Lattice soliton equation hierarchy and associated properties

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As a new subject, soliton theory is shown to be an effective tool for describing and explaining the nonlinear phenomena in nonlinear optics, super conductivity, plasma physics, magnetic fluid, etc. Thus, the study of soliton equations has always been one of the most prominent events in the field of nonlinear science during the past few years. Moreover, it is important to seek lattice soliton equation and study its properties. In this study, firstly, we derive a discrete integrable system by use of Tu model. Then, some properties of the obtained equation hierarchies are discussed.

Keywords: discrete integrable system, Darboux transformation, conservation laws

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1. Introduction

Recently years, discrete integrable system has become the focus of common concern and one of the most important research areas because of its applications in physics, chemistry, and biology. Therefore, more attention should be paid to the generating of discrete integrable systems and its related properties. The method of discrete zero curvature representation is an important way to obtain the lattice soliton equations. According to this method, the lattice soliton equation can be written as a compatibility condition of a suitable discrete spatial spectral problem and the corresponding temporal spectral problem. So far, by use of Tu model,[1] a lot of nonlinear integrable lattice equations have been obtained.[2−5]

At the same time, some systematic methods used for obtaining exact solutions of soliton equations are well established, such as the inverse scattering transform method, the Hirotabilinear transform method, Darboux transformation, and the algebra-geometric method.[6−11] Among the various ways, the Darboux transformation is a powerful tool to obtain explicit solutions of the lattice soliton equation.

One important characteristic of integrable equations is conservation laws which play an important role in mathematics and engineering.[12] Conservation laws are closely related to symmetries of equations, so the existence of an infinite number of conservation laws is a wide definition of complete integrability of equations.

2. Discrete integrable system

Consider the discrete eigenvalue problem

$$\begin{aligned} &E\boldsymbol{\psi}_n = \boldsymbol{U}_n\boldsymbol{\psi}_n, \quad \lambda_t = 0,\\ &\boldsymbol{U}_n(u,\lambda) = \begin{pmatrix} \lambda & \frac{q_n}{p_n} \\ \lambda p_n & 1+q_n \end{pmatrix},\\ &\boldsymbol{\psi}_n = \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}, \boldsymbol{u}_n = \begin{pmatrix} p_n \\ q_n \end{pmatrix}, \end{aligned} \tag{1}$$

where λ is the spectral parameter, and the shift operator E is determined by

$$\begin{aligned} &Ef(n) = f(n+1), \quad E^{-1}f(n) = f(n-1),\\ &(E-1)f(n) = f_{n+1} - f_n, \quad E^k = f^{(k)}. \end{aligned}$$

Let

$$\boldsymbol{\Gamma} = \begin{pmatrix} a_n & b_n \\ \lambda c_n & -a_n \end{pmatrix}. \tag{2}$$

Solving the stationary zero curvature equation

$$(E\boldsymbol{\Gamma})\boldsymbol{U}_n - \boldsymbol{U}_n\boldsymbol{\Gamma} = 0 \tag{3}$$

gives rise to

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$$\begin{cases} \lambda(a_{n+1}-a_n+q_nb_{n+1}-(p_n+\frac{p_n}{q_n})c_n)=0,\\ (p_n+\frac{p_n}{q_n})(a_{n+1}+a_n)+(1+p_n+p_nq_n)b_{n+1}-\lambda b_n=0,\\ \lambda^2c_{n+1}-\lambda q_n(a_{n+1}+a_n)-\lambda(1+p_n+p_nq_n)c_n=0,\\ \lambda(p_n+\frac{p_n}{q_n})c_{n+1}-\lambda q_nb_n-(1+p_n+p_nq_n)(a_{n+1}-a_n)=0. \end{cases} \tag{4}$$

