Control of fractional chaotic system based on fractional-order resistor–capacitor filter*

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We present a new fractional-order resistor–capacitor controller and a novel control method based on the fractional-order controller to control an arbitrary three-dimensional fractional chaotic system. The proposed control method is simple, robust, and theoretically rigorous, and its anti-noise performance is satisfactory. Numerical simulations are given for several fractional chaotic systems to verify the effectiveness and the universality of the proposed control method.

Keywords: fractional chaotic system, chaos control, fractional-order controller, resistor–capacitor filter

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1. Introduction

As a generalization of integer-order dynamic systems, fractional dynamic systems provide better mathematical models for some actual physical and engineering systems.[1,2] Studies have shown that the fractional nonlinear dynamic systems have many dynamic behaviors that are similar to the integer-order systems, such as attractor, bifurcation, and chaos.[3−6] Chaotic phenomenon analysis, synchronization, and control of such systems are widely studied due to their potential applications in information, biology, physics, chemistry, and the other fields.[7−9] Among them, the chaos control is important but also very difficult. Researches in this area are still in the early stage, and there are only a few control methods, such as linear feedback method,[5] backstepping method,[10] sliding mode method,[11] and the Lyapunov equation based method.[12] However, most of the above control methods concentrate on certain fractional systems, and there are few general methods that can be applied to control arbitrary fractional chaotic systems.

On the other hand, in recent years, as a generalization of the traditional integer-order control, the fractional-order control is becoming a matter of concern because of its flexibility and integrity.[2,13−15] The TID controller,[13] the $PI^{\lambda}D^{\mu}$ controller,[14] and the CRONE controller[15] are among the well-known fractional-order controllers. In some of those papers, it is verified that the fractional-order controllers can have better disturbance rejection ratios and less sensitivity to plant parameter variations compared to the traditional controllers.

However, to the best of our knowledge, little attention has been paid to studying the problem of controlling fractional chaotic systems with the use of fractional-order controllers. In this paper, we propose the fractional-order resistor–capacitor (FRC) controller as a generalization of the classical integer-order RC controller for the first time. Then we present the FRC based control law to control the fractional system of the same order to a stable equilibrium point. The proposed FRC controller has the following three advantages. 1) It is general and can be applied to almost all three-dimensional chaotic systems. 2) It is easy to implement and can stabilize the original equilibrium points with great flexibility, little control cost, and no affection on the parameters of the original systems. 3) It has a superior anti-noise performance compared to the classical control methods, and thus is more suitable for engineering applications. Numerical simulation results of the FRC control of the fractional Chen's system, the fractional Lu's system, and the fractional financial system demonstrate the effectiveness and the validity of the proposed control method.

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2. Fractional dynamic systems and stability conditions

In this paper, we consider a three-dimensional fractional nonlinear system of order p ($p \in (0,1]$)

$$\mathrm{D}^p \boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{x}(0) = \boldsymbol{c}. \tag{1}$$

where $\boldsymbol{x} = [x_1(t), x_2(t), x_3(t)]^{\mathrm{T}}$ is the system state variable, $\boldsymbol{c} = [c_1, c_2, c_3]^{\mathrm{T}}$ is the initial value, and D^p denotes the Caputo fractional-order derivative operator.[1] The equilibrium points of system (1) are calculated via solving $\boldsymbol{f}(\boldsymbol{x}^*) = 0$. We have the following conclusion for the stability of these equilibrium points.

Theorem 1[2] For $p \in (0,1]$, the equilibrium point $\boldsymbol{x}^*$ of system (1) is globally asymptotically stable if all the eigenvalues λ_i ($i = 1, 2, 3$) of the Jacobian matrix $\boldsymbol{J} = \partial \boldsymbol{f}/\partial \boldsymbol{x}$ evaluated at $\boldsymbol{x}^*$ satisfy the condition

$$|\arg(\lambda_i)| > p\pi/2. \tag{2}$$

This conclusion can also be extended to n-dimensional fractional-order systems.

The necessary condition with which a three-dimensional nonlinear system can generate chaos is that there exists an unstable equilibrium point, which is also the saddle-focus point (i.e., the equilibrium point has one stable and two unstable eigenvalues).[2] Therefore, a principle for chaos control of fractional nonlinear systems is to eliminate the unstable eigenvalues of the saddle-focus point.

3. Chaos control based on fractional-order RC controller

3.1. Fractional-order resistor–capacitor filter

In this section, we propose a new controller, the fractional-order resistor–capacitor controller, to achieve the chaos control of the fractional chaotic system (1).

The traditional resistor–capacitor controller is a frequently used low-pass filter in engineering systems, which filters out the high-frequency inputs while passing the low frequency inputs. Its transfer function in the one-dimensional form is given by

$$G(s) = \frac{K}{\tau s + 1}, \tag{3}$$

where τ is the filter time constant, and K is the filter pass-band gain.

In this paper, we extend the above RC with $K = \tau$ to a fractional-order form, called the fractional-order resistor–capacitor filter, with a transfer function

$$G_\alpha(s) = \frac{\tau}{\tau s^\alpha + 1} = \frac{1}{s^\alpha + d}, \tag{4}$$

where α ($\alpha \in \mathbf{R}$) is the order, and $d = 1/\tau$ is the control parameter of this controller. When $\alpha = 1$, the FRC filter reduces to the classical RC filter. If $\alpha > 0$, the FRC filter is a low-pass filter and has the ability to filter out the out-of-band noise. Figure 1 shows the amplitude spectra of the FRC filters of various positive orders. It is observed that compared to the integer-order RC ($\alpha = 1$), the FRC of order $0 < \alpha < 1$ has a narrower stopband, so it is closer to an ideal low-pass filter.

