Perturbation method of studying the EI Niño oscillation with two parameters using delay sea–air oscillator model*

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The EI Niño-southern oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific ocean-atmosphere interactions. In this paper, we develop an asymptotic method of solving the nonlinear equation using the ENSO model. Based on a class of oscillator of the ENSO model, a approximate solution of the corresponding problem is studied employing the perturbation method.

Keywords: nonlinear, perturbation method, El Niño-southern oscillation model

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1. Introduction

The El Niño-southern oscillation (ENSO) is a very attractive object of study in the international academic circles. Many scholars have done a great deal of work.[1−9] The authors also have considered some problems of the atmospheric physics, the oceanic weather and so on.[10−18] The essence of the ENSO requires both positive and negative ocean-atmosphere feedbacks. An initial positive sea surface temperature (SST) anomaly in the equatorial eastern Pacific can reduce the zonal SST gradient and hence the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn bring on the ocean circulation changes, thereby reinforcing the SST anomaly. This positive ocean-atmosphere feedback or coupled ocean-atmosphere instability leads the equatorial Pacific to present a never-ending warm state.

In the conceptual oscillator model the variations of both the eastern and western Pacific anomaly patterns should be considered. As will be shown below, the oscillator model can be reduced to a simple oscillator model by further making simplification and assumptions.

Recently, many scholars have investigated the nonlinear problems Approximate methods have been developed, including the method of averaging, the boundary layer method, the matched method and the multiple scales method[19−22] In this paper, we consider a class of ENSO model by using a simple and special perturbation method[23]

2. ENSO model

The western Pacific oscillator emphasizes the role of the western Pacific anomaly patterns in the ENSO. In particular, off-equatorial SST variations initiate equatorial wind anomaly in the western Pacific. These winds anomaly force ocean to proceed eastward to affect anomalies in the equatorial eastern Pacific.

In the delayed oscillator model the effect of the western Pacific on ENSO is not taken into account. It is assumed that winds in the western Pacific do not affect the sea surface temperature anomalies in the eastern Pacific and the atmosphere is in a steady state or

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the Niño-4 (over 5°S–5°N, 160°E–150°W) zonal wind stress anomaly is linearly proportional to the Niño-3 (over 5°S–5°N, 150°E–90°W) SST anomaly. We can obtain the following nonlinear time delay model of equatorial Pacific:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{ad}{R}T - \frac{bd}{R}T(t-\eta) - cf(T), \tag{1}$$

$$T(t) = g, \quad -\eta \le t \le 0, \tag{2}$$

where T is the Niño-3 SST anomaly, η is the time delay for waves to travel to the western boundary and return to the eastern Pacific, a is the coefficient representing the positive feedback between T, b is the coefficient representing the negative feedback due to waves reflection at the western boundary, d is the positive coefficient that relates the Niño-3 SST anomalies to the Niño-4 zonal wind stress anomaly, R is the damping coefficient, g is the initial constant, and $c\,f(T)$ is the disturbed term. And we assume that f is an analytic function and there is a positive constant δ such that $f_T \ge \delta$, and η and c are two small positive parameters and $c/\eta \to 0$ as $\eta \to 0$.

3. Perturbation solution

Expanding $T(t-\eta)$ in η, we have

$$T(t-\eta) = T(t) - \frac{\mathrm{d}T(t)}{\mathrm{d}t}\eta + \frac{1}{2}\frac{\mathrm{d}^2T(t)}{\mathrm{d}t^2}\eta^2 + \cdots. \tag{3}$$

Let $\xi = c/\eta$, i.e., $c = \xi\,\eta$. From Eq. (1), we have

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{ad}{R}T - \frac{bd}{R}T(t-\eta) - \xi\eta\, f(T). \tag{4}$$

Let

$$T \equiv T(t,\xi,\eta) \sim \sum_{i,j=0}^{\infty} T_{ij}(t)\xi^i\eta^j. \tag{5}$$

Substituting Eqs. (3) and (5) into Eq. (4), expanding the nonlinear term in $\xi,\ \eta$, and equating the coefficients of the same powers in $\xi,\ \eta$, for the coefficients of $\xi^i\eta^0$ $(i = 0,\ 1)$, we obtain

$$\frac{\mathrm{d}T_{i0}}{\mathrm{d}t} = \frac{(a-b)d}{R}T_{i0}, \quad i = 0,\ 1, \ldots. \tag{6}$$

From Eq. (6), we obtain

$$T_{i0} = C_{i0}\exp\frac{(a-b)d}{R}t,\ i = 0,\ 1, \ldots, \tag{7}$$

where C_{i0} $(i = 0,\ 1, \cdots)$ are arbitrary constants.

