Fluctuations and dissipations in stochastic energetic resonance*

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By analyzing the fluctuations and dissipations of a Brownian particle colliding with the molecules in a fluid, the work exchanged between the Brownian particle constrained in a bistable potential well and an external periodic force is investigated. Characters of the stochastic energetic resonance are found and studied at different intensities of fluctuations and dissipations. The microscopic mechanism of energy exchange between the Brownian particle and the external force is revealed. The method used in this study provides a novel way of controlling the stochastic energetic resonance.

Keywords: stochastic energetic resonance, fluctuation, dissipation, bistable system

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1. Introduction

Traditionally, a stochastic resonance,[1−5] whose dynamics obeys the Langevin equation, is characterized by quantities such as the signal-to-noise ratio, the spectral power amplification, and the residence time distribution.[6−21] In 2001, Iwai,[22] and Dan and Jayannavar[23] studied the energy transformation in a non-equilibrium process through the stochastic energy analysis, and proposed that the work done by an external force can be a measure of the stochastic resonance in a bistable system. In 2005, Bustamante *et al.*[24] studied the dynamics and fluctuations of small non-equilibrium systems down to the microscopic scale, and proposed that the interaction between the system and the environment is controlled by the thermal fluctuation. In 2007, Saikia *et al.*,[25] Sekimoto,[26] and Seifert[27] investigated the thermodynamics of a Brownian particle driven by an external force, and proposed that the entropy of the system can be defined by a stochastic trajectory. Subsequently, the theory of stochastic energetic resonance was established based on the energetic.[28,29]

There are two kinds of fluctuations, one is around an average value, and the other is not, e.g., the Brownian motion. The Brownian motion depends on the synergistic action of fluctuations and dissipations, i.e., a Brownian particle moves randomly driven by the fluctuation forces and damped by the dissipation forces. Since both the fluctuations and the dissipations are originated from the random collisions of the Brownian particle with the surrounding molecules in the fluid, the Brownian motion must follow the fluctuation and dissipation theorem.

When there exists an external force, the energy exchange between the Brownian particle and the outside occurs. For a particle in a fluid that is constrained in a bistable-potential well and experiencing a periodic external force, the Brownian motion of the particle should also be determined by the fluctuations and the dissipations there[30,31] besides obeying the general rules of dynamics. In this work, through analytic analysis and numerical simulation of the work exchange between an external force and a Brownian particle constrained in a bistable well, we will study the energy absorption processes of the Brownian particle with different intensities of fluctuations and dissipations. Our study reveals the mechanism of the stochastic energetic resonance.

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2. Fluctuations and dissipations in stochastic energetic resonance

2.1. The Brownian particle constrained in a bistable potential well

Consider a Brownian particle in a fluid, which is fettered on the wall of the container through a non-linear spring. The spring provides a bistable potential well, and the mass of the Brownian particle is far heavier than that of the fluid molecules, as shown in Fig. 1. We divide the force $F_{\rm fluid}$ on the Brownian particle from the fluid molecules into two parts. One is the mean force $F_{\rm dissipation}$, which causes the macroscopic viscous damping and embodies the energy dissipation of the ordered movement. The other is the fluctuating force $F_{\rm fluctuation}$, which stems from an irregular force on the Brownian particle from the random thermal movement of the fluid molecules, whose magnitude and direction fluctuate constantly. Suppose that the mass of the Brownian particle is m, the movement equation is

$$\begin{aligned} m\frac{{\rm d}v}{{\rm d}t}+\frac{{\rm d}V(x)}{{\rm d}x} &= F_{\rm fluid}(t) \\ &= F_{\rm fluctuation}(t)+F_{\rm dissipation}(t). \quad (1) \end{aligned}$$

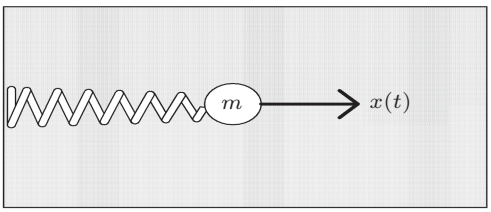


Fig. 1. The Brownian particle of mass m fettered on the wall of a container through a non-linear spring.

