Analytical approximate solution for nonlinear space–time fractional Klein–Gordon equation

Khaled A. Gepreel[a)b)†] and Mohamed S. Mohamed[b)c)]

[a)] *Mathematics Department, Faculty of Science, Zagazig University, Zagazig, Egypt*
[b)] *Mathematics Department, Faculty of Science, Taif University, Taif, Saudi Arabia*
[c)] *Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City* 11884, *Cairo, Egypt*

(Received 29 May 2012; revised manuscript received 26 June 2012)

The fractional derivatives in the sense of Caputo and the homotopy analysis method are used to construct an approximate solution for the nonlinear space–time fractional derivatives Klein–Gordon equation. The numerical results show that the approaches are easy to implement and accurate when applied to the nonlinear space–time fractional derivatives Klein–Gordon equation. This method introduces a promising tool for solving many space–time fractional partial differential equations. This method is efficient and powerful in solving wide classes of nonlinear evolution fractional order equations.

Keywords: homotopy analysis method, nonlinear space–time fractional Klein–Gordon equation, Caputo derivative

PACS: 02.30.Jr **DOI:** 10.1088/1674-1056/22/1/010201

1. Introduction

The seeds of fractional calculus (that is, the theory of integrals and derivatives of arbitrary real or complex orders) were planted over 300 years ago. Fractional differential equations (FDEs) have found applications in many problems in physics and engineering.[1–4] Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Some of the analytical methods for solving nonlinear problems include the Adomian decomposition method (ADM),[5–7] the variational iteration method (VIM),[8] the homotopy-perturbation method (HPM),[9–11] and the homotopy analysis method (HAM).[12–17] The HAM, first proposed in 1992 by Liao,[12] has been successfully applied to solve many problems in physics and science. Deng *et al*.[18,19] used the finite difference methods and the finite element method to solve the fractional Klein–Kramers equations and the fractional Fokker–Planck equation, respectively. Recently Gepreel *et al*.[20] used the fractional complex transformation to obtain the exact solutions for some nonlinear partial fractional differential equations. There are many methods for obtaining analytic approximate solutions for nonlinear fractional differential equations, such as the Adomian decomposition method, the variational iteration method, the homotopy-perturbation method, and the homotopy analysis method. The homotopy anaylsis method is the generalized method and contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of the solution series.

In this article, we use the HAM to calculate the analytic approximate solution to the nonlinear space–time fractional derivative Klein–Gordon equation[21]

$$\begin{aligned}&\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^\beta u}{\partial x^\beta}+u^3\\&=\Gamma(\alpha+1)x^\beta-\Gamma(\beta+1)t^\alpha+x^{3\beta}t^{3\alpha},\\&(t>0,\ 1<\alpha,\ \beta\le 2).\end{aligned}\tag{1}$$

We study the behavior of the approximate solution to the nonlinear space–time fractional derivative Klein–Gordon equation (1) and we determine the convergence region of the approximate solutions.

2. Preliminaries and notations

In this section, we give some definitions and properties of the fractional calculus. Several definitions of fractional calculus have been proposed in the last two centuries. The fractional calculus is the theory of integrals and derivatives of arbitrary order. There are many researchers[1–4] who have developed the fractional calculus and given various definitions of fractional integration and differentiation, such as the Grunwald–Letnikov definition, Riemann–Liouville definition, Caputo's definition, and the generalized function approach.

Definition 1 A real function $h(t), t>0$ is said to be in the space C_μ, $\mu\in\mathbb{R}$ if there exists a real number $p>\mu$ such that $h(t)=t^p h_1(t)$, where $h_1(t)\in \mathrm{C}[0,\infty)$, and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu$, $n\in\mathbb{N}$.

Definition 2 The Riemann–Liouville fractional integral operator J^α of order $\alpha\ge 0$ of function $h\in C_\mu$, $\mu\ge -1$ is defined as

$$\begin{aligned}\mathrm{J}^\alpha h(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(\tau)\,\mathrm{d}\tau,\quad(\alpha>0),\\\mathrm{J}^0 h(t)&=h(t),\end{aligned}\tag{2}$$

†Corresponding author. E-mail: kagepreel@yahoo.com

© 2013 Chinese Physical Society and IOP Publishing Ltd

where $\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of operator J^{α} are

(i) $\mathrm{J}^{\alpha}\mathrm{J}^{\beta}h(t) = \mathrm{J}^{\alpha+\beta}h(t)$,

(ii) $\mathrm{J}^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$, where $\alpha \geqslant 0$, $\gamma \geqslant -1$.

