# Conformal invariance, Noether symmetry, Lie symmetry and conserved quantities of Hamilton systems

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In this paper, the relation of the conformal invariance, the Noether symmetry, and the Lie symmetry for the Hamilton system is discussed in detail. The definition of the conformal invariance for Hamilton systems is given. The relation between the conformal invariance and the Noether symmetry is discussed, the conformal factors of the determining expressions are found by using the Noether symmetry, and the Noether conserved quantity resulted from the conformal invariance is obtained. The relation between the conformal invariance and the Lie symmetry is discussed, the conformal factors are found by using the Lie symmetry, and the Hojman conserved quantity resulted from the conformal invariance of the system is obtained. Two examples are given to illustrate the application of the results.

**Keywords:** Hamilton system, conformal invariance, conformal factor, conserved quantity

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## 1. Introduction

The theory of symmetry is one of the highest level principles in physics. Finding conserved quantities using the symmetries plays an important role in the development of the analytical mechanics.[1−4] The research on the symmetries, including the Noether symmetry, the Lie symmetry, and the Mei symmetry, has been conducted recently.[1,3] The study about other new symmetries, such as the Lagrange symmetry and the conformal invariance, is also in progress. The conformal invariance of the Birkhoff systems was studied by Galiuillin,[4] and the conformal factors and the conserved quantities resulted from the conformal invariance were obtained. The research about the relation between the conformal invariance and the other symmetries has been carried out recently.[5−20] The conformal invariance in the Hamilton systems was discussed, finding the conformal factors by using the Lie symmetry was specially researched, and the Hojman conserved quantities resulted from the conformal invariance were obtained in Ref. [13]. In this paper, the conformal invariance of the Hamilton system under infinitesimal transformations is studied, the conformal factors under the condition that the conformal invariance exists simultaneously with the Noether or the Lie symmetry are obtained, the relation of the conformal invariance, the Noether symmetry, and the Lie symmetry in the Hamilton system is discussed, then the Noether and the Hojman conserved quantities resulted from the conformal invariance are obtained.

## 2. Definition of conformal invariance in Hamilton systems

The Hamilton canonical equations of a system can be written as

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}, \quad (s = 1, \ldots, n), \tag{1}$$

where $q_s$ and $p_s$ are the canonical variables. Equation (1) can be expressed as

$$\dot{q}_s = g_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad \dot{p}_s = h_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad (s = 1, \ldots, n). \tag{2}$$

Introduce infinitesimal transformations for time $t$, generalized coordinates $q_s$, and generalized momenta $p_s$

$$\begin{aligned} &t^* = t + \Delta t, \quad q_s^*(t^*) = q_s(t) + \Delta q_s, \\ &p_s^*(t^*) = p_s(t) + \Delta p_s, \quad (s = 1, \ldots, n). \end{aligned} \tag{3}$$

The expanding forms are

$$\begin{aligned} &t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \boldsymbol{p}), \\ &q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ &p_s^*(t^*) = p_s(t) + \varepsilon\eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \tag{4}$$

where $\varepsilon$ is an infinitesimal parameter, $\xi_0$, $\xi_s$, and $\eta_s$ are called the infinitesimal generators of $t$, $q_s$, and $p_s$ under the infinitesimal transformations (4), respectively.

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The infinitesimal transformation vector of generator is

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_k \frac{\partial}{\partial q_k} + \eta_k \frac{\partial}{\partial p_k}, \tag{5}$$

and its first extension is

$$X^{(1)} = X^{(0)} + (\dot{\xi}_k - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_k} + (\dot{\eta}_k - \dot{p}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{p}_k}. \tag{6}$$