Let $a_n=\sum_{m=0}^{\infty}a_n^{(m)}\lambda^{-m}$, $b_n=\sum_{m=0}^{\infty}b_n^{(m)}\lambda^{-m}$, $c_n=\sum_{m=0}^{\infty}c_n^{(m)}\lambda^{-m}$. From (4) we have the initial data

$$b_n^{(0)}=c_{n+1}^{(0)}=0,\quad a_{n+1}^{(0)}-a_n^{(0)}+q_nb_{n+1}^{(0)}-\left(p_n+\frac{p_n}{q_n}\right)c_n^{(0)}=0. \tag{5}$$

and recursion relation

$$\begin{cases} a_{n+1}^{(m)}-a_n^{(m)}+q_nb_{n+1}^{(m)}-(p_n+\frac{p_n}{q_n})c_n^{(m)}=0,\\ (p_n+\frac{p_n}{q_n})(a_{n+1}^{(m)}+a_n^{(m)})+(1+p_n+p_nq_n)b_{n+1}^{(m)}-b_n^{(m+1)}=0,\\ c_{n+1}^{(m+1)}-q_n(a_{n+1}^{(m)}+a_n^{(m)})-(1+p_n+p_nq_n)c_n^{(m)}=0,\\ (p_n+\frac{p_n}{q_n})c_{n+1}^{(m+1)}-q_nb_n^{(m+1)}-(1+p_n+p_nq_n)(a_{n+1}^{(m)}-a_n^{(m)})=0. \end{cases} \tag{6}$$

We choose $a_n^{(0)}=\frac{1}{2}$, $b_n^{(0)}=c_n^{(0)}=0$ and require $a_j\big|_{[u]=0}=0$, $b_j\big|_{[u]=0}=0$, $c_j\big|_{[u]=0}=0$ $(j\geq 1)$, where $[u]=(u,Eu,E^{-1}u,\ldots)$. Thus, the recursion relation (6) uniquely determines $a_j,b_j,c_j,j\geq 1$ and the first a few quantities are given as follows:

$$\begin{aligned} a_n^{(1)} &= -\left(1+\frac{1}{q_n}\right)p_nq_{n-1},\\ b_n^{(1)} &= p_n+\frac{p_n}{q_n},\quad c_n^{(1)}=q_{n-1},\\ a_n^{(2)} &= \left(1+\frac{1}{q_n}\right)^2p_n^2q_{n-1}^2\\ &\quad -p_nq_{n-2}-p_{n+1}q_{n-1}-\frac{p_nq_{n-2}}{q_n}-\frac{p_{n+1}q_{n-1}}{q_{n+1}},\\ b_n^{(2)} &= p_{n+1}+\frac{p_{n+1}}{q_{n+1}}-\left(1+\frac{1}{q_n}\right)^2p_n^2q_{n-1},\\ c_n^{(2)} &= q_{n-2}-p_nq_{n-1}^2-\frac{p_n}{q_n}q_{n-1}^2,\ldots \end{aligned}$$

For any integer $m\geq 0$, we choose

$$\boldsymbol{\Gamma}_1^{\{m\}}=\sum_{i=0}^{m}\begin{pmatrix} a_n^{(i)}\lambda^{m-i} & b_n^{(i)}\lambda^{m-i}\\ c_n^{(i)}\lambda^{m-i+1} & -a_n^{(i)}\lambda^{m-i}\end{pmatrix}, \tag{7}$$

then the following equation holds:

$$\begin{aligned} &(E\boldsymbol{\Gamma}_1^{\{m\}})\boldsymbol{U}_n-\boldsymbol{U}_n\boldsymbol{\Gamma}_1^{\{m\}}\\ &=\begin{pmatrix} 0 & b_n^{(m+1)}\\ -\lambda c_{n+1}^{(m+1)} & q_nb_n^{(m+1)}-\left(p_n+\frac{p_n}{q_n}\right)c_{n+1}^{(m+1)}\end{pmatrix}. \end{aligned} \tag{8}$$

Then the discrete zero curvature equation admits the following positive hierarchy:

$$\boldsymbol{u}_{n_{t_m}}=\begin{pmatrix} p_n\\ q_n\end{pmatrix}_{t_m}=\begin{pmatrix} \frac{q_nb_n^{(m+1)}}{1+q_n}-\frac{p_nc_{n+1}^{(m+1)}}{q_n(1+q_n)}\\ -c_{n+1}^{(m+1)}\end{pmatrix}. \tag{9}$$