Definition 3 The fractional derivative D^{α} of $h(t)$ in the Caputo's sense is defined as

$$\mathrm{D}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}h^{(n)}(\tau)\,\mathrm{d}\tau, \tag{3}$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$, $h \in C_{-1}^{n}$.

The following are the basic properties of Caputo's fractional derivative:

(i) let $h \in C_{-1}^{n}$, $n \in \mathbb{N}$, then $\mathrm{D}^{\alpha}h$, $0 \leq \alpha \leq n$ is well defined and $\mathrm{D}^{\alpha}h \in C_{-1}$;

(ii) let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $h \in C_{\mu}^{n}$, $\mu \geq -1$, then

$$(\mathrm{J}^{\alpha}\mathrm{D}^{\alpha})h(t) = h(t) - \sum_{k=0}^{n-1} h^{(k)}(0^{+})\frac{t^{k}}{k!}. \tag{4}$$

3. Basic ideas of the homotopy analysis method

To describe the basic ideas of the HAM, we consider the following differential equation:

$$\mathrm{N}[\mathrm{D}_t^{\alpha}u(x,t)] = 0, \tag{5}$$

where N is a nonlinear operator for this problem, D_t^{α} stands for the fractional derivative, x and t denotes the independent variables, and $u(x,t)$ is an unknown function.

By using the HAM, we first construct the zero-order deformation equation

$$\begin{aligned}&(1-q)\mathscr{L}(\phi(x,t;q) - u_0(x,t))\\ &= qhH(t)\mathrm{N}[\mathrm{D}_t^{\alpha}\phi(x,t,q)],\end{aligned} \tag{6}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $\mathscr{L}$ is an auxiliary linear operator, and $u_0(x,t)$ is the initial guess. Obviously, when $q = 0$ and $q = 1$, we have

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t). \tag{7}$$

Liao[12,13] expanded $\phi(x,t;q)$ in the Taylor series with respect to the embedding parameter q

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{8}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}|_{q=0}. \tag{9}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function $H(t)$ are selected such that series (8) is convergent at $q = 1$, then we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{10}$$

Let us define the vector

$$\boldsymbol{u}_n(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\}. \tag{11}$$

Differentiating Eq. (6) m times with respect to q, then setting $q = 0$ and dividing then by $m!$, we have the m-th order deformation equation

$$\mathscr{L}(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathscr{R}_m(\vec{u_{m-1}}), \tag{12}$$

where

$$\mathscr{R}_m(\vec{u_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{N}[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{13}$$

$$\varkappa_m = \begin{cases}0, & m \leq 1,\\ 1, & m > 1.\end{cases} \tag{14}$$

Applying the Riemann–Liouville integral operator J^{α} on both sides of Eq. (12), we have

$$\begin{aligned}u_m(x,t) \;=\; &\varkappa_m u_{m-1}(x,t) - \varkappa_m\sum_{i=0}^{n-1} u_{m-1}^{i}(0^{+})\frac{t^{i}}{i!}\\ &+ hH(t)\mathrm{J}^{\alpha}\mathscr{R}_m(\boldsymbol{u}_{m-1}).\end{aligned} \tag{15}$$

The m-th order deformation Eq. (12) is linear and thus can be easily solved, especially by means of symbolic computation software such as Mathematica, Maple, MathLab.