Let

$$F'_s = \dot{q}_s - \frac{\partial H}{\partial p_s}, \tag{7}$$

$$F''_s = \dot{p}_s + \frac{\partial H}{\partial q_s}, \tag{8}$$

$$\boldsymbol{F}_s = \begin{pmatrix} F'_s \\ F''_s \end{pmatrix}. \tag{9}$$

**Definition** In the infinitesimal transformations of infinitesimal generators $\xi_0(t, \boldsymbol{q}, \boldsymbol{p})$, $\xi_s(t, \boldsymbol{q}, \boldsymbol{p})$, and $\eta_s(t, \boldsymbol{q}, \boldsymbol{p})$, if $\boldsymbol{F}_s$ satisfies the following determining expression:

$$X^{(1)} \boldsymbol{F}_s = \boldsymbol{\ell}_s^k \boldsymbol{F}_k, \tag{10}$$

then the infinitesimal transformations have conformal invariance for the corresponding first order differential equations, this kind of invariance is called the conformal invariance, and $\boldsymbol{\ell}_s^k$ is called the conformal factor.

## 3. Conformal invariance and Noether symmetry of Hamilton systems

For Hamilton system (1), under the infinitesimal transformations (4), the Noether identity can be written in the form

$$p_s \dot{\xi}_s - \frac{\partial H}{\partial t} \xi_0 - \frac{\partial H}{\partial q_s} \xi_s - H \dot{\xi}_0 + \dot{G}_{\rm N} = 0, \tag{11}$$

where $G_{\rm N} = G_{\rm N}(t, \boldsymbol{q}, \boldsymbol{p})$ is a gauge function.

**Proposition 1** For canonical Eqs. (7) and (8), if the generators of the infinitesimal transformations satisfy the determining Eq. (10), the conditions that their conformal invariance exists simultaneously with the Noether symmetry under the action of infinitesimal transformations are that there exists the following conformal factor:

$$\boldsymbol{\ell}_s^k = \begin{pmatrix} \ell_s'^k & 0 \\ 0 & \ell_s''^k \end{pmatrix}, \tag{12}$$

where

$$\ell_s'^k = \frac{\partial}{\partial p_s}(\xi_k - \dot{q}_k \xi_0),$$
$$\ell_s''^k = -\frac{\partial}{\partial q_s}(\xi_k - \dot{q}_k \xi_0), \quad (s, k = 1, \ldots, n). \tag{13}$$

**Proof** As

$$\begin{aligned} & X^{(1)} F'_s - \frac{\partial}{\partial p_s}\left( p_k \dot{\xi}_k - \frac{\partial H}{\partial t} \xi_0 \right. \\ & \left. - \frac{\partial H}{\partial q_k} \xi_k - H \dot{\xi}_0 + \dot{G}_{\rm N} \right) \\ = & X^{(1)} \left( \dot{q}_s - \frac{\partial H}{\partial p_s} \right) - \frac{\partial}{\partial p_s} \left( p_k \dot{\xi}_k \right. \\ & \left. - \frac{\partial H}{\partial t} \xi_0 - \frac{\partial H}{\partial q_k} \xi_k - H \dot{\xi}_0 + \dot{G}_{\rm N} \right) \\ = & -\frac{\rm d}{{\rm d}t} \left( p_k \frac{\partial \xi_k}{\partial p_s} - H \frac{\partial \xi_0}{\partial p_s} + \frac{\partial G_{\rm N}}{\partial p_s} \right) \\ & + \frac{\partial}{\partial p_s} (\xi_k - \dot{q}_k \xi_0) \left( \dot{q}_k - \frac{\partial H}{\partial p_k} \right) \\ = & \frac{\partial}{\partial p_s} (\xi_k - \dot{q}_k \xi_0) F'_k, \end{aligned} \tag{14}$$

$$\begin{aligned} & X^{(1)} F''_s + \frac{\partial}{\partial q_s}\left( p_k \dot{\xi}_k - \frac{\partial H}{\partial t} \xi_0 \right. \\ & \left. - \frac{\partial H}{\partial q_k} \xi_k - H \dot{\xi}_0 + \dot{G}_{\rm N} \right) \\ = & X^{(1)} \left( \dot{p}_s + \frac{\partial H}{\partial q_s} \right) + \frac{\partial}{\partial q_s} \left( p_k \dot{\xi}_k \right. \\ & \left. - \frac{\partial H}{\partial t} \xi_0 - \frac{\partial H}{\partial q_k} \xi_k - H \dot{\xi}_0 + \dot{G}_{\rm N} \right) \\ = & \frac{\rm d}{{\rm d}t} \left( p_k \frac{\partial \xi_k}{\partial q_s} - H \frac{\partial \xi_0}{\partial q_s} + \eta_s + \frac{\partial G_{\rm N}}{\partial p_s} \right) \\ & - \frac{\partial}{\partial q_s} (\xi_k - \dot{q}_k \xi_0) \left( \dot{p}_k + \frac{\partial H}{\partial q_k} \right) \\ = & -\frac{\partial}{\partial q_s} (\xi_k - \dot{q}_k \xi_0) F''_k, \end{aligned} \tag{15}$$

