Topological aspect of vortex lines in two-dimensional Gross–Pitaevskii theory*

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Using the ϕ-mapping topological theory, we study the topological structure of vortex lines in a two-dimensional generalized Gross–Pitaevskii theory in (3+1)-dimensional space-time. We obtain the reduced dynamic equation in the framework of two-dimensional Gross–Pitaevskii theory, from which a conserved dynamic quantity is derived on the stable vortex lines. Such equations can also be used to discuss Bose–Einstein condensates in heterogeneous and highly nonlinear systems. We obtain an exact dynamic equation with a topological term, which is ignored in traditional hydrodynamic equations. The explicit expression of vorticity as a function of the order parameter is derived, where the δ function indicates that the vortices can only be generated from the zero points of $\boldsymbol{\Phi}$ and are quantized in terms of the Hopf indices and Brouwer degrees. The ϕ-mapping topological current theory also provides a reasonable way to study the bifurcation theory of vortex lines in the two-dimensional Gross–Pitaevskii theory.

Keywords: Gross–Pitaevskii equation, Bose–Einstein condensate, vortex line, bifurcation theory

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1. Introduction

The nonlinear quantum mechanics has a practical importance in different fields, such as condensed matter,[1] quantum optics,[2] and atomic and molecular physics.[3] Especially, quantum gravity may be related to nonlinear quantum mechanics,[4−7] and quantum computing is another important example.[8−11] Intensive studies, both theoretical and experimental, have been carried out since the advent of Bose–Einstein condensate (BEC),[12−14] which has recently been observed in dilute atomic vapors.[14,15] Various aspects of the condensate have been widely investigated. Well-known vortex states in superfluid and superconductor systems also play an important role in characterizing the superfluid properties of BEC systems.[16,17] The quasi-two-dimensional dipolar vortex streets are generated by a moving momentum source in the stratified fluid.[18] The quantized vortices in superfluids have been the center of fundamental theoretical and experimental work.[19,20] It is known that vortex-like excitations exist in high-temperature superconductors,[21] neutron stars,[22] and even rotating nuclei.[23] Some other topological defects that could exist in dilute gas condensates include textures found in Fermi superfluid, ^{3}He,[24] skyrmions,[25] and spin monopoles.[26] The quantum aspects of the vortex system in the BEC system are governed by a nonlinear Schrödinger equation which is equivalent to a Gross–Pitaevskii (GP) equation.[27,28]

Recently, much interest has been paid to low-dimensional BEC. Kolomeisky *et al.* have shown that an essential modification of the GP theory is necessary in two-dimensional space.[29] A relatively simple nonlinear Schrödinger equation (i.e., the GP equation)[27,28] with some higher-order terms of the Lagrangian density describes precisely the atomic condensates and their dynamics (at least at low temperatures). The field Lagrangian L is a function of field amplitude $\psi(\boldsymbol{r},t)$, which can usually be expressed as an integral over all space of a Lagrangian density $\mathscr{L}$

$$L=\int \mathscr{L}(\psi,\boldsymbol{\nabla}\psi,\dot{\psi},t)\mathrm{d}^3r. \tag{1}$$

According to variational principle, we can derive the

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Euler–Lagrange equation

$$\frac{\partial \mathscr{L}}{\partial \psi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{\psi}}\right) - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{L}}{\partial(\partial\psi/\partial x_i)}\right) = 0. \quad (2)$$

By substituting the following Lagrangian density of Schrödinger field

$$\mathscr{L} = \mathrm{i}\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - V_{\mathrm{ext}}\psi^*\psi \quad (3)$$

into Euler–Lagrange equation, the well-known linear Schrödinger equation can be derived. Here, the V_{ext} is the external potential. In Ref. [30], it is suggested that when the Lagrangian density of Schrödinger field includes the terms $\psi^*\dot{\psi}$, $\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi$, and $\psi^*\psi$, one can obtain the linear Schrödinger equation. If the Lagrangian density includes higher-order terms, e.g., $(\boldsymbol{\nabla}\psi^*) \cdot \boldsymbol{\nabla}\psi(\psi^*\psi)$, $(\boldsymbol{\nabla}\psi^*)^2 \cdot (\boldsymbol{\nabla}\psi)^2$, $(\psi^*\psi)^m$ $(m > 2)$, $\psi\psi^* \ln(\psi\psi^*)$, $\ln(\psi^*\psi)^n$, we obtain some nonlinear Schrödinger equations, such as the famous GP equation, and logarithmic-form equation.[31] In the present work, using the ϕ-mapping topological current theory,[32,33] we will confirm that in the system described by the generalized nonlinear Schrödinger equation, there is an elementary vortex topological current constructed by the order parameter. We reveal the intrinsic relation between vorticity and condensate wave function, as well as the topological structure of vortices. The reduced dynamic equation and a conserved dynamic quantity on stable vortex lines are derived. We also find that there exists a crucial case of branch process. The vortex lines cross, split, or merge at the bifurcation points.