If an external periodic force $F_{\rm external}$ is applied to the Brownian particle, equation (1) can be written as

$$\begin{aligned} &m\frac{{\rm d}v}{{\rm d}t}+\frac{{\rm d}V(x)}{{\rm d}x}-F_{\rm external} \\ &= F_{\rm fluid}(t)=F_{\rm fluctuation}(t)+F_{\rm dissipation}(t), \quad (2) \end{aligned}$$

where $x(t)$ and $v(t)$ denote the displacement and the velocity of the Brownian particle, respectively, $V(x)=-\frac{1}{2}x^2+\frac{x^4}{4}$ is the double-well potential, $F_{\rm external}=A\cos(\omega t)$ ($A\ll 1$ is the amplitude, ω is the frequency) is the applied periodic force, $F_{\rm dissipation}(t)=-\gamma v(t)$ with γ being the damping coefficient, $F_{\rm fluctuation}(t)=\sqrt{2D}\xi(t)$, $\xi(t)$ denotes a Gaussian white noise with the statistical correlation functions $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(0)\rangle=\delta(t)$, and D is the fluctuation intensity. Then equation (2) can be expressed as

$$m\frac{{\rm d}v}{{\rm d}t}+\frac{{\rm d}V(x)}{{\rm d}x}-A\cos(\omega t)=\sqrt{2D}\xi(t)-\gamma v. \quad (3)$$

Two terms in the right side of Eq. (3) represent two basic elements of the Brownian motion, which are the fluctuation and the dissipation. They are caused by random collisions between the surrounding fluid molecules and the Brownian particle. The first term represents the energy from the random motion of the fluid molecules, which drives the particle to do the Brownian motion; the second term represents the dissipation during the Brownian motion in a given direction. These two parts are not independent. The fluctuation intensity D depends on the temperature T and the damping coefficient γ of the fluid via the relationship $D=\gamma k_{\rm B}T/m^2$, i.e., the fluctuation and dissipation theorem, where $k_{\rm B}$ is the Boltzmann constant.

2.2. Stochastic energetic resonance

When both sides of Eq. (3) are multiplied by ${\rm d}x$ in dot product, we obtain the following equation:

$$\begin{aligned} &m\frac{{\rm d}v}{{\rm d}t}{\rm d}x-\left(-\frac{{\rm d}V(x)}{{\rm d}x}+A\cos\omega t\right){\rm d}x \\ &=\sqrt{2D}\xi(t){\rm d}x-\gamma v{\rm d}x, \quad (4) \end{aligned}$$

where $m\frac{{\rm d}v}{{\rm d}t}{\rm d}x={\rm d}(\frac{1}{2}mv^2)$ is the variation of the kinetic energy. Considering the contribution of the input periodic force $A\cos(\omega t)$, we can write the potential function of the system as

$$U(x,t)=V(x)-xA\cos(\omega t). \quad (5)$$

Then, the differential representation of $U(x,t)$ is

$$\begin{aligned} {\rm d}U(x,t) &= \frac{\partial U(x,t)}{\partial x}{\rm d}x+\frac{\partial U(x,t)}{\partial t}{\rm d}t \\ &= -\left(-\frac{{\rm d}V(x)}{{\rm d}x}+A\cos\omega t\right){\rm d}x \\ &\quad +xA\omega\sin\omega t{\rm d}t. \quad (6) \end{aligned}$$

According to Eq. (4), the equation of energy balance is

$${\rm d}E+{\rm d}'W={\rm d}'Q_1-{\rm d}'Q_2, \quad (7)$$

where $E=mv^2/2+U(x,t)$ is the sum of the potential energy and the kinetic energy of the Brownian particle, ${\rm d}'W=xA\omega\sin\omega t{\rm d}t$ is the elementary work on the Brownian particle from the external macroscopic

force, which describes the capacity of the energy exchange between the Brownian particle and the outside, $\mathrm{d}'Q_1 = \sqrt{2D}\xi(t)\mathrm{d}x$ is the microscopic work done by the fluctuation force, and $\mathrm{d}'Q_2 = \gamma v\,\mathrm{d}x$ is the thermal energy transferred to the environment during the transformation of the ordered motion into the disordered one.