Taking $m=0$ in (9), the system (9) reduces to

$$\begin{cases} p_{n_{t_0}}=-p_n,\\ q_{n_{t_0}}=0. \end{cases} \tag{10}$$

Taking $m=1$ in (9), the system (9) reduces to

$$\begin{cases} p_{n_{t_1}}=\dfrac{q_np_{n+1}(1+q_{n+1})(1+p_n)}{q_{n+1}(1+q_n)}\\ \qquad -\dfrac{p_nq_{n-1}}{q_n(1+q_n)}-\dfrac{(1+q_n)p_n^2q_{n-1}}{q_n},\\ q_{n_{t_1}}=p_{n+1}q_n^2+\dfrac{p_{n+1}}{q_{n+1}}q_n^2-q_{n-1}. \end{cases} \tag{11}$$

3. Darboux transformation

Taking $m=1$ in (7) the spectral problem can be written as

$$\boldsymbol{Y}_{n+1}=\boldsymbol{U}_n\boldsymbol{Y}_n,\ \boldsymbol{Y}_{n_t}=\boldsymbol{V}_n\boldsymbol{Y}_n, \tag{12}$$

$$\boldsymbol{U}_n=\begin{pmatrix} \lambda & \frac{q_n}{p_n}\\ \lambda p_n & 1+q_n\end{pmatrix},$$

$$\boldsymbol{V}_n=\boldsymbol{\Gamma}^{\{1\}}=\begin{pmatrix} \frac{1}{2}\lambda-\frac{p_{n-1}q_n}{p_n} & \frac{q_n}{p_n}\\ \lambda p_{n-1} & \frac{p_{n-1}q_n}{p_n}-\frac{1}{2}\lambda\end{pmatrix}. \tag{13}$$

Let $\boldsymbol{\varphi} = (\varphi^1, \varphi^2)^{\mathrm{T}}$, $\boldsymbol{\phi} = (\phi^1, \phi^2)^{\mathrm{T}}$ are two basic solutions of spectral problem (12) and use $(\boldsymbol{\varphi}, \boldsymbol{\phi})$ to define a matrix $\boldsymbol{T}_n$ by

$$\boldsymbol{T}_n = \begin{pmatrix} \lambda + t_{11}(n) & t_{12}(n) \\ \lambda t_{21}(n) & \lambda + t_{22}(n) \end{pmatrix}, \tag{14}$$

with

$$\begin{aligned} t_{11}(n) &= \frac{\lambda_1\alpha_2(n) - \lambda_2\alpha_1(n)}{\alpha_1(n) - \alpha_2(n)}, \\ t_{12}(n) &= \frac{\lambda_2 - \lambda_1}{\alpha_1(n) - \alpha_2(n)}, \\ t_{21}(n) &= \frac{(\lambda_1 - \lambda_2)\alpha_1(n)\alpha_2(n)}{\lambda_2\alpha_1(n) - \lambda_1\alpha_2(n)}, \\ t_{22}(n) &= \frac{\lambda_1\lambda_2(\alpha_2 - \alpha_1)}{\lambda_2\alpha_1(n) - \lambda_1\alpha_2(n)}, \end{aligned} \tag{15}$$

$$\alpha_i(n) = \frac{\varphi_n^2(\lambda_i) - \gamma_i\phi_n^2(\lambda_i)}{\varphi_n^1(\lambda_i) - \gamma_i\phi_n^1(\lambda_i)}, \quad (i = 1, 2), \tag{16}$$

where parameter λ_i, and $\gamma_i (\lambda_1 \neq \lambda_2, \gamma_1 \neq \gamma_2)$ are suitably chosen such that all the denominators in Eqs. (15) and (16) are nonzero. From Eq. (14), we have

$$\det \boldsymbol{T}_n = (\lambda - \lambda_1)(\lambda - \lambda_2). \tag{17}$$

Now, we assume that there is a gauge transformation

$$\tilde{\boldsymbol{Y}}_n = \boldsymbol{T}_n \boldsymbol{Y}_n \tag{18}$$

that can transform the spectral problem (12) into spectral problems of $\tilde{Y}_n$ in $\lambda \neq \lambda_i$ $(i = 1, 2)$ as follows:

$$\tilde{\boldsymbol{Y}}_{n+1} = \tilde{\boldsymbol{U}}_n \tilde{\boldsymbol{Y}}_n, \ \tilde{\boldsymbol{Y}}_t = \tilde{\boldsymbol{V}}_n \tilde{\boldsymbol{Y}}_n, \tag{19}$$