This paper is organized as follows. In Section 2, we investigate the reduced dynamic equation and derive a conserved dynamic quantity on stable vortex lines. In Section 3, we discuss the basic formulation of mean-field theory (more precisely, the time-dependent GP equation) that describes the dilute BEC in the low-temperature limit. In Section 4, we study the bifurcation points of ψ, and show that there exist four branch processes (intersection, splitting, and mergence). We present our concluding remark in Section 5.

2. Two-dimensional Gross–Pitarvskii equation

We begin with a Landau–Ginzburg effective Lagrangian in the following form:

$$L = \int \mathrm{d}^3 r\Big(\mathrm{i}\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\boldsymbol{\nabla}\psi - V_{\mathrm{ext}}\psi^*\psi + e\,\|\psi\|^4\Big), \quad (4)$$

where the higher-order term $e\,\|\psi\|^4$ is added to the Lagrangian density $\mathscr{L}$ of the Schrödinger filed; $V_{\mathrm{ext}}(\boldsymbol{x})$ is the external potential which prevents the condensate from spreading out to infinity; e is the nonlinear parameter reflecting the interaction extent between coherent atoms. We can determine the constant e with dimension analysis. The Lagrangian density $\mathscr{L}$ of Schrödinger field has the dimension of energy density, i.e., $[E][L]^{-3}$, and ψ has the dimension of $[L]^{-3/2}$. In the above equation, the dimension of constant e is $[E][L]^3$. Then, the constant e can be written as

$$e = g\frac{\hbar^3}{m^2 c}, \quad (5)$$

where the constant g is dimensionless. This Lagrangian can be associated to a standard closed-form equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + V_{\mathrm{ext}}\psi + e\|\psi\|^2\psi, \quad (6)$$

which takes the form of the GP equation. We note in particular that for two-dimensional (2D) systems, Kolomeisky *et al.*[29] have suggested that in many experimentally interesting cases, the nonlinearity will be cubic or even logarithmic for $\|\psi\|^2$. Now we consider the generalized GP equation, where the quadratic $e\|\psi\|^2$ is replaced by an arbitrary nonlinearity $f(\psi^*\psi) = f(\|\psi\|^2)$, i.e.,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = \Big(-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_{\mathrm{ext}} + f(\|\psi\|^2)\Big)\psi. \quad (7)$$

It is well known that the wave function $\psi(\boldsymbol{x}, t)$ can be considered to be a section of a complex line bundle with base manifold M ($M = R^3 \otimes R$). The condensed wave function $\psi(\boldsymbol{x}, t)$ reads

$$\psi(\boldsymbol{x}, t) = \phi^1(\boldsymbol{x}, t) + \mathrm{i}\phi^2(\boldsymbol{x}, t), \quad (8)$$

where $\phi^1(\boldsymbol{x}, t)$ and $\phi^2(\boldsymbol{x}, t)$ are two components of a two-dimensional vector field $\boldsymbol{\Phi}(\boldsymbol{x}, t) = (\phi^1(\boldsymbol{x}, t), \phi^2(\boldsymbol{x}, t))$ in the (3+1)-dimensional spacetime. Substituting Eq. (8) into Eq. (6), we obtain

$$-\hbar\frac{\partial\phi^2}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\phi^1 + V_{\mathrm{ext}}\phi^1 + f(\|\psi\|^2)\phi^1, \quad (9)$$

$$\hbar\frac{\partial\phi^1}{\partial t} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\phi^2 + V_{\mathrm{ext}}\phi^2 + f(\|\psi\|^2)\phi^2. \quad (10)$$

Adding Eq. (9) to Eq. (10), we obtain the dynamic equation of the condensed atoms

$$\hbar\frac{1}{\|\phi\|^2}\epsilon_{ab}\phi^a\frac{\partial\phi^b}{\partial t} = \frac{\hbar^2}{2m}\frac{1}{\|\phi\|^2}\phi^a\boldsymbol{\nabla}^2\phi^a - (V_{\mathrm{ext}} + W),$$

$$a, b = 1, 2. \tag{11}$$

where $W = f(\|\psi\|^2)$. Based on Eqs. (9) and (10), it is easy to find that the continuity equation of the condensates describes the probability density ρ of the condensed atoms. It can be expressed as

$$\frac{\partial}{\partial t}(\|\phi\|^2) + \frac{\hbar}{m}\boldsymbol{\nabla}\cdot(\phi^1\boldsymbol{\nabla}\phi^2 - \boldsymbol{\nabla}\phi^1\phi^2) = 0, \tag{12}$$

where

$$\|\phi\|^2 = \phi^a\phi^a = \psi^*\psi, \tag{13}$$

and

$$\boldsymbol{j} = \frac{\hbar}{m}(\phi^1\boldsymbol{\nabla}\phi^2 - \boldsymbol{\nabla}\phi^1\phi^2) \tag{14}$$

is known as the probability current density vector. Equation (11) is the basic equation of the line-bundle representation of quantum mechanics, which can be regarded as the dynamic equation of a particle with mass m in the external potential field $V_{\rm ext}(\boldsymbol{x})$. It is clear that the continuity equation (12) can be rewritten in a simple and compact form