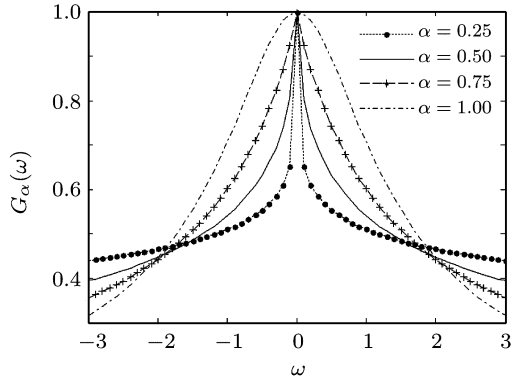


Fig. 1. Amplitude spectra of FRC filters of various positive orders, with $d = 1$.

Consider an FRC controller of order α with input x, output u, and state variable w, its dynamics equation and output equation can be written as

$$\mathrm{D}^\alpha w = -dw + x, \quad u = w. \tag{5}$$

Obviously, the FRC has an eigenvalue $\lambda = -d$ at the equilibrium point $u^* = x/d$, so it is in a steady state if $d > 0$ and an unstable state if $d < 0$. In this paper, we consider the case of $d > 0$.

3.2. Controller design

The controlled system under the action of the FRC filter can be expressed as

$$\begin{aligned} \mathrm{D}^p \boldsymbol{x} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(u), \\ \mathrm{D}^p u &= x_1 - du, \end{aligned} \tag{6}$$

where $d > 0$, $\boldsymbol{x} = [x_1(t), x_2(t), x_3(t)]^{\mathrm{T}}$ is the system state variable, and u is the output of the FRC

with input x_1. Each component of the control function $\boldsymbol{g}(u) = [g_1(u), g_2(u), g_3(u)]^{\mathrm{T}}$ is a function of the control variable u.

Assume that $\boldsymbol{x}^* = (x_1^*, x_2^*, x_3^*)$ is a saddle-focus point of system (1), which has one stable eigenvalue λ_1 and two unstable eigenvalues λ_2, λ_3 satisfying $|\arg(\lambda_1)| > p\pi/2$ and $|\arg(\lambda_{2,3})| < p\pi/2$. To keep the original equilibrium points, the controlled system (6) should satisfy the following conditions at the equilibrium point $\tilde{\boldsymbol{x}}^* = (x_1^*, x_2^*, x_3^*, u^*)$:

$$\begin{cases} f_i(x_1^*, x_2^*, x_3^*) + g_i(u^*) = 0, \quad i = 1, 2, 3, \\ x_1^* - du^* = 0. \end{cases} \tag{7}$$

Besides, $f_i\ (x_1^*, x_2^*, x_3^*) = 0\ (i = 1, 2, 3)$, so we obtain $u^* = x_1^*/d$ and $g_i(u^*) = 0$.

The characteristic equation of the Jacobian matrix at $\tilde{\boldsymbol{x}}^*$ is given by

$$\begin{aligned} F(\lambda) = {} & \lambda^4 + (d - f_{11} - f_{22} - f_{33})\,\lambda^3 \\ & + [-\dot{g}_1(u^*) - (f_{11}+f_{33}+f_{22})d + b_2]\,\lambda^2 \\ & + [(f_{22} + f_{33})\dot{g}_1(u^*) - f_{12}\dot{g}_2(u^*) \\ & - f_{13}\dot{g}_3(u^*) + b_2 d + b_1]\lambda \\ & + (f_{32}f_{23} - f_{22}f_{33})\,\dot{g}_1(u^*) \\ & + (f_{12}f_{33} - f_{32}f_{13})\,\dot{g}_2(u^*) \\ & + (f_{22}f_{13} - f_{12}f_{23})\,\dot{g}_3(u^*) + b_1 d, \end{aligned} \tag{8}$$

where

$$\begin{aligned} f_{ij} = {} & (\partial f_i/\partial x_j)|_{x_1=x_1^*, x_2=x_2^*, x_3=x_3^*}, \\ b_1 = {} & -f_{31}f_{12}f_{23} + f_{11}f_{23}f_{32} + f_{31}f_{13}f_{22} \\ & - f_{11}f_{22}f_{33} - f_{21}f_{32}f_{13} + f_{21}f_{12}f_{33}, \\ b_2 = {} & f_{22}f_{33} - f_{31}f_{13} + f_{11}f_{22} + f_{11}f_{33} \\ & - f_{21}f_{12} - f_{23}f_{32}. \end{aligned} \tag{9}$$