4. The HAM for the nonlinear fractional-order cubically nonlinear Klein–Gordon equation

In this section, to demonstrate the effectiveness of our approach, we apply the HAM to construct approximate solutions for the nonlinear space–time fractional derivative Klein–Gordon equation

$$\begin{aligned}&\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - \frac{\partial^{\beta}u}{\partial x^{\beta}} + u^3\\ &= \Gamma(\alpha+1)x^{\beta} - \Gamma(\beta+1)t^{\alpha} + x^{3\beta}t^{3\alpha},\\ &(t > 0,\ 1 < \alpha,\ \beta \leq 2),\end{aligned} \tag{16}$$

with the initial condition

$$u(x,0) = 0, \quad u_t(x,0) = 0.$$

The exact solution is

$$u(x,t) = x^{\beta}t^{\alpha}. \tag{17}$$

By means of HAM, we choose the linear operator

$$\mathscr{L}[\phi(x,t;q)] = \frac{\partial\phi(x,t;q)}{\partial t}, \tag{18}$$

with property $\mathscr{L}[c] = 0$, where c is a constant. We define a nonlinear operator as

$$\mathrm{N}[\phi(x,t;q)] \;=\; \frac{\partial^{\alpha}\phi(x,t;q)}{\partial t^{\alpha}} - \frac{\partial^{\beta}\phi(x,t;q)}{\partial x^{\beta}} + \phi^3(x,t;q)$$

$$-\Gamma(\alpha+1)x^{\beta}+\Gamma(\beta+1)t^{\alpha}-x^{3\beta}t^{3\alpha}. \quad (19)$$

We construct the zeroth-order deformation equation

$$(1-q)\mathscr{L}(\phi(x,t;q)-u_0(x,t))=qhH(t)\mathrm{N}[\phi(x,t,q)].$$

For $q=0$ and $q=1$, we have

$$\begin{aligned}\phi(x,t;0) &= u_0(x,t)=u(x,0),\\ \phi(x,t;1) &= u(x,t).\end{aligned} \quad (20)$$

Thus, we obtain the m-th order deformation equation

$$\mathscr{L}(u_m(x,t)-\varkappa_m u_{m-1}(x,t))=hH(t)\mathscr{R}_m(u_{m-1}),$$

where

$$\begin{aligned}\mathscr{R}_m &= \mathrm{D}_t^{\alpha}(u_{m-1})-\frac{\partial^{\beta}u_{m-1}}{\partial x^{\beta}}+\sum_{i=0}^{m-1}\sum_{j=0}^{i}u_j u_{i-j}u_{m-1-i}\\ &\quad -(\Gamma(\alpha+1)x^{\beta}-\Gamma(\beta+1)t^{\alpha}+x^{3\beta}t^{3\alpha})(1-\varkappa_m). \quad (21)\end{aligned}$$

The auxiliary function can be determined uniquely $H(t)=1$.

Now the solution of the m-th order deformation equation (21) for $m\geq 1$ becomes

$$u_m(x,t)=\varkappa_m u_{m-1}(x,t)+h\mathrm{J}^{\alpha}\mathscr{R}_m(u_{m-1}), \quad (22)$$

and so on. We substitute the initial condition (17) into the system (22) with the aid of Maple; the approximate solutions of Eq. (16) take the following form:

$$\begin{aligned}u_1(x,t) &= h\left[-t^{\alpha}x^{\beta}+\frac{\Gamma(\beta+1)\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}\right.\\ &\quad \left.-\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}x^{3\beta}t^{4\alpha}\right], \quad (23)\\ u_2(x,t) &= h(1+h)\left[-t^{\alpha}x^{\beta}+\frac{\Gamma(\beta+1)\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}\right.\\ &\quad \left.-\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}x^{3\beta}t^{4\alpha}\right]+h^2\left[\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}\right.\\ &\quad \left.+\frac{\Gamma(3\beta+1)\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)\Gamma(2\beta+1)}x^{2\beta}t^{5\alpha}\right],\\ &\quad \cdots \quad (24)\end{aligned}$$

According to the HAM, we can conclude that

$$\begin{aligned}u_{\mathrm{app}} &= u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots\\ &= h\left[-t^{\alpha}x^{\beta}+\frac{\Gamma(\beta+1)\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}\right.\\ &\quad \left.-\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}x^{3\beta}t^{4\alpha}\right]+h(1+h)\\ &\quad \times\left[-t^{\alpha}x^{\beta}+\frac{\Gamma(\beta+1)\Gamma(\alpha+1)t^{2\alpha}}{\Gamma(2\alpha+1)}\right.\\ &\quad \left.-\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}x^{3\beta}t^{4\alpha}\right]+h^2\left[\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}\right.\\ &\quad \left.+\frac{\Gamma(3\beta+1)\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)\Gamma(2\beta+1)}x^{2\beta}t^{5\alpha}\right]. \quad (25)\end{aligned}$$

In the special case when we set $h=-1$, we get accurate approximation solutions of Eq. (16) in the following forms.