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0, \tag{15}$$

which reflects the conservation of the total probability of the condensed atoms. The corresponding current density $\boldsymbol{j}$ automatically takes a hydrodynamic form

$$\boldsymbol{j} = \rho\boldsymbol{u}, \tag{16}$$

with a flow velocity

$$\boldsymbol{u} = \frac{\boldsymbol{j}}{\rho} = \frac{\hbar}{m}(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*)/\|\psi\|^2. \tag{17}$$

Define a two-dimensional unit vector (n^1, n^2) as the two-dimensional unit vector field of the complex scalar field

$$n^a = \frac{\phi^a}{\|\phi\|}, \quad \|\phi\|^2 = \phi^a\phi^a, \quad a = 1, 2. \tag{18}$$

This is a reasonable representation, and ϕ^a is a two-component vector field related to the order parameter field ϕ. Obviously, it can be regarded as a smooth mapping between the (3+1)-dimensional space X (with the local coordinate x) and the two-dimensional Euclidean space R^2, i.e., $\phi : x \to \boldsymbol{\Phi}(\boldsymbol{x}) \in \mathrm{R}^2$, and n^a is a section of the sphere bundle $S^1(X)$. In the following, we will discuss the first term on the right-hand side of the dynamic equation (11), which is related to the quantum potential. Using the relation (18), we obtain

$$\frac{\phi^a\boldsymbol{\nabla}^2\phi^a}{\|\phi\|^2} = \frac{\boldsymbol{\nabla}^2\|\phi\|}{\|\phi\|} - \boldsymbol{\nabla}n^a\cdot\boldsymbol{\nabla}n^a. \tag{19}$$

Then Eq. (11) can be expressed as

$$\begin{aligned}&\hbar\frac{1}{\|\phi\|^2}\epsilon_{ab}\phi^a\frac{\partial\phi^b}{\partial t}\\ &= -\frac{\hbar^2}{2m}\boldsymbol{\nabla}n^a\cdot\boldsymbol{\nabla}n^a - (V_{\rm ext} + W + U),\end{aligned} \tag{20}$$

where $U = -(\hbar^2/2m)(\boldsymbol{\nabla}^2\|\phi\|/\|\phi\|)$ is the Bohm quantum potential.[34] Differentiating Eq. (20) with respect to coordinates x^μ, we have

$$\begin{aligned}&\hbar\epsilon_{ab}\partial_\mu\left[\frac{1}{\|\phi\|^2}\phi^a\frac{\partial\phi^b}{\partial t}\right]\\ &= m\epsilon_{\mu\nu\lambda}u^\nu(\boldsymbol{\nabla}\times\boldsymbol{u})^\lambda\\ &\quad - \partial_\mu\left(\frac{1}{2}mu^2 + U + V_{\rm ext} + W\right),\end{aligned} \tag{21}$$

$$\begin{aligned}&\hbar\epsilon_{ab}\partial_\mu\left[\frac{1}{\|\phi\|^2}\phi^a\frac{\partial\phi^b}{\partial t}\right]\\ &= m\frac{\partial u_\mu}{\partial t} - m\epsilon_{\lambda\mu\nu}\frac{\partial x^\nu}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{u})^\lambda.\end{aligned} \tag{22}$$

Subtracting Eq. (22)) from Eq. (21), we obtain the reduced dynamic equation

$$\begin{aligned}m\frac{\partial u_\mu}{\partial t} &= m\epsilon_{\lambda\mu\nu}\left(u^\nu + \frac{\partial x^\nu}{\partial t}\right)(\boldsymbol{\nabla}\times\boldsymbol{u})^\lambda\\ &\quad - \partial_\mu\left(\frac{1}{2}mu^2 + U + V_{\rm ext} + W\right).\end{aligned} \tag{23}$$

The angular velocity is given by

$$\boldsymbol{w} = \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{u}, \tag{24}$$

then we have

$$\begin{aligned}m\frac{\partial u_\mu}{\partial t} &= 2m\epsilon_{\lambda\mu\nu}\left(u^\nu + \frac{\partial x^\nu}{\partial t}\right)w^\lambda\\ &\quad - \partial_\mu\left(\frac{1}{2}mu^2 + U + V_{\rm ext} + W\right).\end{aligned} \tag{25}$$

In the case of the stable vortex lines, it is easy to obtain

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}t} = 0, \tag{26}$$

which directly leads to

$$\begin{aligned}&2m\epsilon_{\lambda\mu\nu}\left(u^\nu + \frac{\partial x^\nu}{\partial t}\right)w^\lambda\\ &= \partial_\mu\left(\frac{1}{2}mu^2 + U + V_{\rm ext} + W\right).\end{aligned} \tag{27}$$

Multiplying Eq. (27) with w^μ, and considering that w^μ is proportional to $\mathrm{d}x^\mu$, we obtain

$$\mathrm{d}x^\mu\partial_\mu\left(\frac{1}{2}mu^2 + U + V_{\rm ext} + W\right) = 0. \tag{28}$$

Thus, on the stable vortex lines, we derive a conserved dynamic quantity in the following form

$$\frac{1}{2}mu^2 + U + V_{\rm ext} + W = \text{const.} \tag{29}$$

It should be pointed out that the above equations do not depend on the concrete form of the nonlinear terms in Eq. (6). Whichever form the nonlinear term takes, even a logarithmic form, all the conclusions are the same.