Suppose $F(\lambda)$ has four eigenvalues, λ_0, λ_1, $\tilde{\lambda}_2$, and $\tilde{\lambda}_3$, where λ_1 is a stable eigenvalue of system (1), then from the Vieta theorem of polynomial equations, we have

$$\begin{cases} \lambda_0 + \lambda_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 = -d + f_{11}+f_{22}+f_{33}, \\ \lambda_0(\lambda_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3) + \lambda_1(\tilde{\lambda}_2 + \tilde{\lambda}_3) + \tilde{\lambda}_2\tilde{\lambda}_3 \\ \quad = -\dot{g}_1(u^*) - (f_{11}+f_{33}+f_{22})d + b_2, \\ \lambda_0\lambda_1(\tilde{\lambda}_2 + \tilde{\lambda}_3) + \lambda_0\tilde{\lambda}_2\tilde{\lambda}_3 + \lambda_1\tilde{\lambda}_2\tilde{\lambda}_3 \\ \quad = -(f_{22}+f_{33})\dot{g}_1(u^*) + f_{12}\dot{g}_2(u^*) \\ \qquad + f_{13}\dot{g}_3(u^*) - b_2 d - b_1, \\ \lambda_0\lambda_1\tilde{\lambda}_2\tilde{\lambda}_3 = (f_{32}f_{23} - f_{22}f_{33})\dot{g}_1(u^*) \\ \quad + (f_{12}f_{33} - f_{32}f_{13})\dot{g}_2(u^*) \\ \quad + (f_{22}f_{13} - f_{12}f_{23})\dot{g}_3(u^*) + b_1 d. \end{cases} \tag{10}$$

Suppose $k = \dot{g}_1(u^*)$ is the other control parameter, then the above equations can be seen as linear equations of $\tilde{\lambda}_2 + \tilde{\lambda}_3$, $\tilde{\lambda}_2\tilde{\lambda}_3$, $\dot{g}_2(u^*)$, and $\dot{g}_3(u^*)$. Solving these equations gives the following expressions of $\tilde{\lambda}_2 + \tilde{\lambda}_3$, $\tilde{\lambda}_2\tilde{\lambda}_3$, $\dot{g}_2(u^*)$, $\dot{g}_3(u^*)$:

$$\begin{bmatrix} \tilde{\lambda}_2 + \tilde{\lambda}_3 \\ \tilde{\lambda}_2\tilde{\lambda}_3 \end{bmatrix} = \boldsymbol{C} \begin{bmatrix} d \\ k \end{bmatrix} + \boldsymbol{B}, \tag{11}$$

$$\begin{aligned} \begin{bmatrix} \dot{g}_2(u^*) \\ \dot{g}_3(u^*) \end{bmatrix} = \boldsymbol{F} \times \Bigg\{ \Bigg(& \begin{bmatrix} \lambda_0\lambda_1 & \lambda_0 + \lambda_1 \\ 0 & \lambda_0\lambda_1 \end{bmatrix} \boldsymbol{C} \\ & + \begin{bmatrix} b_2 & f_{22}+f_{33} \\ -b_1 & f_{22}f_{33} - f_{32}f_{23} \end{bmatrix} \Bigg) \begin{bmatrix} d \\ k \end{bmatrix} \\ & + \begin{bmatrix} \lambda_0\lambda_1 & \lambda_0 + \lambda_1 \\ 0 & \lambda_0\lambda_1 \end{bmatrix} \boldsymbol{B} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \Bigg\}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \boldsymbol{B} = {} & \begin{bmatrix} f_{11}+f_{22}+f_{33} - (\lambda_0 + \lambda_1) \\ b_2 - (f_{11} + f_{22}+f_{33})(\lambda_0 + \lambda_1) \\ +(\lambda_0 + \lambda_1)^2 - \lambda_0\lambda_1 \end{bmatrix}, \\ \boldsymbol{C} = {} & \begin{bmatrix} -1 & 0 \\ \lambda_0 + \lambda_1 - f_{11} - f_{22} - f_{33} & -1 \end{bmatrix}, \\ \boldsymbol{F} = {} & \begin{bmatrix} f_{12} & f_{13} \\ f_{12}f_{33} - f_{32}f_{13} & f_{22}f_{13} - f_{12}f_{23} \end{bmatrix}^{-1}. \end{aligned} \tag{13}$$

To ensure the controlled system (6) is still a real system, we assume $\dot{g}_i(u^*)$, $i = 1, 2, 3$ to be real, thus $\lambda_0 + \lambda_1$ and $\lambda_0\lambda_1$ are both real numbers. So λ_0 and λ_1 satisfy the conjugate relation $\lambda_0 = \bar{\lambda}_1$. Since $|\arg(\lambda_1)| > p\pi/2$, then from the symmetry of λ_0 and λ_1 in the complex plane, we also have $|\arg(\lambda_0)| > p\pi/2$.

Moreover, from Eq. (11), we can see that $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$ are the roots of equation $\lambda^2 + s_1\lambda + s_2 = 0$, where

$$\begin{aligned} s_1 = {} & d - (f_{11}+f_{22}+f_{33}) + (\lambda_0 + \lambda_1), \\ s_2 = {} & (\lambda_0 + \lambda_1 - f_{11} - f_{22} - f_{33})d - k + b_2 \\ & - (f_{11}+f_{22}+f_{33})(\lambda_0 + \lambda_1) + (\lambda_0 + \lambda_1)^2 \\ & - \lambda_0\lambda_1. \end{aligned} \tag{14}$$

To determine the stability of $\tilde{\lambda}_2$ and $\tilde{\lambda}_3$, we derive the following lemma.

Lemma 1 A necessary and sufficient condition for roots y_1, y_2 of the quadratic polynomial equation

$y^2+q_1y+q_2=0$ to satisfy $|\arg(y_{1,2})|>p\pi/2$ is $q_1>0$, $q_2>0$ or $q_1<0$, $q_2>q_1^2\left[1+\mathrm{tg}^2(p\pi/2)\right]/4$.

Proof The discriminant of the above equation is $\Delta=q_1^2-4q_2$.

If $\Delta\geq 0$, that is $q_2\leq q_1^2/4$. In this case, the equation has two real roots, and they are both negative. So $|\arg(y_{1,2})|>p\pi/2$ if and only if $q_1>0$, $q_2>0$.