Case 1 When $\alpha\in(1,2]$ and $\beta=2$, the approximation solution (19) takes the following form:

$$\begin{aligned}u_{\mathrm{app}} &= t^{\alpha}x^2-2t^{2\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}+t^{4\alpha}x^6\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\\ &\quad +30t^{5\alpha}x^4\frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}+2t^{2\alpha}\frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}+\cdots. \quad (26)\end{aligned}$$

Case 2 When $\beta\ \varepsilon(1,2]$ and $\alpha=2$, the approximation solution (19) takes the following form:

$$\begin{aligned}u_{\mathrm{app}} &= t^2x^{\beta}+\frac{1}{56}t^8x^{3\beta}-\frac{1}{12}\Gamma(\beta+1)t^4\\ &\quad +\frac{\Gamma(3\beta+1)}{(5040)\Gamma(2\beta+1)}t^{10}x^{2\beta}\\ &\quad +\frac{1}{12}\Gamma(\beta+1)t^4+\cdots. \quad (27)\end{aligned}$$

Case 3 When both α and β belong to $(1,2]$, the approximation solution (19) takes the following form:

$$\begin{aligned}u_{\mathrm{app}} &= t^{\alpha}x^{\beta}+t^{4\alpha}x^{3\beta}\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}-t^{2\alpha}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(4\alpha+1)}\\ &\quad +t^{5\alpha}x^{2\beta}\frac{\Gamma(3\alpha+1)\Gamma(3\beta+1)}{\Gamma(5\alpha+1)\Gamma(2\beta+1)}\\ &\quad +t^{2\alpha}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(2\alpha+1)}+\cdots. \quad (28)\end{aligned}$$

The results (26)–(28) are the same approximate solutions obtained by El-Sayed.[21]

To investigate the influence of h on the convergence of the solution series given by the HAM, we first plot the so-called h-curve of $u(0.5,0.5)$. According to the h-curve, it is easy to discover the valid region of h. We use 3 terms in evaluating the approximate solution $u(x;t)=\sum_{i=0}^{2}u_i(x,t)$. Note that the solution series contains the auxiliary parameter h which provides us with a simple way to adjust and control the convergence of the solution series. As pointed out by Liao,[12] the valid region of h is a horizontal line segment. Therefore, it is straightforward to choose an appropriate range for h that ensures the convergence of the solution series. We stretch the h-curve of $u(0.5,0.5)$ in Fig. 1, which shows that the solution series is convergent when $-1.45<h<-0.9$.

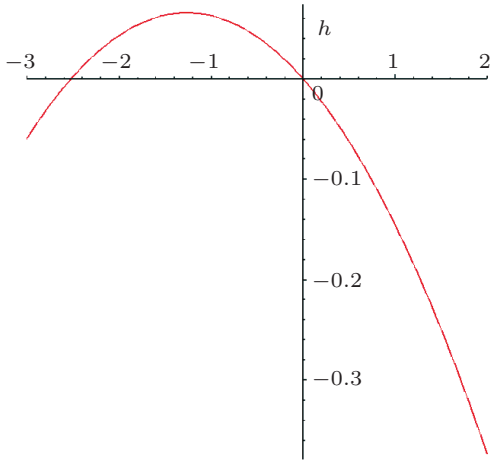


Fig. 1. (color online) The h curve of the third order approximate solution (26) when $x = 0.5, t = 0.5, \alpha = 1.99$, and $\beta = 2$.

