is obtained

$$\upsilon(x,t) = \pm ic\sqrt{\frac{1}{D}}\frac{C_1}{C_1(x-ct)+C_2}, \tag{83}$$

$$\omega(x,t) = \pm ic\sqrt{D}\frac{C_1}{C_1(x-ct)+C_2}, \tag{84}$$

$$u(x,t) = c\frac{C_1}{C_1(x-ct)+C_2}+C_3, \tag{85}$$

where C_1 and C_2 are arbitrary constants.

It is worth noting that the rational solutions (34)–(36) and (83)–(85) are presented here for the first time.

4. Conclusion

The solutions of these nonlinear evolution equations have many potential applications in physics. The performance of this method is reliable, simple and gives many new exact solutions. Though the obtained solutions represent only a small part of the large variety of possible solutions for the equations considered, they might serve as seeding solutions for a class of localized structures existing in the physical systems. Our solutions are in more general forms, and many known solutions to these equations are only special cases of them. With the aid of Maple and Mathematica, we have verified the correctness of the obtained solutions by putting them back into the original equation. The solutions contain free parameters. These solutions will be very useful in various physical situations where these equations arise. We have noted that the (G'/G)-expansion method changes the given difficult problems into simple problems which can be solved easily. We hope that they will be helpful for further studies in applied sciences.

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