Here, we introduce W as the work done by the applied force $A\cos(\omega t)$ to the Brownian particle

$$W = \frac{1}{n}\int_{\tau_0}^{n\tau+\tau_0} A\omega\sin(\omega t)\langle x(t)\rangle\,\mathrm{d}t, \tag{8}$$

where $\tau = 2\pi/\omega$ is the period of the applied force, τ_0 is the initial transient time, and $\langle\ \rangle$ denotes the ensemble average. Under the adiabatic approximation ($A \ll 1$, $\omega \ll 1$, $D \ll 1$), the statistical average of the stochastic variable $x(t)$ can be expressed as[3]

$$\langle x(t)\rangle = X(D,\gamma)\cos[\omega t + \varphi(D,\gamma)]. \tag{9}$$

The amplitude $X(D,\gamma)$ and the phase $\varphi(D,\gamma)$ are

$$X(D,\gamma) = \frac{A\gamma^2}{D}\frac{2r_{\mathrm{K}}}{\sqrt{4r_{\mathrm{K}}^2+\omega^2}},$$
$$\varphi(D,\gamma) = -\arctan\left(\frac{\omega}{2r_{\mathrm{K}}}\right),$$

where r_{K} is the Kramers escape rate. According to the Kramers escape theory, when the damping coefficient of the system is large, the Kramers escape rate of a particle is approximately[30]

$$r_{\mathrm{K}}(D,\gamma) = \left(\sqrt{\frac{\gamma^2}{4}+\omega_{\mathrm{b}}^2}-\frac{\gamma}{2}\right)\frac{\omega_0}{2\pi\omega_{\mathrm{b}}} \times \exp\left(-\frac{\gamma^2}{4D}\right). \tag{10}$$

In Eq. (10), the angular frequencies of the bistable system at the minimum and the maximum are $\omega_0 = \sqrt{2}$ and $\omega_{\mathrm{b}} = 1$, respectively. When equation (9) is substituted into Eq. (8), the work done by the applied force to the Brownian particle is obtained as

$$W = -\pi A X(D,\gamma)\sin\varphi(D,\gamma). \tag{11}$$

Equation (11) indicates that the work W is a function of the fluctuation intensity D and the damping coefficient γ. With $A = 0.2$, $\omega = 0.02\pi$, $\gamma = 1$ in Eq. (11), the theoretical curve of the work W versus the intensity is shown in Fig. 2. The work W obviously peaks at a certain intensity D, indicating the happening of the stochastic energetic resonance.

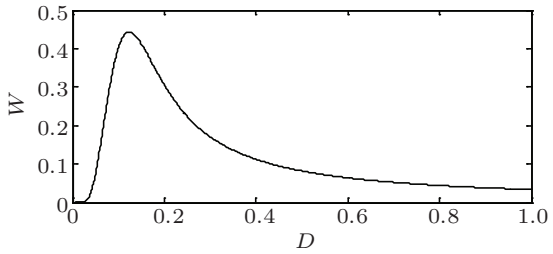


Fig. 2. Work W as a function of intensity D, showing the happening of the stochastic energetic resonance.

2.3. Fluctuations and dissipations

Equation (11) shows that the work done by an external force to the Brownian particle relies on the collisions between that particle and the fluid molecules, as W is a function of D and γ. The D describes the fluctuation intensity of the stochastic force on the Brownian particle, and γ describes the damping of the Brownian motion in the fluid, which causes the dissipation. As the Brownian particle in the fluid dissipates energy, the external force is introduced to keep the movement of the Brownian particle. In fact, the work done to the Brownian particle by an external force is different for different types of liquid molecules because of their different characters in fluctuations and dissipations. The theoretical relationship between W and D is plotted in Fig. 3 at the damping coefficients γ of 0.5, 1.0, and 2.0 respectively. It can be seen that the height of the stochastic energetic resonance peak increases with the decreasing γ, and that the position of the peak moves to larger intensity D with the increasing γ. Figure 4 shows the relationship between work W and damping coefficient γ at the intensities D of 0.3, 0.6, and 0.9 respectively. Again, the resonance peak is found. The peak position shifts to the side of larger damping coefficient γ with the increasing D.