If $\Delta<0$, that is $q_2>q_1^2/4$. In this case, the equation has a pair of conjugate complex roots. If $q_1>0$, these two roots are in the left half part of the complex plane, so $|\arg(y_{1,2})|>p\pi/2$. If $q_1<0$, then $|\arg(y_{1,2})|>p\pi/2$ if and only if $q_2>q_1^2\left[1+\mathrm{tg}^2(p\pi/2)\right]/4$.

Thus, from the above discussion and Lemma 1, we have the following conclusion.

Theorem 2 Assume that $\boldsymbol{x}^*=(x_1^*,x_2^*,x_3^*)$ is a saddle-focus point of system (1) with eigenvalues λ_1, λ_2, λ_3, and the control function $\boldsymbol{g}(u)$ satisfies $g_i(x_1^*/d)=0$, $i=2,3$, $\dot{g}_1(x_1^*/d)=k$, and Eq. (12). Then the necessary and sufficient conditions with which an FRC filter can control the system responses to the asymptotically stable equilibrium point $\boldsymbol{x}^*$ are the control parameters d and k satisfying one of the following conditions:

$$\begin{aligned}
1)\ & d>\max\{f_{11}+f_{22}+f_{33}-(\lambda_0+\lambda_1),0\},\\
& k<(\lambda_0+\lambda_1-f_{11}-f_{22}-f_{33})d+b_2\\
&\quad -(f_{11}+f_{22}+f_{33})(\lambda_0+\lambda_1)+(\lambda_0+\lambda_1)^2\\
&\quad -\lambda_0\lambda_1; \qquad (15)\\
2)\ & 0<d<(f_{11}+f_{22}+f_{33})-(\lambda_0+\lambda_1),\\
& k<-\left[1+\mathrm{tg}^2(p\pi/2)\right]d^2/4\\
&\quad +\left[1-\mathrm{tg}^2(p\pi/2)\right][\lambda_0+\lambda_1\\
&\quad -(f_{11}+f_{22}+f_{33})]d/2\\
&\quad -\left[1+\mathrm{tg}^2(p\pi/2)\right](f_{11}+f_{22}+f_{33})^2/4\\
&\quad +\left[3-\mathrm{tg}^2(p\pi/2)\right](\lambda_0+\lambda_1)^2/4\\
&\quad +\left[\mathrm{tg}^2(p\pi/2)-1\right](f_{11}+f_{22}+f_{33})(\lambda_0+\lambda_1)/2\\
&\quad -\lambda_0\lambda_1+b_2, \qquad (16)
\end{aligned}$$

where $\lambda_0+\lambda_1=2\mathrm{Re}(\lambda_1)$, and $\lambda_0\lambda_1=|\lambda_1|^2$.

Substituting the expressions of s_1, s_2 in Eq. (14) into Lemma 1 gives the proof of Theorem 2. From Theorem 2, we can see that the asymptotically stability of the controlled system at the equilibrium point is only related to the control parameter of the FRC controller, besides the values of the control function and its first-order derivative function at the equilibrium point. Thereby, the control function related to the proposed FRC controller may has a lot of forms, such as the linear function, the polynomial function, and the fraction function.

4. Simulation and analysis

In this section, we present three illustrative examples to verify and demonstrate the effectiveness of the proposed control method, including the FRC controls of the fractional Chen's system, the fractional Lu's system, and the fractional financial system.

4.1. The fractional Chen's system

The expressions for the fractional Chen's system of order p ($p\in(0,1]$) are given by[2]

$$\begin{aligned}
&\mathrm{D}^p x_1=a(x_2-x_1),\\
&\mathrm{D}^p x_2=(c-a)x_1-x_1x_3+cx_2,\\
&\mathrm{D}^p x_3=x_1x_2-bx_3. \qquad (17)
\end{aligned}$$

When $a=35$, $b=3$, $c=28$, this system has three equilibrium points at $(0,0,0)$, $(-3\sqrt{7},-3\sqrt{7},21)$, and $(3\sqrt{7},3\sqrt{7},21)$. The first equilibrium point is a saddle point; the other two are saddle-focus points and surrounded by a chaotic double scroll attractor. The smallest order of this system to generate chaos is 0.8244.[2] The simulation is performed with initial value $(-2,-1,20)$, simulation time 10 s, time step 0.001 s, and order $p=0.9$. In this case, the system exhibits the chaotic behavior, as shown in Fig. 2. Now we apply the proposed FRC controller with order $p=0.9$ to this system. From Theorem 2, we have

$$\begin{aligned}
&\dot{g}_1(u^*)=k,\\
&\dot{g}_2(u^*)=20.978d+1.767k-386.575,\\
&\dot{g}_3(u^*)=40.775d+0.63k-751.4, \qquad (18)
\end{aligned}$$

and

$$d>26.856,\quad k<-26.856d+734.214,$$

or

$$\begin{aligned}
&0<d<26.856,\\
&k<-10.216d^2+521.859d-6633.925. \qquad (19)
\end{aligned}$$

Here, we consider $g_i(u)$ to be the polynomial functions

$$g_i(u)=\dot{g}_i(u)(u-x_1^*/d)+h_i(u-x_1^*/d)^2, \qquad (20)$$

or the fraction functions

$$g_i(u) = \frac{\dot{g}_i(u)(u - x_1^*/d) + h_i(u - x_1^*/d)^2}{l_{i1}(u - x_1^*/d) + l_{i2}(u - x_1^*/d)^2 + 1}, \quad (21)$$

where h_i, l_{ij} $(i = 1, 2, 3, j = 1, 2)$ are the adjustable parameters. In fact, from the previous discussion, we can see that the high-order terms have little affection on the control results, so we only need to consider the quadratic case.