3. Topological structure of vortex in BEC

BEC in optical lattices is a very active field of research, both from the theoretical and experimental aspect. The optical lattice is created by the interference of laser beams, and the lattice potential's strength can be modulated from weak to strong. In Refs. [27] and [28], Gross and Pitaevskii have independently investigated an inhomogeneous dilute Bose gas, which has the possibility of nonuniform states. The GP mean-field theory is an indispensable tool of both analyzing and predicting the outcome of experiments.

In mean-field approximation, the full BEC dynamic properties are well described by the time-dependent GP equation[35]

$$\begin{aligned} &\mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} + \frac{\hbar^2\boldsymbol{\nabla}^2}{2m}\psi(x,t) - V(r)\psi(x,t) \\ &- g\|\psi(x,t)\|^2\psi(x,t) = 0, \end{aligned} \tag{30}$$

where $\psi(x,t)$, g, and $V(r)$ are, respectively, the 2D wave function ("order parameter"), interaction strength, and external trapping potential. The quantity $\|\psi\|^2$ determines the atomic number density. The effective interaction parameter $g = 4\pi\hbar^2 a_s/m$ is tunable (even its sign), because the s-wave scattering length a_s can be adjusted using magnetic fields in the vicinity of a Feshbach resonance.[36] Numerous research groups are investigating the intermediate regime between molecular condensates and degenerate Fermi gases (the so-called "BEC–BCS" crossover regime[37]). Theoretical algorithms for manipulating $a_{\rm s}$ have been develop by analogy with "dispersion management" in nonlinear optics. Interactions between atoms are repulsive when $a_{\rm s} > 0$ and attractive when $a_{\rm s} < 0$.[39] For a dilute ideal gas, $a_{\rm s} \approx 0$. In a uniform bulk system, $a_{\rm s}$ must be helpful to prevent an instability leading to a collapse, but a Bose condensate in an external confining trap can remain stable for $a_{\rm s} < 0$ as long as the number of condensed atoms $N = \int \mathrm{d}r\|\psi\|^2$ remains below a critical value. The scattering lengths for the atomic species of Bose condensations used in the experiments is:[38] $a_{\rm s} = 2.75$ nm for ^{23}Na, $a_{\rm s} = 5.77$ nm for ^{87}Rb, and $a_{\rm s} = -1.45$ nm for ^{7}Li. In early experiments, only harmonic potentials were employed. The external trapping potential $V(r)$ of a periodic potential and a harmonic oscillator trap are modeled.[39] Recently, a variety of potentials have been constructed experimentally. Apart from the harmonic traps, these include double-well traps, periodic lattices, and superpositions of lattices or superlattices with harmonic traps.

In order to obtain the equations resembling the hydrodynamics for the condensate wave function ψ, we establish ψ in an equivalent "quantum-hydrodynamics" form[40]

$$\psi = \|\psi\|\ \mathrm{e}^{\mathrm{i}\Theta(\boldsymbol{x})}, \tag{31}$$

and the velocity field $\boldsymbol{u}$ becomes the gradient of phase factor $\Theta(\boldsymbol{x})$

$$\boldsymbol{u} = \frac{\hbar}{m}\boldsymbol{\nabla}\Theta, \tag{32}$$

which inevitably leads to the following irrotational result:

$$\boldsymbol{\nabla}\times\boldsymbol{u} = 0. \tag{33}$$

However, Onsager and Feynman[19] found that the equality $\boldsymbol{\nabla}\times\boldsymbol{u} = 0$ must be modified. Landau predicted that $\boldsymbol{\nabla}\times\boldsymbol{u}$ can be non-zero at a singular vortex line, and behaves like a δ function. Actually, we ignore an essential fact implied in Eq. (32) that the phase of the order parameter, Θ, is undefined at the vortex cores, i.e., zero points of the order parameter. Therefore, we cannot express the wave function in the form of Eq. (31) at the zero points of ψ. Instead of adopting the traditional expression (31), we shall rewrite it as

$$\psi = \phi^1 + \mathrm{i}\phi^2 = \|\phi\|(n^1 + \mathrm{i}n^2). \tag{34}$$

In the following, we will explore the exact expression for $\boldsymbol{\nabla}\times\boldsymbol{u}$ in the topological quantum field theory based on the expression (34) of the order parameter field by substituting Eq. (34) into the time-dependent GP equation (30).