with $\tilde{\boldsymbol{U}}_n = \boldsymbol{T}_{n+1}\boldsymbol{U}_n\boldsymbol{T}_n^{-1}$, $\tilde{\boldsymbol{V}}_n = (\boldsymbol{T}_{nt} + \boldsymbol{T}_n\boldsymbol{V}_n)\boldsymbol{T}_n^{-1}$. From Eqs. (15) and (16), we have

$$\alpha_i(n+1) = \frac{\mu_i(n)}{\nu_i(n)}, i = 1, 2, \tag{20}$$

with

$$\begin{aligned} \mu_i(n) &= \lambda_i q_n + (1 + q_n)\alpha_i(n), \\ \nu_i(n) &= \lambda_i + \frac{q_n}{p_n}\alpha_i(n). \end{aligned} \tag{21}$$

By using Eqs. (15) and (20), we have

$$\begin{aligned} t_{11}(n+1) &= \frac{\mu_2(n)\nu_1(n)\lambda_1 - \mu_1(n)\nu_2(n)\lambda_2}{\mu_1(n)\nu_2(n) - \mu_2(n)\nu_1(n)}, \\ t_{12}(n+1) &= \frac{\nu_1(n)\nu_2(n)(\lambda_2 - \lambda_1)}{\mu_1(n)\nu_2(n) - \mu_2(n)\nu_1(n)}, \\ t_{21}(n+1) &= \frac{\mu_1(n)\mu_2(n)(\lambda_1 - \lambda_2)}{\mu_1(n)\nu_2(n)\lambda_2 - \mu_2(n)\nu_1(n)\lambda_1}, \\ t_{22}(n+1) &= \frac{(\mu_2(n)\nu_1(n) - \mu_1(n)\nu_2(n))\lambda_1\lambda_2}{\mu_1(n)\nu_2(n)\lambda_2 - \mu_2(n)\nu_1(n)\lambda_1}. \end{aligned} \tag{22}$$

From all of these, we can prove the following assertion.

Proposition 1 The matrix $\tilde{\boldsymbol{U}}_n = \boldsymbol{T}_{n+1}\boldsymbol{U}_n\boldsymbol{T}_n^{-1}$ has the same form as $\boldsymbol{U}_n$, that is,

$$\tilde{\boldsymbol{U}}_n = \begin{pmatrix} \lambda & \dfrac{\tilde{q}_n}{\tilde{p}_n} \\ \lambda\tilde{p}_n & 1 + \tilde{q}_n \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} \tilde{p}_n &= p_n + t_{21}(n+1), \\ \tilde{q}_n &= (p_n + t_{21}(n+1))\left(\frac{q_n}{p_n} - t_{12}(n)\right). \end{aligned} \tag{24}$$

The transformation (18) and (24): $(\boldsymbol{Y}_n; p_n, q_n \rightarrow \tilde{\boldsymbol{Y}}_n; \tilde{p}_n, \tilde{q}_n)$ is usually called a Darboux transformation(DT) of the eigenvalue problem (12). Equation (24) is the so-called Bäcklund transformation (BT) between the new and old potentials.

Proof Let

$$\begin{aligned} \boldsymbol{T}_n^{-1} &= \frac{\boldsymbol{T}_n^*}{\det \boldsymbol{T}_n}, \\ \boldsymbol{T}_{n+1}\boldsymbol{U}_n\boldsymbol{T}_n^* &= \begin{pmatrix} f_{11}(\lambda, n) & f_{12}(\lambda, n) \\ f_{21}(\lambda, n) & f_{22}(\lambda, n) \end{pmatrix}. \end{aligned} \tag{25}$$

It is easy to verify that $f_{11}(\lambda, n), f_{12}(\lambda, n), f_{21}(\lambda, n)$, and $f_{22}(\lambda, n)$ are 3-order polynomials in λ. By virtue of Eqs. (12) and (15), it can be verified that $f_{kl}(\lambda_i, n) = 0 (i, k, l = 1, 2)$. Based on the above results, we can suppose

$$\boldsymbol{T}_{n+1}\boldsymbol{U}_n\boldsymbol{T}_n^* = (\det \boldsymbol{T}_n)\boldsymbol{S}_n, \tag{26}$$

with

$$\boldsymbol{S}_n = \begin{pmatrix} s_{11}^0 + \lambda s_{11}^1 & s_{12}^0 + \lambda s_{12}^1 \\ s_{21}^0 + \lambda s_{21}^1 & s_{22}^0 + \lambda s_{22}^1 \end{pmatrix}. \tag{27}$$