so

$$\ell_s'^k = \frac{\partial}{\partial p_s}(\xi_k - \dot{q}_k \xi_0),$$
$$\ell_s''^k = -\frac{\partial}{\partial q_s}(\xi_k - \dot{q}_k \xi_0), \quad (s, k = 1, \ldots, n). \tag{16}$$

We obtain

$$\begin{aligned} & X^{(1)} F'_s - \frac{\partial}{\partial p_s} (\xi_k - \dot{q}_k \xi_0) F'_k \\ = & \frac{\partial}{\partial p_s} \left( p_k \dot{\xi}_k - \frac{\partial H}{\partial t} \xi_0 - \frac{\partial H}{\partial q_k} \xi_k - H \dot{\xi}_0 + \dot{G}_{\rm N} \right), \end{aligned} \tag{17}$$

$$\begin{aligned} & X^{(1)} F''_s + \frac{\partial}{\partial q_s} (\xi_k - \dot{q}_k \xi_0) F''_k \\ = & -\frac{\partial}{\partial q_s} \left( p_k \dot{\xi}_k - \frac{\partial H}{\partial t} \xi_0 - \frac{\partial H}{\partial q_k} \xi_k - H \dot{\xi}_0 + \dot{G}_{\rm N} \right). \end{aligned} \tag{18}$$

Therefore, the Hamilton system has the conformal invariance and the Noether symmetry simultaneously

under the action of infinitesimal transformations (4), which satisfies the Noether identity (11), we have

$$X^{(1)}\boldsymbol{F}_s = \boldsymbol{\ell}_s^k \boldsymbol{F}_k. \tag{19}$$

**Deduction 1** For Hamilton systems, if the infinitesimal transformations (4) are conformal invariant, the infinitesimal transformations are Noether symmetrical as well. If the infinitesimal transformations are Noether symmetrical, the infinitesimal transformations must be Lie symmetrical.[1] So for Hamilton systems, if the infinitesimal transformations are conformal invariant, the infinitesimal transformations must be Lie symmetrical.

**Proposition 2** For Hamilton system (1), under the action of infinitesimal transformations that are conformal invariant and Noether symmetrical simultaneously, besides the infinitesimal generators $\xi_0$, $\xi_s$, and $\eta_s$ satisfing the conformal factors (12) and the gauge function $G_{\rm N} = G_{\rm N}(t, \boldsymbol{q}, \boldsymbol{p})$ satisfying the Noether identity (11), there exists the Noether conserved quantity

$$I_{\rm N} = p_s\xi_s - H\xi_0 + G_{\rm N} = \text{const.} \tag{20}$$

**Example 1** Consider a system of two degrees with the following Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + q_2, \tag{21}$$

and the equations of motion are

$$\boldsymbol{F}_s = \begin{pmatrix} \dot{q}_1 - p_1 \\ \dot{q}_2 - p_2 \\ \dot{p}_1 \\ \dot{p}_2 + 1 \end{pmatrix}. \tag{22}$$