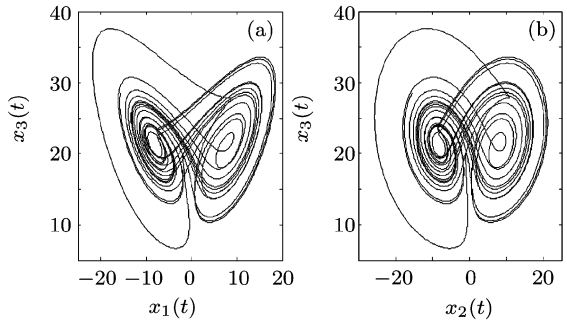


Fig. 2. State trajectories of the fractional Chen's system in chaotic state shown in in (a) x_3–x_1 and (b) x_3–x_2 planes, and $p = 0.9$.

4.1.1. The $g_i(u)$ being polynomial functions

Set the control functions to be the polynomial functions given in Eq. (20) with $d = 30$, $k = -271.4$, and $h_i = 1$, $i = 1, 2, 3$. The simulation results are shown in Fig. 3, in which the control signal is added at $t = 0.2$ s. It is observed that the time waveforms of the state variables of the controlled system are quickly controlled to the equilibrium point $(-3\sqrt{7}, -3\sqrt{7}, 21)$, and the control variable u also approaches to $x_1^*/d = -0.2646$. This shows the effectiveness of the proposed method. Moreover, the control energy is smaller with a larger d. This indicates the small cost of the FRC control.

Secondly, we fixe d at 30 and adjust the value of k under the first condition of expression (19). Figure 4(a) shows the time waveforms of $x_1(t)$ with various k. We can find that $x_1(t)$ arrives at the equilibrium point more quickly with a smaller k. We then fix k at -3000 and adjust the value of d. Figure 4(b) shows the time waveforms of $x_1(t)$ with various d. It is observed that $x_1(t)$ arrives at the equilibrium point more quickly with a larger d.

4.1.2. The $g_i(u)$ being fraction functions

Set the control functions to be the fraction functions given in Eq. (21) with $d = 30$, $k = -271$, and $h_i = l_{ij} = 1$ $(i = 1, 2, 3,\ j = 1, 2)$. The simulation results are shown in Fig. 5, in which the control signal is added at $t = 0.5$ s. It is observed that the time waveforms of the state variables of the controlled system are quickly controlled to the equilibrium point.

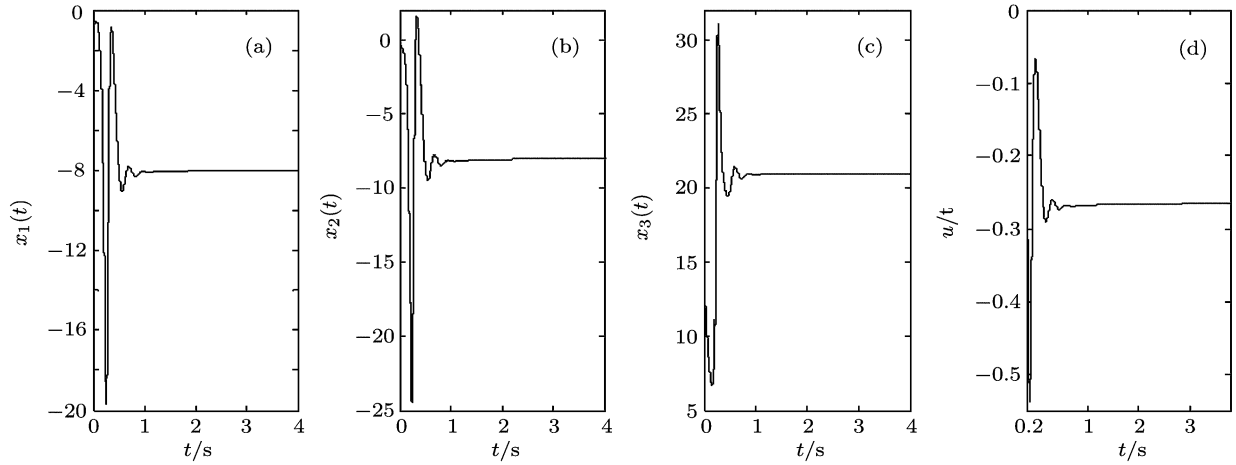


Fig. 3. Time waveforms of state variables (a) x_1, (b) x_2, (c) x_3, and (d) u of the controlled system with the polynomial control function.

Moreover, we fix d at 30 and k at –400, and change the values of h_i and l_{ij}, the time waveforms of $x_1(t)$ are shown in Fig. 6. It is observed that the changes of h_i and l_{ij} do not change the stability of the controlled system, which is consistent with the previous theoretical analysis.