That is,

$$\boldsymbol{T}_{n+1}\boldsymbol{U}_n = \boldsymbol{S}_n\boldsymbol{T}_n, \tag{28}$$

where s_{ij}^l, $(i, j = 1, 2; l = 0, 1)$ are undetermined functions independent of λ. By comparing the coefficients of λ^i $(i = 0, 1, 2)$ in both sides of Eq. (28), we obtain

$$\begin{aligned} & s_{11}^0 = 0, \quad s_{12}^1 = 0, \quad s_{21}^0 = 0, \quad s_{22}^1 = 0, \\ & s_{11}^1 = 1, \quad s_{12}^0 = \frac{q_n}{p_n} - t_{12}(n) = \frac{\tilde{q}_n}{\tilde{p}_n}, \\ & s_{21}^1 = p_n + t_{21}(n+1) = \tilde{p}_n, \\ & s_{22}^0 = 1 + (p_n + t_{21}(n+1))\left(\frac{q_n}{p_n} - t_{12}(n)\right) \\ & \qquad = 1 + \tilde{q}_n. \end{aligned} \tag{29}$$

From (29), we have $\tilde{p}_n = p_n + t_{21}(n+1)$, $\tilde{q}_n = (p_n + t_{21}(n+1))\,(q_n/p_n - t_{12}(n))$.
Thus, we complete the proof.

Proposition 2 Under the gauge transformation (18), the matrix $\tilde{\boldsymbol{V}}_n$ has the same form as $\boldsymbol{V}_n$, that is

$$\tilde{\boldsymbol{V}}_n = \begin{pmatrix} \frac{1}{2}\lambda - \frac{\tilde{p}_{n-1}\tilde{q}_n}{\tilde{p}_n} & \frac{\tilde{q}_n}{\tilde{p}_n} \\ \lambda\tilde{p}_{n-1} & \frac{\tilde{p}_{n-1}\tilde{q}_n}{\tilde{p}_n} - \frac{1}{2}\lambda \end{pmatrix}. \quad (30)$$

Proof Let

$$\boldsymbol{T}_n^{-1} = \frac{\boldsymbol{T}_n^*}{\det \boldsymbol{T}_n},$$
$$(\boldsymbol{T}_{nt} + \boldsymbol{T}_n V_n)\boldsymbol{T}_n^* = \begin{pmatrix} g_{11}(\lambda,n) & g_{12}(\lambda,n) \\ g_{21}(\lambda,n) & g_{22}(\lambda,n) \end{pmatrix}. \quad (31)$$

It is easy to verify that $g_{11}(\lambda,n), g_{12}(\lambda,n), g_{21}(\lambda,n)$, and $g_{22}(\lambda,n)$ are cubic polynomials in λ. By virtue of Eqs.(12) and (15), it can be verified that $g_{kl}(\lambda_i,n) = 0$ $(i,k,l = 1,2)$. Therefore,we have

$$(\boldsymbol{T}_{nt} + \boldsymbol{T}_n\boldsymbol{V}_n)\boldsymbol{T}_n^* = (\det \boldsymbol{T}_n)\boldsymbol{R}_n, \quad (32)$$

with

$$\boldsymbol{R}_n = \begin{pmatrix} r_{11}^0 + \lambda r_{11}^1 & r_{12}^0 + \lambda r_{12}^1 \\ r_{21}^0 + \lambda r_{21}^1 & r_{22}^0 + \lambda r_{22}^1 \end{pmatrix}. \quad (33)$$

That is,

$$\boldsymbol{T}_{nt} + \boldsymbol{T}_n\boldsymbol{V}_n = \boldsymbol{R}_n\boldsymbol{T}_n, \quad (34)$$

where r_{ij}^l, $(i,j = 1,2;\ l = 0,1)$ are independent of λ. Comparing the coefficients of λ^i $(i = 0,1,2)$ in both sides of Eq. (34), we have

$$\begin{aligned} r_{12}^1 &= 0, \quad r_{21}^0 = 0, \quad r_{11}^1 = -r_{22}^1 = \frac{1}{2}, \\ r_{12}^0 &= \frac{q_n}{p_n} - t_{12}(n) = \frac{\tilde{q}_n}{\tilde{p}_n}, \\ r_{21}^1 &= p_{n-1} + t_{21}(n) = \tilde{p}_{n-1}, \\ r_{11}^0 &= -r_{22}^0 = (p_{n-1} + t_{21}(n))\left(t_{12}(n) - \frac{q_n}{p_n}\right) \\ &= -\frac{\tilde{q}_n\tilde{p}_{n-1}}{\tilde{p}_n}. \end{aligned} \quad (35)$$

From the above propositions we come to the following conclusion.