When the infinitesimal generators are

$$\xi_0 = p_2 + t, \quad \xi_1 = 0, \quad \xi_2 = (p_2 + t)\left(q_2 + \frac{1}{2}t^2\right),$$
$$\eta_1 = 0, \quad \eta_2 = (p_2 + t)^2, \tag{23}$$

we have

$$\begin{aligned} & X^{(1)}\boldsymbol{F}_s \\ &= \begin{pmatrix} 0 \\ -\dot{q}_1(\dot{q}_1 - p_1) + \left(q_2 + \frac{1}{2}t^2 - \dot{q}_2\right)(\dot{q}_2 - p_2) \\ 0 \\ -(p_2 + t)(\dot{p}_2 + 1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\dot{q}_1 & q_2 + \frac{1}{2}t^2 - \dot{q}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(p_2 + t) \end{pmatrix} \\ &\quad \times \begin{pmatrix} \dot{q}_1 - p_1 \\ \dot{q}_2 - p_2 \\ \dot{p}_1 \\ \dot{p}_2 + 1 \end{pmatrix}. \end{aligned} \tag{24}$$

Then we obtain the conformal factor

$$\boldsymbol{\ell}_s^k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\dot{q}_1 & q_2 + \frac{1}{2}t^2 - \dot{q}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(p_2 + t) \end{pmatrix}. \tag{25}$$

Utilizing Proposition 1, we can also obtain the conformal factor

$$\begin{aligned} \boldsymbol{\ell}_s^k &= \begin{pmatrix} \frac{\partial}{\partial p_s}(\xi_k - \dot{q}_k\xi_0) & 0 \\ 0 & -\frac{\partial}{\partial q_s}(\xi_k - \dot{q}_k\xi_0) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\dot{q}_1 & q_2 + \frac{1}{2}t^2 - \dot{q}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(p_2 + t) \end{pmatrix}. \end{aligned} \tag{26}$$

Apparently, the result is the same to Eq. (26). Therefore, equation (23) under the action of Eq. (24) is conformal invariant and Noether symmetrical simultaneously.

Utilizing Proposition 2, the giving infinitesimal generator Eq. (24) satisfies the Noether identity (11), we obtain one solution of the gauge function

$$G_{\rm N} = \frac{1}{2}\left(q_2 + \frac{1}{2}t^2\right)^2 - \frac{1}{3}p_2(p_2 + t)^3. \tag{27}$$

Then the Noether conserved quantity (20) gives

$$\begin{aligned} I_{\rm N} = {} & p_2(p_2 + t)\left(q_2 + \frac{1}{2}t^2\right) + \frac{1}{2}\left(q_2 + \frac{1}{2}t^2\right)^2 \\ & - \frac{1}{3}p_2(p_2 + t)^3 = \text{const.} \end{aligned} \tag{28}$$

# 4. Conformal invariance and Lie symmetry of Hamilton systems

For canonical Eqs. (7) and (8), if the generators of the infinitesimal transformations satisfy the determining Eq. (10), the necessary and sufficient conditions

that the conformal invariance exists simultaneously with the Lie symmetry under the action of infinitesimal transformations are that there exist the following conformal factors:[13]

$$\boldsymbol{\ell}_s^m = \boldsymbol{B}_s^m = \begin{pmatrix} \dfrac{\partial \xi_s}{\partial q_m} - \dot{\xi}_0 \delta_m^s & \dfrac{\partial \xi_s}{\partial p_m} \\ \dfrac{\partial \eta_s}{\partial q_m} & \dfrac{\partial \eta_s}{\partial p_m} - \dot{\xi}_0 \delta_m^s \end{pmatrix}. \quad (29)$$

**Deduction 2** For Hamilton systems, if the infinitesimal transformations (4) are conformal invariant, the infinitesimal transformations are Lie symmetrical as well. If the infinitesimal transformations are Lie symmetrical, besides there exists a gauge function $G_{\rm N} = G_{\rm N}(t, \boldsymbol{q}, \boldsymbol{p})$ satisfying the Noether identity (11), the infinitesimal transformations must be Noether symmetrical. Otherwise, they would not be Noether symmetrical.[1] So if the infinitesimal transformations are conformal invariant and satisfy the Noether identity (11), the infinitesimal transformations must be Noether symmetrical.