5. Discussion

In this section we compare the approximate solutions and the exact solution of the nonlinear space–time fractional derivative Klein–Gordon equation. We show in Fig. 2 the behavior of the approximate solution (26) in comparison with that of the exact solution (17) when α is a natural number and β is a rational number ($\alpha = 2$, $\beta = 1.9$, and $h = 1.2$). Figure 3 illustrates the comparison between the approximate solution (27) and the exact solution for the nonlinear Klein–Gordon partial differential equation when α, β are natural numbers ($\alpha = 2, \beta = 2, h = 1.2$). In Fig. 4, the behavior of the approximate solution (26) is compared with that of the exact solution (17) when α is a rational number and β is a natural number ($\alpha = 1.9$, $\beta = 2$, and $h = 1.2$). Figure 5 illustrates the comparison between the approximate solution (25) and the exact solution for the nonlinear Klein–Gordon partial differential equation when α, β are rational numbers ($\alpha = 0.99$, $\beta = 0.99$, $h = 1.2$). Figure 6 illustrates the projection of the exact solution (17) when $\alpha, \beta = 2$, $t = 0.5$ and the approximate solution (26) when $\beta = 2$ and $\alpha = 1.9$, 1.95, 1.85. Figure 7 illustrates the projection of the exact solution (17) when α, $\beta = 2, t = 0.5$ and the approximate solution (26) when $\alpha = 2$ and $\beta = 1.9$, 1.95, 1.85. Figure 8 illustrates the projection of the exact solution (17) when α, $\beta = 2, t = 0.5$ and the approximate solutions (25) when α, β are rational numbers. From Figs. 1–8, we deduce that the behavior of the approximate solutions is the same as that of the exact solution at different values of α, β. Consequently, we deduce that the approximate solution is a rapidly convergent series as the exact solutions.

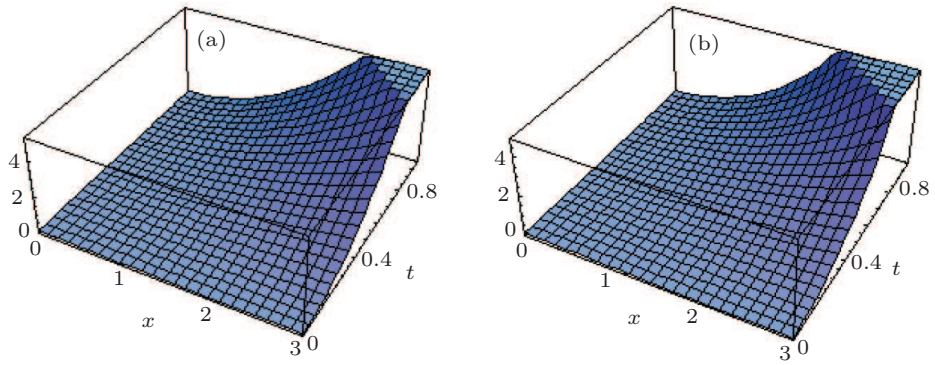


Fig. 2. (color online) Panels (a) and (b) give the exact solution (17) and the approximate solution (26), respectively, when $h = -1.2, \alpha = 2, \beta = 1.9, 0 \le x \le 3$, and $0 \le t \le 1$.

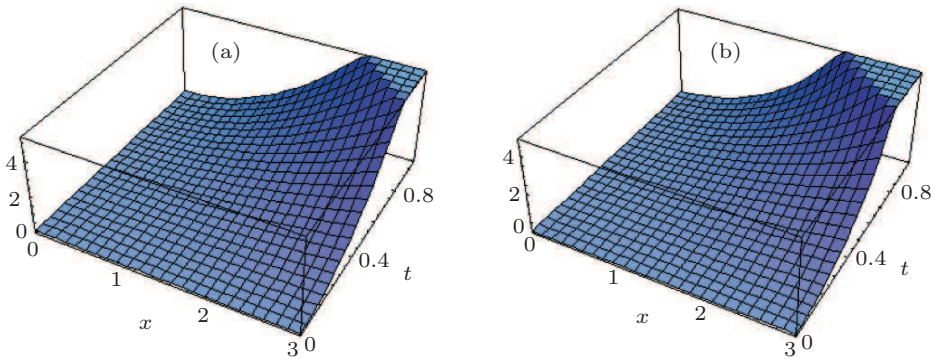


Fig. 3. (color online) Panels (a) and (b) give the exact solution (17) and the approximate solution (27), respectively, when $h = -1.2$, $\alpha = 2$, $\beta = 2$, $0 \le x \le 3$, and $0 \le t \le 1$.