Then the real part constitutes the analog of the Bernoulli equation

$$\begin{aligned} &-\hbar\epsilon_{ab}\phi^a\frac{\partial\phi^b}{\partial t} \\ &= -\frac{\hbar^2}{2m}\phi^a\boldsymbol{\nabla}^2\phi^a + g\|\phi\|^2 + (V-\mu)\|\phi\|^2. \end{aligned} \tag{35}$$

Correspondingly, the imaginary part yields the familiar continuity equation for compressible flow

$$\frac{\partial}{\partial t}\|\phi\|^2+\frac{\hbar}{m}\boldsymbol{\nabla}\cdot\left(\frac{\phi^1\boldsymbol{\nabla}\phi^2-\boldsymbol{\nabla}\phi^1\phi^2}{\|\phi\|^2}\|\phi\|^2\right)=0, \tag{36}$$

where the condensed atom density and the corresponding current density, respectively, are

$$\begin{aligned}\rho &= \|\phi\|^2=\phi^*\phi,\\ \boldsymbol{j} &= \frac{\hbar}{m}(\phi^1\boldsymbol{\nabla}\phi^2-\boldsymbol{\nabla}\phi^1\phi^2),\end{aligned} \tag{37}$$

which automatically assume a hydrodynamic form $\boldsymbol{j}(r,t)=\rho(r,t)\boldsymbol{u}(r,t)$ with the velocity field

$$\boldsymbol{u}=\frac{\phi^1\boldsymbol{\nabla}\phi^2-\boldsymbol{\nabla}\phi^1\phi^2}{\|\phi\|^2}. \tag{38}$$

Noting that

$$\partial_i n^a=\frac{\partial_i\phi^a}{\|\phi\|}+\phi^a\partial_i\frac{1}{\|\phi\|}, \tag{39}$$

and using the Green function relation in ϕ-space

$$\triangle_\phi \ln\|\phi\|=2\pi\delta^2(\Phi),\quad \triangle_\phi=\frac{\partial}{\partial\phi^a}\frac{\partial}{\partial\phi^a}, \tag{40}$$

it can be proved that[32]

$$\boldsymbol{\nabla}\times\boldsymbol{u}=\frac{h}{m}\delta^2(\boldsymbol{\Phi})\boldsymbol{D}\left(\frac{\phi}{x}\right), \tag{41}$$

where the vector Jacobians $\boldsymbol{D}(\phi/x)=(D^1(\phi/x),\ D^2(\phi/x),\ D^3(\phi/x))$ are defined as

$$\begin{aligned}D^1\left(\frac{\phi}{x}\right)&=\det\begin{pmatrix}\partial_y\phi^1 & \partial_z\phi^1\\ \partial_y\phi^2 & \partial_z\phi^2\end{pmatrix},\\ D^2\left(\frac{\phi}{x}\right)&=\det\begin{pmatrix}\partial_z\phi^1 & \partial_x\phi^1\\ \partial_z\phi^2 & \partial_x\phi^2\end{pmatrix},\\ D^3\left(\frac{\phi}{x}\right)&=\det\begin{pmatrix}\partial_x\phi^1 & \partial_y\phi^1\\ \partial_x\phi^2 & \partial_y\phi^2\end{pmatrix}.\end{aligned} \tag{42}$$

Equation (41) with singular function $\delta^2(\boldsymbol{\Phi})$ is just the precise topological expression for $\boldsymbol{\nabla}\times\boldsymbol{u}$ that Landau and Feynman expected to find out. We can reach an important conclusion from this expression that $\boldsymbol{\nabla}\times\boldsymbol{u}$ is equal to zero if and only if $\boldsymbol{\Phi}\neq 0$. Moreover, $\boldsymbol{\nabla}\times\boldsymbol{u}$ does not vanish but approaches infinity at the zero points of ϕ^a $(a=1,2)$, i.e.,

$$\phi^1(x,y,z,t)=0,\ \ \phi^2(x,y,z,t)=0. \tag{43}$$

This makes the solution of $\phi^a(x,y,z,t)=0$ extremely important.

The implicit function theory[41] suggests that, under the regular condition

$$D^i(\phi/x)\neq 0, \tag{44}$$

the general solutions of Eq. (43) can be expressed as

$$x=x_j(s,t),\ \ y=y_j(s,t),\ \ z=z_j(s,t), \tag{45}$$

which represent the evolution surfaces of N isolated strings L_j $(j=1,2,\ldots,N)$ moving in the (3+1)-dimensional sapcetime X with s being the string parameter. These singular strings are just the vortex lines in BEC systems. The location and direction of the i-th vortex are respectively determined by the i-th singular string L_i and the vector Jacobian $\boldsymbol{D}(\phi/x)\neq 0$ on L_i.