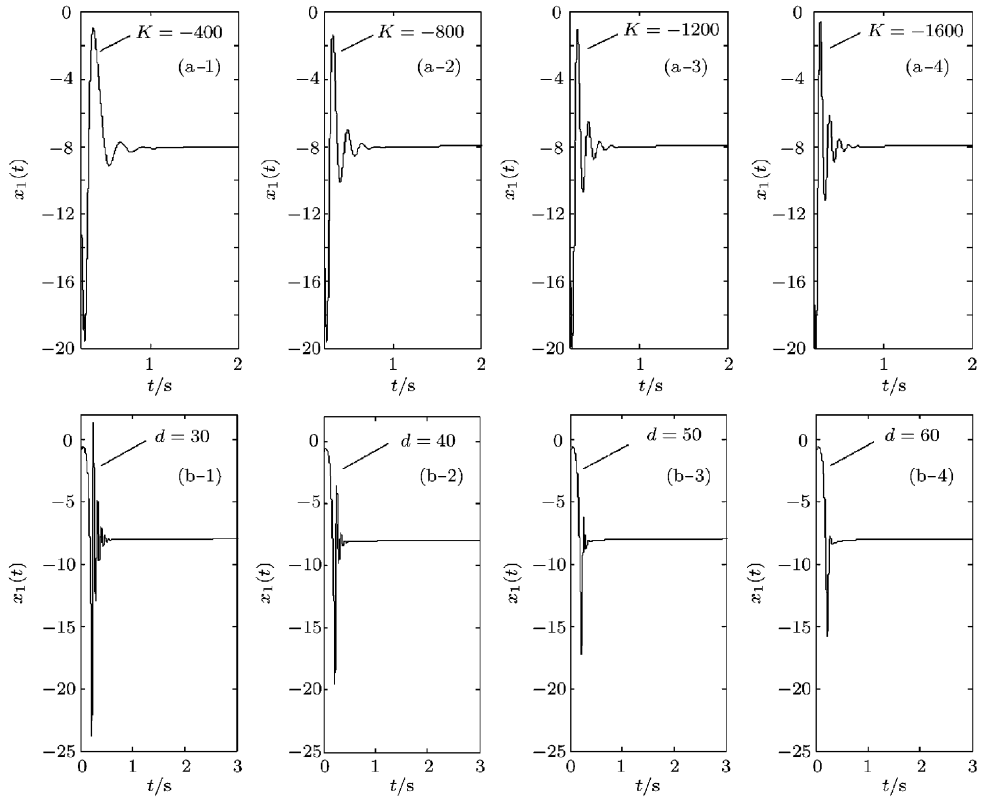


Fig. 4. Time waveforms of $x_1(t)$ (a) for various k at $d = 30$ and (b) for various d at $k = -3000$.

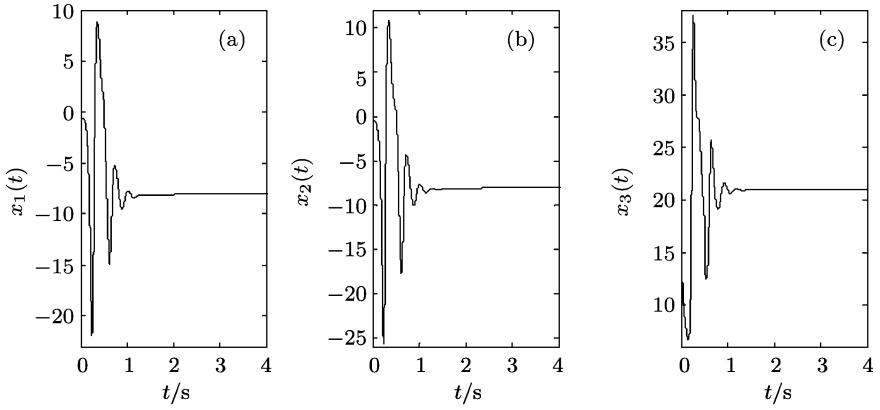


Fig. 5. Time waveforms of state variables (a) x_1, (b) x_2, and (c) x_3 of the controlled system with the fraction control function.

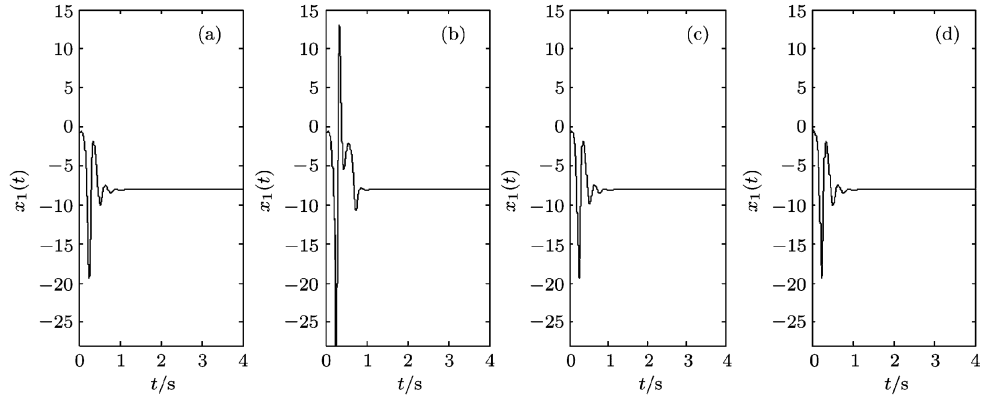


Fig. 6. Time waveforms of $x_1(t)$ with (a) $h_i = 1$, $l_{ij} = 1$, (b) $h_1 = 10$, $h_2 = h_3 = 1$, $l_{ij} = 1$, (c) $h_1 = 1$, $h_2 = -1$, $h_3 = 20$, $l_{ij} = 2$, and (d) $h_1 = 1$, $h_2 = 10$, $h_3 = 2$, $l_{ij} = 1$, where $i = 1, 2, 3$, $j = 1, 2$, $d = 30$, $k = -400$.