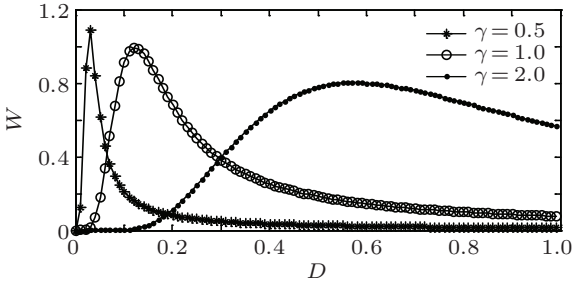


Fig. 3. Theoretical stochastic energetic resonances at $\gamma = 0.5$, 1.0, 2.0.

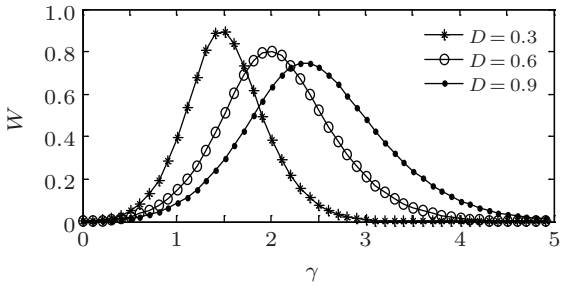


Fig. 4. Theoretical relationships between W and γ at $D = 0.3$, 0.6, 0.9.

3. Numerical simulation and analysis

The Brownian particle will exchange energy with the fluid if a weak periodic force is added to the particle, and the energy absorption depends on the intensity D and the damping coefficient γ of the fluid. We employ the fourth-order Runge–Kutta method to solve numerically Eq. (3) with the parameters $\tau_0 = 20\tau$, $n = 50$, $m = 1$, $A = 0.3$, $\omega = 0.02\pi$. The work done by the external force to the Brownian particle can be deduced. It changes with the varying fluctuation intensity D. Figure 5 shows the results of simulation at the damping coefficients of 0.5, 1.0, and 2.0 respectively. The results of numerical simulation shown in Fig. 5 are in good agreement with the analytic predictions shown in Fig. 3. The simulated relationships between work W and damping coefficient γ at the intensities D of 0.3, 0.6, and 0.9 respectively are shown in Fig. 6. It also agrees well with the analytic predictions shown in Fig. 4.

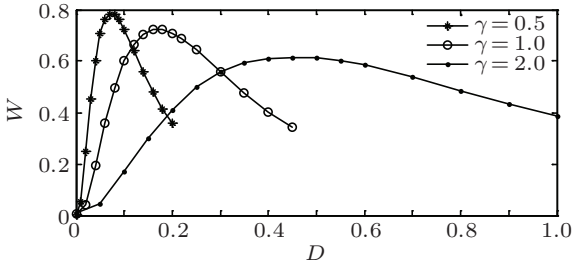


Fig. 5. Numerical simulations of work W as a function of the intensity D ($\omega = 0.02\pi$, $A = 0.3$).

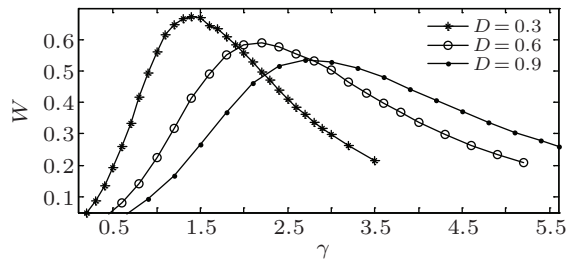


Fig. 6. Numerical simulations of work W as a function of the damping coefficient γ ($\omega = 0.02\pi$, $A = 0.3$).

4. Conclusion

Based on the microscopic dynamics of collisions between the Brownian particle and the fluid molecules, we have analyzed the influence of fluctuation and dissipation strengths on the characters of stochastic energetic resonance in the Brownian motion. Both the analytic analysis and the numerical simulation show that the work exerted by an applied periodic force on the Brownian particle exhibits a resonant response with the varying damping coefficient and fluctuation intensity. Moreover, the height and the position of the resonance peak change with the damping coefficient and the fluctuation intensity. In the presence of a weak periodic force, fluctuations of individual molecules in the fluid behave collectively instead of being independent and irrelevant, which influences strongly on the behavior of the Brownian motion at the macroscopic scale.

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