For a fixed t, we can obtain the following result from δ-function theory as:[42]

$$\begin{aligned}&\delta^2(\Phi)=\sum_{j=1}^{N}\beta_j\int_{L_j}\frac{\delta^3(\boldsymbol{x}-\boldsymbol{x}_j(s))}{\left|D\left(\frac{\phi}{u}\right)\right|_{\Sigma_j}}\mathrm{d}s,\\ &i=1,2,3,\end{aligned} \tag{46}$$

where

$$D\left(\frac{\phi}{u}\right)_{\Sigma_j}=\left(\frac{1}{2}\epsilon^{lk}\epsilon_{mn}\frac{\partial\phi^m}{\partial u^l}\frac{\partial\phi^n}{\partial u^k}\right),$$

and Σ_j is the j-th planar element transversal to L_j with local coordinates (u^1,u^2). The positive integer β_j is the Hopf index of ϕ-mapping. Since the direction vector of L_j is given by

$$\left.\frac{\mathrm{d}x^i}{\mathrm{d}s}\right|_{x_j}=\left.\frac{D^i(\phi/x)}{D(\phi/u)}\right|_{x_j}. \tag{47}$$

From Eqs. (46) and (47), we find the important inner topological structure of $\boldsymbol{\nabla}\times\boldsymbol{u}$

$$\begin{aligned}(\boldsymbol{\nabla}\times\boldsymbol{u})^i &= \frac{h}{m}j^i\\ &= \frac{h}{m}\sum_{j=1}^{N}\beta_j\eta_j\int_{L_j}\frac{\mathrm{d}x^i}{\mathrm{d}s}\delta^3(\boldsymbol{x}-\boldsymbol{x}_j(s))\mathrm{d}s,\end{aligned} \tag{48}$$

where η_j is the Brouwer degree, and $\eta_j=\mathrm{sgn}[D(\phi/u)]=\pm 1$. From Eq. (48), the winding number of $\boldsymbol{\Phi}$ around L_j is

$$W_j=\beta_j\eta_j. \tag{49}$$

Thus, the vorticity of vortex line L_j is

$$\Gamma_j=\int_{\Sigma_j}(\boldsymbol{\nabla}\times\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{s}=\frac{h}{m}W_j, \tag{50}$$

which describes an infinite straight vortex line with quantized circulation as suggested by Onsager and

Feynman. The total vorticity on a surface Σ should be

$$\Gamma = \int_{\Sigma} (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{s} = \frac{h}{m} \sum_{j=1}^{N} W_j. \tag{51}$$

It is obvious that the vorticity in a nonlinear system characterizes the density of topological charges for vortices. The vorticity is quantized by W_j, which is called the winding number. This is the topological essence of the vorticity in topological quantum mechanics.

In the weakly interacting BEC regime, the coefficients in the time-dependent GP equation (30) are almost the same as those given by Eq. (7), except that the coupling constant g has a physical meaning of the interaction strength. As a result, at any finite temperature below or near T_{c}, the dynamic equation of the condensed atoms is quite similar to the traditional dynamic equation (23).

The time-dependent GP equations (35) and (36) necessarily reproduce all the standard hydrodynamic behavior for both the vortex line and irrotational compressible flow. Using the relation that

$$\frac{\phi^a \boldsymbol{\nabla}^2 \phi^a}{\|\phi\|^2} = \frac{\boldsymbol{\nabla}^2 \|\phi\|^2}{\|\phi\|} - \left(\boldsymbol{\nabla} \frac{\phi^a}{\|\phi\|^2} \right), \tag{52}$$

and differentiating Eq. (35) with respect to coordinates, then we obtain the reduced dynamic equation

$$m \frac{\partial u_\mu}{\partial t} = m \epsilon_{\lambda\mu\nu} \left(u^\nu + \frac{\partial x^\nu}{\partial t} \right) (\boldsymbol{\nabla} \times \boldsymbol{u})^\lambda - \partial_\mu \left(\frac{1}{2} m u^2 + U + V + g \|\psi\|^2 \right). \tag{53}$$

Furthermore, for the case of the stable vortex line, the reduced dynamic equation yields a conserved dynamic quantity in the following form:

$$\frac{1}{2} m u^2 + U + V(r) + g \|\psi\|^2 = \text{const.}, \tag{54}$$

where $mu^2/2$ is the kinetic energy of the Bose atoms. In the limit $Na_{\mathrm{s}}/d_0 \gg 1$, which is relevant to current experiments on trapped Bose condensates, the repulsive interactions significantly expand the condensate, so that the kinetic energy associated with the density variation becomes negligible as compared with the trap energy $V(r)$ and interaction energy. If we take traditional expression $\psi = \|\psi\| \, \mathrm{e}^{\mathrm{i}\Theta(\boldsymbol{x})}$, the GP equation (30) is transferred into the traditional hydrodynamic equations[43]

$$\frac{\partial}{\partial t} \rho + \boldsymbol{\nabla} [\rho(r,t) v(r,t)] = 0, \tag{55}$$

$$m \frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla} \left(\frac{1}{2} m u^2 + T_{qp} + V + g \|\psi\|^2 \right), \tag{56}$$

where

$$T_{qp} = -\hbar^2 \frac{\boldsymbol{\nabla}^2 \sqrt{\rho(r,t)}}{2m \sqrt{\rho(r,t)}}$$

is the quantum pressure term, which has the same form as the Bohm quantum potential U. By comparing Eq. (53) with Eq. (56), we find that the first term on the RHS of Eq. (53) has been ignored all the time. The above results, obviously, indicate that this term describes the topological properties of the BEC system which takes the form of δ-like function. When the vector field has no zero values, the $\delta(\Phi)$ is zero and equation (53) becomes Eq. (56). At the zero points of ϕ^a, this term does not vanish but approaches infinity. Therefore, as for the usual equation (56), it only describes the non-topological properties of the system, while equation (53) describes both the vortex state and the irrotational state.