Proposition 3 The transformation $(\boldsymbol{Y}_n; p_n, q_n \to \tilde{\boldsymbol{Y}}_n; \tilde{p}_n, \tilde{q}_n)$ is the Darboux transformation of the spectral problem (12), under the transformation

$$\begin{aligned} \tilde{p}_n &= p_n + t_{21}(n+1), \\ \tilde{q}_n &= (p_n + t_{21}(n+1))\left(\frac{q_n}{p_n} - t_{12}(n)\right), \end{aligned} \quad (36)$$

where the solutions p_n, q_n are mapped into the new solutions $\tilde{p}_n, \tilde{q}_n$.

Substituting the trivial solutions $p_n = 1$, $q_n = 1$ of Eq. (11) into (12) leads to

$$\begin{aligned} \boldsymbol{Y}_{n+1} &= \begin{pmatrix} \lambda & 1 \\ \lambda & 2 \end{pmatrix}\boldsymbol{Y}_n, \\ \boldsymbol{Y}_{n_t} &= \begin{pmatrix} \frac{1}{2}\lambda - 1 & 1 \\ \lambda & -\frac{1}{2}\lambda + 1 \end{pmatrix}\boldsymbol{Y}_n, \end{aligned} \quad (37)$$

from which two basic solutions of Eq. (11) are presented as

$$\begin{aligned} \boldsymbol{\varphi}_n &= \left(1 + \frac{\sqrt{\lambda^2+4} + \lambda}{2}\right)^n \\ &\quad \times \mathrm{e}^{\frac{\sqrt{\lambda^2+4}}{2}t}\begin{pmatrix} 1 \\ 1 + \frac{\sqrt{\lambda^2+4} - \lambda}{2} \end{pmatrix}, \\ \boldsymbol{\phi}_n &= \left(1 - \frac{\sqrt{\lambda^2+4} - \lambda}{2}\right)^n \\ &\quad \times \mathrm{e}^{-\frac{\sqrt{\lambda^2+4}}{2}t}\begin{pmatrix} 1 \\ 1 - \frac{\sqrt{\lambda^2+4} + \lambda}{2} \end{pmatrix}. \end{aligned} \quad (38)$$

Substituting (38) into (16), we have

$$\alpha_i(n) = \frac{\omega_i\,\mathrm{e}^{\sqrt{\lambda_i^2+4}t}\left(1 + \frac{\sqrt{\lambda_i^2+4} - \lambda_i}{2}\right) - \gamma_i\left(1 - \frac{\sqrt{\lambda_i^2+4} + \lambda_i}{2}\right)}{\omega_i\,\mathrm{e}^{\sqrt{\lambda_i^2+4}t} - \gamma_i}, (i = 1,2), \quad (39)$$

where $\omega_i = (2 + \lambda_i + \sqrt{\lambda_i^2+4})^{2n}/4\lambda_i^n$. Hence, with the help of DT (18) and (36) the new solutions of Eq. (11) are given as follows:

$$\begin{aligned} \tilde{p}_n &= 1 + \frac{(\lambda_1 + 2\alpha_1(n))(\lambda_2 + 2\alpha_2(n))(\lambda_1 - \lambda_2)}{\lambda_2(\lambda_1 + 2\alpha_1(n))(\lambda_2 + \alpha_2(n)) - \lambda_1(\lambda_1 + \alpha_1(n))(\lambda_2 + 2\alpha_2(n))}, \\ \tilde{q}_n &= \left(1 + \frac{(\lambda_1 + 2\alpha_1(n))(\lambda_2 + 2\alpha_2(n))(\lambda_1 - \lambda_2)}{\lambda_2(\lambda_1 + 2\alpha_1(n))(\lambda_2 + \alpha_2(n)) - \lambda_1(\lambda_1 + \alpha_1(n))(\lambda_2 + 2\alpha_2(n))}\right)\left(1 - \frac{\lambda_2 - \lambda_1}{\alpha_1(n) - \alpha_2(n)}\right). \end{aligned} \quad (40)$$

If we choose $\lambda_1 = 4.01$, $\lambda_2 = 4.05$, $\gamma_1 = 0.11$, $\gamma_2 = 0.1$, by use of Matlab the plots of $\tilde{p}_n$ (see Fig.1), $\tilde{q}_n$ (see Fig.2) are presented.