4.1.3. FRC control in the presence of measurement noise

Further, in order to study the control performance of the proposed FRC controller in the presence of measurement noise, we add the white noise with variance $D = 4$ into the first formula of the controlled fractional Chen's system. The FRC control results of $x_1(t)$ without and with noise are given in Figs. 7(a)–7(c). We also apply the linear feedback (LF) control method in Ref. [5] to control the system to its equilibrium point (0,0,0), the white noise with variance $D = 4$ is also added to the first formula of the controlled system in Ref. [5]. The LF control results of $x_1(t)$ without and with noise are given in Figs. 7(d)–7(f). In all the figures, the control signals are added at $t = 2.5$ s. It is observed that the control error (the control result without noise − the control result with noise) of the FRC controller is in the range from –0.2 to 0.2, which is much smaller than that of the LF controller. This is because the FRC is a low-pass filter, which can effectively filter out the out-of-band noise and thus reduce the noise effect on the control. This also shows the superiority of the proposed FRC controller.

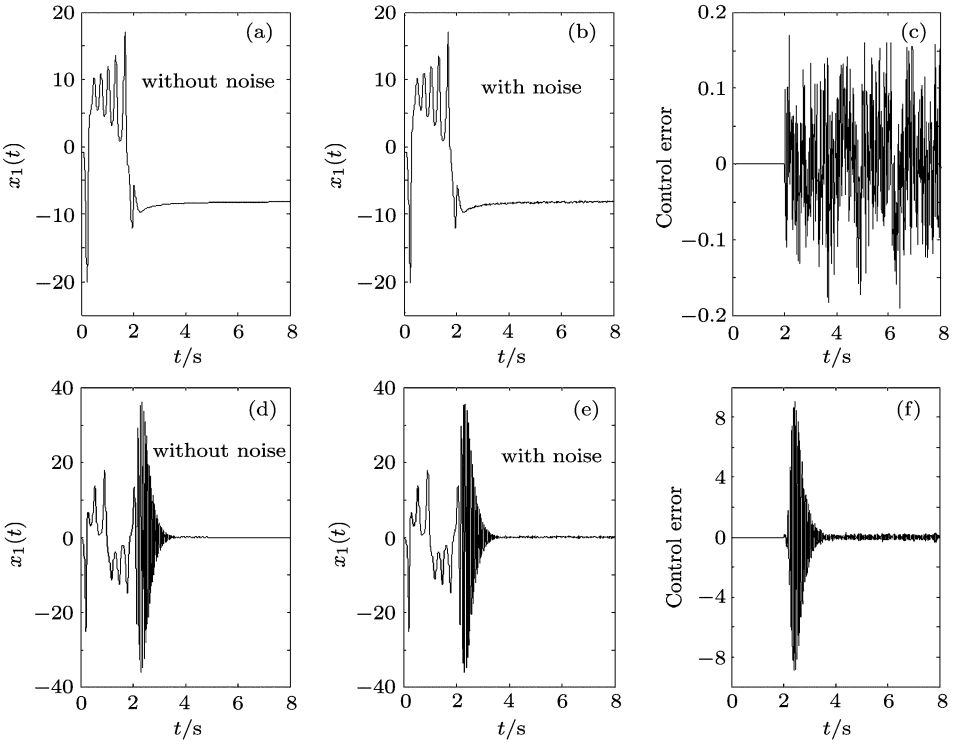


Fig. 7. Comparison of the time waveforms of $x_1(t)$ without and with noise, panels (a)–(c) and (d)–(f) show the results of the FRC control and the LF control, respectively.

4.2. The fractional Lu's system

The fractional Lu's system of order p ($p \in (0, 1]$) is given by[2]

$$\begin{aligned} \mathrm{D}^p x_1 &= -a x_1 + a x_2, \\ \mathrm{D}^p x_2 &= c x_2 - x_1 x_3, \\ \mathrm{D}^p x_3 &= x_1 x_2 - b x_3. \end{aligned} \tag{22}$$

When $a = 36$, $b = 3$, $c = 20$, this system has three equilibrium points at $(0, 0, 0)$, $(2\sqrt{15}, 2\sqrt{15}, 20)$, and $(-2\sqrt{15}, -2\sqrt{15}, 20)$. The last two equilibrium points are saddle-focus points. The smallest order of this system to generate chaos is 0.9156.[2] This system exhibits the chaotic behavior with initial value $(0.1, 0.2, 0.5)$ and order $p = 0.95$, as shown in Fig. 8.

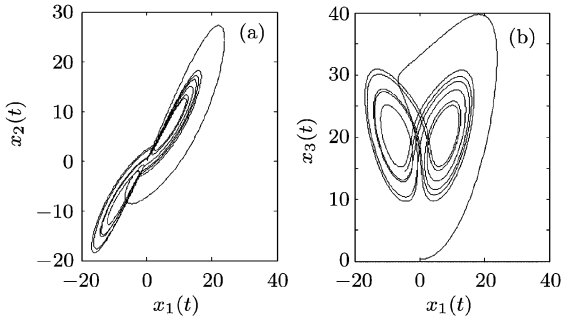


Fig. 8. State trajectories of the fractional Lu's system in chaotic state shown in (a) x_2–x_1 and (b) x_3–x_1 planes, and $p = 0.95$.

Applying the FRC controller with $d = 40$, $k = -276$ to this Lu's system, the simulation results are shown in Fig. 9, in which the control signals are added at $t = 0.5$ s. As can be seen from this figure, the proposed FRC controller can effectively control the fractional order Lu's systems to its equilibrium.