4. Intersection, splitting, and mergence of vortex lines

As discussed above, the zeros of the condensate wave function ψ play an important role in studying the vortex lines in the BEC system in the framework of the GP theory. Now we begin to study the properties of the zero lines (locations of line defects), i.e., the properties of the solutions of Eq. (43). In particular, we focus on the evolution processes of the vortex lines in the neighborhood of the critical point. Such vortex lines would no longer be stable, and they would evolve. In other words, generation and annihilation, splitting, and mergence of vortex lines would take place. For simplicity, we fix the z coordinate and consider an arbitrary cross section normal to the z axis. Namely, we consider $(2+1)$-dimensional space-time with coordinates $x_1 = x$, $x_2 = y$, and $x_0 = t$.

The velocity of the intersection point between L_j and the cross section is given by

$$v_j^i = \left. \frac{\mathrm{d}x^i}{\mathrm{d}t} \right|_{\boldsymbol{x}_j} = \left. \frac{D^i(\phi/x)}{D^0(\phi/x)} \right|_{\boldsymbol{x}_j}, \quad i = 1, 2. \tag{57}$$

As we know, the solution (45) of Eq. (43) is based on the condition that the Jacobian $\boldsymbol{D}(\phi/x) \neq 0$. When the condition fails, the above result (45) will change in some way. Let us turn to the case that the restrictions of Eq. (43) are

$$D^\mu(\phi/x) = 0, \quad \mu = 0, 1, 2. \tag{58}$$

These three restrictive conditions will lead to an important fact that the functional relationship between t and x or t and y is not unique in the neighborhood of $(\boldsymbol{x}_j^*, t^*)$. Due to Eq. (58), the velocity vector of the vortex lines at the bifurcation point $(\boldsymbol{x}_j^*, t^*)$ cannot be determined by Eq. (57). In general, there will be more than one velocity vector at the bifurcation point. Now we search for different directions of all branch curves or velocity fields of the line defects at the bifurcation point. It is known that, at the bifurcation point (x_j^{*i}, t^*), the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is smaller than 2 (for $D^i(\phi/x) = 0, \quad i = 0, 1, 2$).[43] Then, we suppose that the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is 1, $J_2(\phi/x) = \partial\phi^1/\partial x^2$, and

$$\det[J_2(\phi/x)] = \frac{\partial\phi^1}{\partial y} \neq 0. \tag{59}$$

Then, from the implicit function theorem,[41] it follows that there is a unique continuous solution to the following equation:

$$\phi^1(x^1, x^2, t) = 0. \tag{60}$$

This solution can be expressed as

$$x^2 = f^1(x^1, t). \tag{61}$$

Substituting Eq. (61) into Eq. (60), we obtain

$$\phi^1(x^1, f^1(x^1, t), t) \equiv 0. \tag{62}$$

Based on the differentiations of Eq. (62), and with the Gaussian elimination method, we can derive the first and second partial derivatives of $f^1(x^1, t)$. Substituting Eq. (61) into $\phi^2(\boldsymbol{x}, t) = 0$, we obtain

$$F(x^1, t) \equiv \phi^2(x^1, f^1(x^1, t), t). \tag{63}$$

From the definition of the branch point, we obtain

$$F(\boldsymbol{x}_j^*, t^*) = 0. \tag{64}$$

Using the partial derivatives of f^2 and f^3, as well as the Cramer rule, it can be proved that

$$\left.\frac{\partial F}{\partial x^1}\right|_{(\boldsymbol{x}_j^*, t^*)} = 0, \quad \left.\frac{\partial F}{\partial t}\right|_{(\boldsymbol{x}_j^*, t^*)} = 0. \tag{65}$$

According to Duan's topological current theory, the Taylor expansion of Eq. (63) in the neighborhood of the bifurcation point $(\boldsymbol{x}_j^*, t^*)$ reads

$$F(x^1, t) = \frac{1}{2}A(x^1 - x_j^{1*})^2 + B(x^1 - x_j^{1*})(t - t^*) + \frac{1}{2}C(t - t^*)^2 + \cdots = 0, \tag{66}$$

where A, B, and C are three constants given by the two-order partial derivative terms

$$A = \left.\frac{\partial^2 F}{(\partial x^1)^2}\right|_{(x_j^{1*}, t^*)}, \quad B = \left.\frac{\partial^2 F}{\partial x^1 \partial t}\right|_{(x_j^{1*}, t^*)}, \quad C = \left.\frac{\partial^2 F}{(\partial t)^2}\right|_{(x_j^{1*}, t^*)}. \tag{67}$$