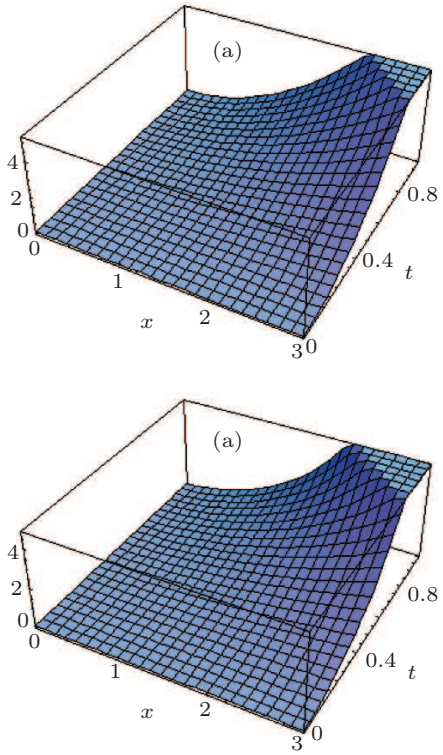


Fig. 4. (color online) Panels (a) and (b) give the exact solution (17) and the approximate solution (26), respectively, when $h = -1.2$, $\alpha = 1.9$, $\beta = 2, 0 \le x \le 3$, and $0 \le t \le 1$.

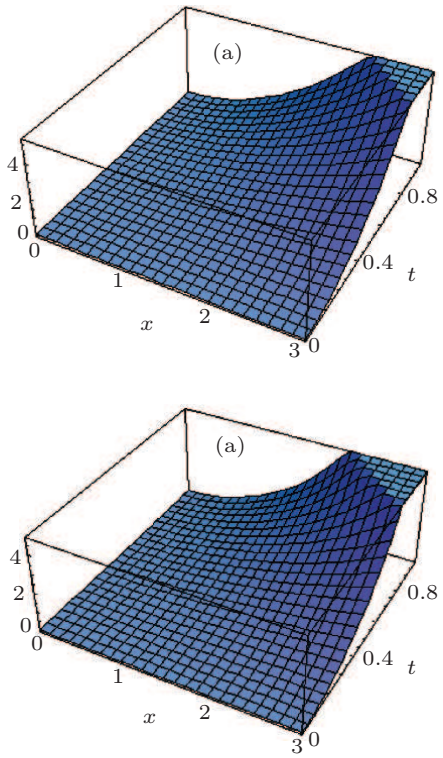


Fig. 5. (color online) Panels (a) and (b) give the exact solution (17) and the approximate solution (25), respectively, when $h = -1.2$, $\alpha = 0.99$, $\beta = .99, 0 \le x \le 3$, and $0 \le t \le 1$.

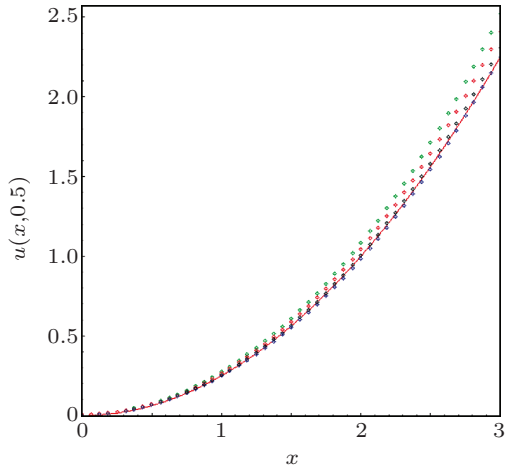


Fig. 6. (color online) The exact solution (17) is compared with the approximate solution (25) when $h = -1.2, \beta = 2$, $x = 0.5$, and $t = 0.5$.

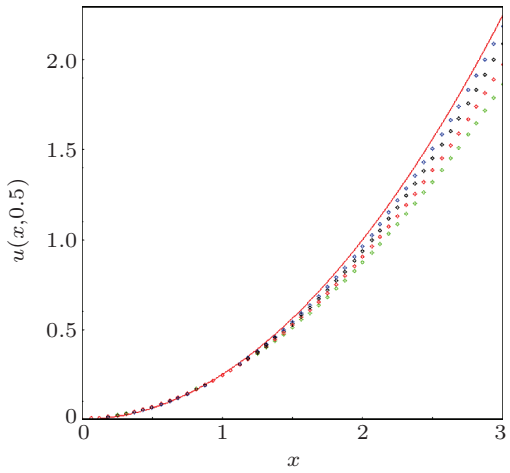


Fig. 7. (color online) The exact solution (17) is compared with the approximate solution (25) when $h = -1.2$, $\alpha = 2$, $x = 0.5$, and $t = 0.5$.