For Hamilton system (1), under the action of special infinitesimal transformations that are conformal invariant and Lie symmetrical simultaneously, there exists the Hojman conserved quantity[13]

$$I_{\rm H} = \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s} = \text{const.} \quad (30)$$

**Example 2** We study the relation between the conformal invariance and the Lie symmetry, and the Hojman conserved quantities for the Hamilton systems. Consider a system with the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2)\,\mathrm{e}^{\gamma t}, \quad (31)$$

where $\gamma$ is a constant. The equations of motion are

$$\boldsymbol{F}_s = \begin{pmatrix} \dot{q}_1 - p_1\,\mathrm{e}^{\gamma t} \\ \dot{q}_2 - p_2\,\mathrm{e}^{\gamma t} \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix}. \quad (32)$$

When the infinitesimal generators are

$$\xi_0 = t, \quad \xi_1 = p_1\left(q_1 - \frac{1}{\gamma}p_1 e^{\gamma t}\right),$$
$$\xi_2 = p_2\left(q_2 - \frac{1}{\gamma}p_2 e^{\gamma t}\right), \quad \eta_1 = 0, \quad \eta_2 = 0, \quad (33)$$

we have

$$X^{(1)}\boldsymbol{F}_s = \begin{pmatrix} (p_1-1)(\dot{q}_1 - p_1\,\mathrm{e}^{\gamma t}) + \dot{p}_1\left(q_1 - \dfrac{2}{\gamma}p_1\,\mathrm{e}^{\gamma t}\right) \\ (p_2-1)(\dot{q}_2 - p_2\,\mathrm{e}^{\gamma t}) + \dot{p}_2\left(q_2 - \dfrac{2}{\gamma}p_2\,\mathrm{e}^{\gamma t}\right) \\ -\dot{p}_1 \\ -\dot{p}_2 \end{pmatrix}$$
$$= \begin{pmatrix} p_1 - 1 & 0 & q_1 - \dfrac{2}{\gamma}p_1\,\mathrm{e}^{\gamma t} & 0 \\ 0 & p_2 - 1 & 0 & q_2 - \dfrac{2}{\gamma}p_2\,\mathrm{e}^{\gamma t} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \times \begin{pmatrix} \dot{q}_1 - p_1\,\mathrm{e}^{\gamma t} \\ \dot{q}_2 - p_2\,\mathrm{e}^{\gamma t} \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix}. \quad (34)$$

Evidently, we obtain the conformal factors

$$\boldsymbol{\ell}_s^m = \begin{pmatrix} p_1 - 1 & 0 & q_1 - \dfrac{2}{\gamma}p_1\,\mathrm{e}^{\gamma t} & 0 \\ 0 & p_2 - 1 & 0 & q_2 - \dfrac{2}{\gamma}p_2\,\mathrm{e}^{\gamma t} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (35)$$

which are the necessary and sufficient conditions that the system is conformal invariant and Lie symmetrical simultaneously. The giving infinitesimal generator Eq. (34) satisfies another form of the determining equations of the Lie symmetry under the action of special infinitesimal transformations, the Hojman conserved quantity Eq. (31) gives

$$I_{\rm H} = p_1 + p_2 = \text{const.} \quad (36)$$

# 5. Conclusion

For Hamilton systems, the infinitesimal transformations have the conformal invariance for the corresponding first order differential equations. The conformal factors of the determining expressions can be found by uisng the Noether symmetry, which are the conditions that the system's conformal invariance exists simultaneously with the Noether symmetry under the action of infinitesimal transformations. The conformal factors can be found by using the Lie symmetry as well, which are the necessary and sufficient conditions that the system's conformal invariance exists

simultaneously with the Lie symmetry. Besides, for Hamilton systems, if the infinitesimal transformations are conformal invariant and the conformal factors satisfy condition (12), the infinitesimal transformations are Lie symmetrical as well. If the conformal factors satisfy the necessary and sufficient conditions (30), the infinitesimal transformations are Noether symmetrical as well. Therefore, the relation of the conformal invariance, the Noether symmetry, and the Lie symmetry for the Hamilton system has been found. And we obtain the Noether conserved quantities and the Hojman conserved quantities resulted from the conformal invariance, find the relation between the conformal invariance and the conserved quantity.

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