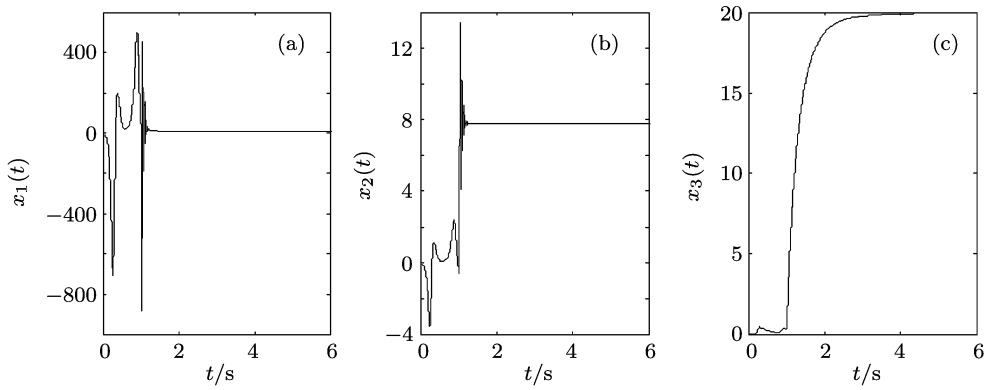


Fig. 9. Time waveforms of state variables (a) x_1, (b) x_2, and (c) x_3 of the controlled fractional Lu's system.

4.3. The fractional financial system

The fractional financial system of order p ($p \in (0, 1]$) is given by[2]

$$\begin{aligned} \mathrm{D}^p x_1 &= x_3 + (x_2 - a)x_1, \\ \mathrm{D}^p x_2 &= 1 - bx_2 - x_1^2, \\ \mathrm{D}^p x_3 &= -x_1 - cx_3. \end{aligned} \tag{23}$$

When $a = 1$, $b = 0.1$, $c = 1$, this system has three equilibrium points at (0,10,0), (0.894427, 2, −0.894427), and (−0.894427, 2, 0.894427). The last two equilibrium points are saddle-focus points. The smallest order of this system to generate chaos is 0.8536.[2] This system exhibits the chaotic behavior with initial value $(2, -1, -1)$ and order $p = 0.9$, as shown in Fig. 10.

Applying the FRC controller with $d = 3$, $k = -11.8$ to this fractional financial system, the simulation results in Fig. 11 show that this controller can also stabilize the fractional system to the equilibrium, in which the control signals are added at $t = 10$ s. In other words, the simulation results show that the obtained theoretic results in Section 3 are feasible and efficient for controlling the fractional chaotic systems.

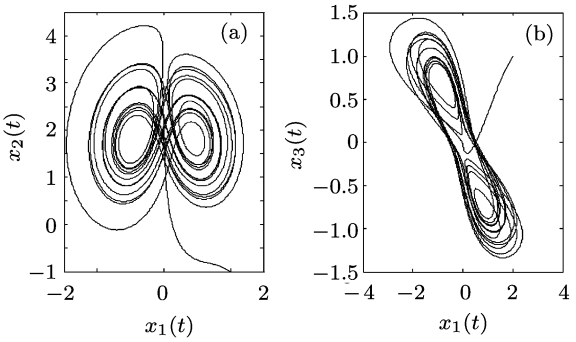


Fig. 10. State trajectories of the fractional financial system in chaotic state shown in (a) x_2–x_1 and (b) x_3–x_1 planes, and $p = 0.9$.

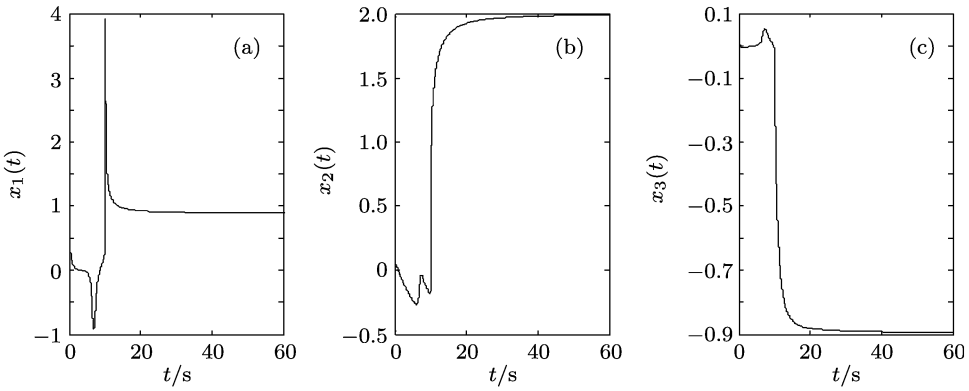


Fig. 11. Time waveforms of state variables (a) x_1, (b) x_2, and (c) x_3 of the controlled fractional financial system.

5. Conclusion

We proposed a fractional-order resistor–capacitor (FRC) controller and presented an FRC control law for arbitrary three-dimensional and fractional systems. According to the stability theorem of the fractional systems, the control law achieved to stabilize the fractional chaotic systems. The numerical simulation results indicated that the proposed control method can effectively control the fractional chaotic systems, which provides a theoretical basis for the applications of the fractional-order control in fractional chaotic systems. Future works regarding this topic include a theoretical analysis of the stability of the fractional chaotic systems and in-depth studies on the chaos control, especially the optimal control of the fractional chaotic systems.

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