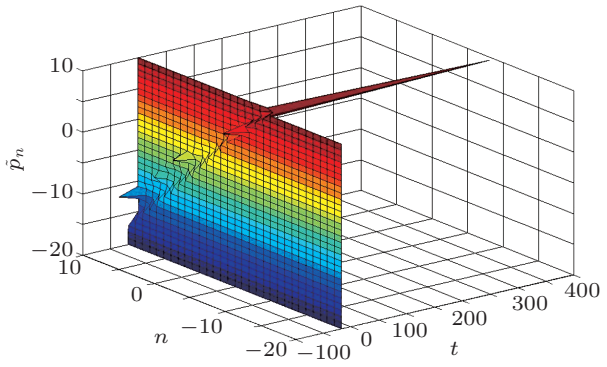


Fig. 1. (colour online) Plot of $\tilde{p}_n$.

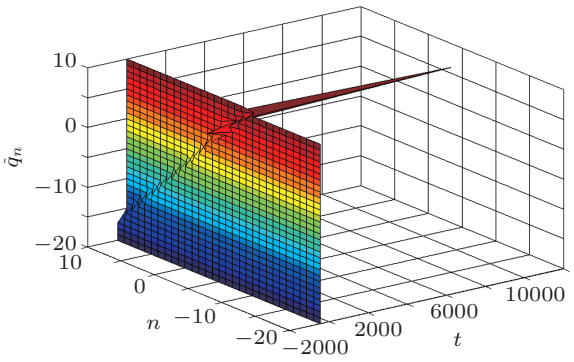


Fig. 2. (colour online) Plot of $\tilde{q}_n$.

In addition, if the resulting solutions (40) are taken as a new starting point, we can make the DT once again by Eq. (36) and engender another set of soliton solutions. This procedure can be done continuously and will usually yield a series of new solutions for Eq. (11).

4. Conservation laws

Consider the following Lax pair:

$$\boldsymbol{\varphi}_{n+1} = \begin{pmatrix} \lambda & \dfrac{q_n}{p_n} \\ \lambda p_n & 1+q_n \end{pmatrix} \boldsymbol{\varphi}_n,$$

$$\boldsymbol{\varphi}_{n_t} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \boldsymbol{\varphi}_n, \quad \boldsymbol{\varphi}_n = \begin{pmatrix} \varphi_n^1 \\ \varphi_n^2 \end{pmatrix}. \quad (41)$$

Direct calculation yields

$$\varphi_{n+1}^2 = \lambda p_n \frac{\varphi_n^2 - \varphi_{n-1}^2}{p_{n-1}} + (1+q_n)\varphi_n^2, \quad (42)$$

and

$$\varphi_{n_t}^2 = \left(\frac{C_n}{p_{n-1}} + D_n\right)\varphi_n^2 - \frac{C_n}{p_{n-1}}\varphi_{n-1}^2. \quad (43)$$

Let $\theta(n) = \theta_n = \varphi_n^2/\varphi_{n+1}^2$, then Eqs. (42) and (43) can be written as

$$1 = \frac{\lambda p_n}{p_{n-1}}(\theta_n - \theta_{n-1}\theta_n) + (1+q_n)\theta_n, \quad (44)$$

and

$$-(\ln\theta_n)_t = (E-1)\left(\frac{C_n}{p_{n-1}} + D_n - \frac{C_n}{p_{n-1}}\theta_{n-1}\right). \quad (45)$$