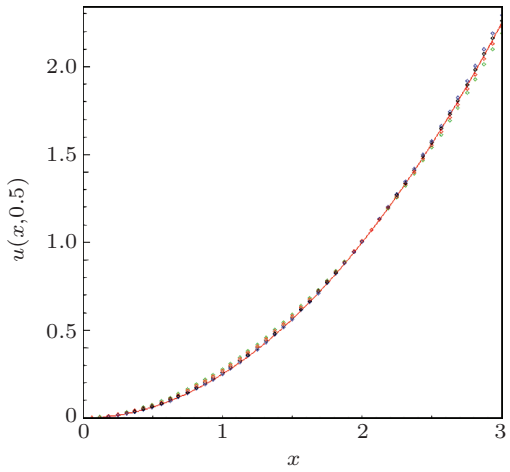


Fig. 8. (color online) The exact solution (17) is compared with the approximate solution (25) when $h = -1.2$, $x = 0.5$, and $t = 0.5$.

6. Conclusion

In this paper, the homotopy analysis method has been successfully applied to obtain the numerical solutions of the nonlinear space–time fractional derivative Klein–Gordon equation. The reliability of this method and the reduction in computation give this method a wider applicability. The HAM contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region and the rate of convergence of the series solution. It is also demonstrated that the Adomian decomposition method and the homotopy perturbation method are special cases of the HAM. The HAM is clearly a very efficient and powerful technique for finding the numerical solutions of the proposed equation. It therefore provides more realistic series solutions that generally converge very rapidly for real physical problems.

References

[1] Oldham K B and Spanier J 1974 *The Fractional Calculus* (New York: Academic Press)

[2] Miller K S and Ross B 1993 *An Introduction to the Fractional and Fractional Differential Equations* (New York: John Wiley and Sons)

[3] LuchkoY and Gorenflo R 1998 *The Initial-value Problem for Some Fractional Differential Equations with Caputo Derivative* (Berlin: Fachbereich Mathematik and Informatic, Freie Universitat)

[4] Podlubny I 1999 *Fractional Differential Equations* (New York: Academic Press)

[5] Daftardar-Gejji V and Bhalekar S 2008 *App. Math. Comput.* **202** 113

[6] Daftardar-Gejji V and Jafari H 2007 *Appl. Math. Comput.* **189** 541

[7] Herzallah M A E and Gepreel K A 2012 *Appl. Math. Modeling* (in-press)

[8] Sweilam N H, Khader M M and Al-Bar R F 2007 *Phys. Lett.* A **371** 26

[9] Golbabai A and Sayevand K 2011 *Comput. Math. Appl.* **61** 2227

[10] Golbabai A and Sayevand K 2010 *Nonlinear Sci. Lett.* A **2** 147

[11] Gepreel K 2011 *Appl. Math. Lett.* **24** 1428

[12] Liao S J 1992 *The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problem* (Ph.D thesis) (Shanghai: Shanghai Jiao Tong University)

[13] Liao S J 1995 *Int. J. Nonlinear Mech.* **30** 371

[14] Hemida K and Saad M 2010 *Journal of Applied Functional Analysis* **5** 344

[15] Saad M 2012 *Journal of Applied Functional Analysis* **7** (in press)

[16] Saad M 2012 *Journal of Advanced Research in Scientific Computing* **3** 56

[17] Saad M and Ghany H 2011 *Journal of Advanced Research in Dynamical and Control Systems* **3** 1

[18] Deng W H and Li C 2011 *Numerical Methods for Partial Differential Equations* **27** 1561

[19] Deng W H 2008 *SIAM Journal on Numerical Analysis* **47** 204

[20] Gepreel K and Omran S 2012 *Chin. Phys.* B (Accepted for publication)

[21] El-Sayed A M A, Elsaid A and Hammad D 2012 *Journal of Applied Mathematics* **2012** 1