By dividing Eq. (66) by $(t - t^*)^2$, equation (66) under the limits that $x^1 \to x_j^{1*}$ and $t \to t^*$ leads to

$$A\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + 2B\frac{\mathrm{d}x^1}{\mathrm{d}t} + C = 0, \tag{68}$$

or

$$C\left(\frac{\mathrm{d}t}{\mathrm{d}x^1}\right)^2 + 2B\frac{\mathrm{d}t}{\mathrm{d}x^1} + A = 0. \tag{69}$$

From Eq. (68) or Eq. (69), we can obtain the velocity component $\mathrm{d}x^1/\mathrm{d}t$. The remaining component $\mathrm{d}x^2/\mathrm{d}t$ can be given by[43]

$$\frac{\mathrm{d}x^2}{\mathrm{d}t} = f_{x^1}^{x^2}\frac{\mathrm{d}x^1}{\mathrm{d}t} + f_t^{x^2},$$

where the partial derivative coefficients $f_{x^1}^{x^2}$ and $f_t^{x^2}$ have been calculated using Eq. (60).

The solutions of Eq. (68) give different directions of the branch curves, which are the zero lines of the two-component vector order parameter. According to the different values of A, B, and C, there are four important cases showing the evolution of the vortex lines in detail: besides the encountering of the vortex lines, it also includes splitting and merging of the vortex lines. When a multi-charged line defect passes through the bifurcation point in three-dimensional space, it may split into several line defects along different branch curves. Moreover, several line defects can merge into one line defect at the bifurcation point. No matter which process takes place, from the following equality:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = 0, \tag{70}$$

one can always reach the conclusion that the winding number $\beta\eta$ is conserved. Now the topological structure of the vorticity at the bifurcation point should be written as

$$\boldsymbol{\nabla}\times\boldsymbol{u}=\begin{cases}\dfrac{h}{m}\displaystyle\sum_{l_{\rm i}}W_{l_{\rm i}}\int_{l_{\rm i}}\delta^3(\boldsymbol{x}-\boldsymbol{x}_{l_{\rm i}}(s))\mathrm{d}\boldsymbol{x}, & t<t^*,\\ \dfrac{h}{m}\displaystyle\sum_{l_{\rm f}}W_{l_{\rm f}}\int_{l_{\rm f}}\delta^3(\boldsymbol{x}-\boldsymbol{x}_{l_{\rm f}}(s))\mathrm{d}\boldsymbol{x}, & t>t^*,\end{cases}\tag{71}$$

where $l_{\rm i}$ denotes the initial vortex lines, and $l_{\rm f}$ denotes the finial vortex lines. Equation (71) is another version ($(\boldsymbol{D}(\phi/x)=0)$) of vorticity (48). In the above discussions, we consider only the case that two vortex lines pass through the bifurcation point. The configurations in which more vortex lines pass through the bifurcation point are high-energy unstable states, and will fall apart as quickly as they form. If the condition (59) fails, all other sub-Jacobians vanish at the bifurcation point, and we need to discuss the evolution processes at the higher-order bifurcation point. The procedure of deduction will be more complicated. However, the method will be similar to that we have used above.

5. Discussion and conclusion

By making use of Duan's topological current theory, we have clarified the issue on the generation of vortices in the nonlinear systems governed by the nonlinear GP equation. We find that the vortices originate from zeros of Φ naturally in the framework of Duan's topological current theory. Moreover, the reduced dynamic equation of the nonlinear systems is derived, from which we obtain a conserved dynamic quantity on the stable vortex lines. It is found that in BEC, the vorticity is just the topological current constructed by the condensate wave function. The explicit expression of the vorticity $\boldsymbol{\nabla}\times\boldsymbol{u}$ is given, which is no longer zero but takes the form of δ-like function. We reveal the intrinsic relation between the vorticity and the order parameter field, and prove in an straightforward manner that the vorticity represents the density of topological charges for vortices in 2D generalized GP equation. Using Duan's topological current theory, an exact reduced dynamic equation with a topological term is derived, which also behaves like δ-like function, and is always ignored in the traditional equation. Furthermore, we obtain a conserved dynamic quantity on the stable vortex lines.

These vortex lines are unstable at these branch points of the vector order parameter field. The branch processes including the spatial intersection, splitting, and mergence of vortex lines are discussed in detail. We should point out that when several vortices cross, split, or merge at a certain time t^*, the branch conditions $\boldsymbol{D}(\phi/x)|_{(\boldsymbol{x}_l^*,\,t^*)}=0$ are still valid, although these configurations are unstable. The results could be very conducive for further theoretical and experimental investigations of BEC. At last, it must be pointed out when the vortex lines cross, split, or merge at zero points of the condensate wave function, the branch conditions are valid not only in GP theory, but also in London theory, Ginzburg–Landau theory, etc. We hope these results in this paper will be helpful for simulating vortex dynamics in the 2D GP theory.

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