Then, expanding

$$\theta_n = \sum_{j=1}^{\infty} \frac{\theta_n^{(j)}}{\lambda^j}. \quad (46)$$

Substituting Eq.(46) into Eq.(44) leads to

$$1 = \frac{\lambda p_n}{p_{n-1}}\left[\sum_{j=1}^{\infty}\frac{\theta_n^{(j)}}{\lambda^j} - \left(\sum_{j=1}^{\infty}\frac{\theta_{n-1}^{(j)}}{\lambda^j}\right)\left(\sum_{j=1}^{\infty}\frac{\theta_n^{(j)}}{\lambda^j}\right)\right] + (1+q_n)\left(\sum_{j=1}^{\infty}\frac{\theta_n^{(j)}}{\lambda^j}\right). \quad (47)$$

From (47) we obtain the following recursion relation:

$$\theta_n^{(1)} = \frac{p_{n-1}}{p_n}, \theta_n^{(j+1)} = \sum_{i=1}^{j}\theta_{n-1}^{(i)}\theta_n^{(j-i+1)} - \frac{p_{n-1}(1+q_n)}{p_n}\theta_n^{(j)}. \quad (48)$$

At this moment, we have

$$\theta_n^{(1)} = \frac{p_{n-1}}{p_n}, \quad \theta_n^{(2)} = \frac{p_{n-2}}{p_n} - \frac{p_{n-1}^2(1+q_n)}{p_n^2},$$
$$\theta_n^{(3)} = \frac{p_{n-3}}{p_n} - \frac{2p_{n-1}p_{n-2}(1+q_n)}{p_n^2} - \frac{q_{n-1}p_{n-2}^2}{p_n p_{n-1}} + \frac{p_{n-1}^3(1+q_n)^2}{p_n^3}, \ldots \quad (49)$$

Substituting (46) into (45) and $\lambda_t = 0$, we obtain

$$\left(\sum_{i=1}^{\infty}(-1)^i\frac{1}{i}\left(\sum_{j=1}^{\infty}\frac{\theta_n^{(j+1)}}{\lambda^j}\right)^i\right)_t = (E-1)\left(\frac{C_n}{q_{n-1}} + D_n - \frac{C_n}{q_{n-1}}\left(\sum_{j=1}^{\infty}\frac{\theta_{n-1}^{(j)}}{\lambda^j}\right)\right). \quad (50)$$

From (13), we have $C_n = \lambda p_{n-1}$, $D_n = p_{n-1}q_n/p_n - \frac{1}{2}\lambda$. Comparing the coefficients of λ^{-j} in Eq. (50) we can obtain an infinite number conservation laws of Eq. (11). The first two are pointed out as

$$\left(\frac{q_{n-2}}{q_n} - \frac{q_{n-1}^2(1+p_n+p_nq_n)}{q_n^2}\right)_t$$

$$= (E-1)\left(\frac{q_{n-3}}{q_{n-1}} - \frac{q_{n-2}^2(1+p_{n-1}+p_{n-1}q_{n-1})}{q_{n-1}^2}\right), \quad (51)$$

$$\left(\frac{q_{n-3}}{q_n} - \frac{2q_{n-1}q_{n-2}(1+p_n+p_nq_n)}{q_n^2} - \frac{p_{n-1}q_{n-2}^2(1+q_{n-1})}{q_nq_{n-1}} + \frac{q_{n-1}^3(1+p_n+p_nq_n)^2}{q_n^3} - \frac{q_{n-2}^2}{2q_n^2} - \frac{q_{n-1}^4(1+p_n+p_nq_n)^2}{2q_n^4} + \frac{q_{n-2}q_{n-1}^2(1+p_n+p_nq_n)}{q_n^3}\right)_t$$

$$= (E-1)\left(\frac{q_{n-4}}{q_{n-1}} - \frac{2q_{n-2}q_{n-3}(1+p_{n-1}+p_{n-1}q_{n-1})}{q_{n-1}^2} - \frac{p_{n-2}q_{n-3}^2(1+q_{n-2})}{q_{n-1}q_{n-2}} + \frac{q_{n-2}^3(1+p_{n-1}+p_{n-1}q_{n-1})^2}{q_{n-1}^3}\right). \quad (52)$$

Similarly, we can obtain the conservation laws of the other lattice equations in the hierarchy (11).

5. Conclusions

In this paper, firstly a discrete spectral problem is constructed, then integrable hierarchies of lattice equations are obtained. Furthermore, we discussed the Darboux transformation, soliton solutions, and the conservation laws of the obtained hierarchy. However, how to work out the Hamiltonian structures and integrable couplings of the obtained system is still a question. We will discuss these problems in the future.

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