Substituting Eqs. (3) and (5) into Eq. (4), for the coefficients of $\xi^i\eta^1$ $(i = 0,\ 1, \ldots)$, we have

$$\frac{\mathrm{d}T_{i1}}{\mathrm{d}t} = \frac{(a-b)d}{R}T_{i1} - \frac{\mathrm{d}T_{i0}}{\mathrm{d}t} - F_{(i-1)0}. \tag{8}$$

From Eq. (8), we have

$$T_{i1} = C_{i1}\exp\frac{(a-b)d}{R}t - \int_0^t \left[\frac{\mathrm{d}T_{i0}}{\mathrm{d}\tau} + F_{(i-1)0}\right] \times \exp\frac{(a-b)d}{R}(t-\tau)\,\mathrm{d}\tau,\ i = 0,\ 1, \cdots \tag{9}$$

Analogously, for the coefficients of $\xi^i\eta^j$ $(i = 0,\ 1, \cdots,\ j = 2, 3, \cdots)$ we have

$$\frac{\mathrm{d}T_{ij}}{\mathrm{d}t} = \frac{(a-b)d}{R}T_{ij} - G_{ij} - F_{(i-1)(j-1)}, \quad i = 0,\ 1, \cdots, j = 2,\ 3, \cdots, \tag{10}$$

where

$$G_{ij} = \sum_{k=1}^{j}\frac{1}{k\,!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}T_{i(j-k)}(t)$$

$$F_{ij} = \frac{1}{i}\frac{1}{j}\frac{\partial^{i+j}}{\partial\xi^i\partial\eta^j}\left[f\left(\sum_{i,j=0}^{\infty}T_{ij}(t)\xi^i\eta^j\right)\right]_{\xi=\eta=0}.$$

From Eq. (10), we have

$$T_{ij} = C_{ij}\exp\frac{(a-b)d}{R}t - \int_0^t [G_{ij} + F_{ij}] \times \exp\frac{(a-b)d}{R}(t-\tau)\,\mathrm{d}\tau. \tag{11}$$

Substituting Eqs. (7), (9), and (11) into Eq. (5), and observing Eq. (2), we obtain the formal solution of Eq. (4) as

$$\begin{aligned} T(t,\xi,\eta) \sim\ & g\ \exp\frac{(a-b)d}{R}t \\ & - \left[\sum_{i=1}^{\infty}\int_0^t\left[\frac{\mathrm{d}T_{i0}}{\mathrm{d}\tau} + F_{(i-1)0}\right] \times \exp\frac{(a-b)d}{R}(t-\tau)\,\mathrm{d}\tau\right]\xi^i\eta \\ & - \left[\sum_{i=0,j=1}^{\infty}\int_0^t [G_{ij} + F_{(i-1)(j-1)}] \times \exp\frac{(a-b)d}{R}(t-\tau)\,\mathrm{d}\tau\right]\xi^i\eta^j, \end{aligned} \tag{12}$$

where the terms of negative subscript are zero and g is the initial condition of the SST anomalies $T(t)$.

From Eq. (12), we know that: 1) Because coefficient a representing positive feedback between T is less than coefficient b representing negative feedback due to waves reflection at the western boundary, SST

anomaly T is an exponential decay function. Thus, SST anomaly T is stable. 2) Because coefficient a representing positive feedback between T is greater than coefficient b representing the negative feedback due to waves reflection at the western boundary, SST anomaly T is an exponential increase function. Thus, SST anomaly T is unstable.

4. Uniform validity and accuracy of solution

Now we consider the uniformly valid expansion of Eq. (12) for small parameters ξ and η.

We construct the auxiliary functions α and β as follows:

$$\alpha = Z_m - r\zeta,\ \beta = Z_m + r\zeta,\ 0 \le t \le T_0, \quad (13)$$

where r is a positive constant, which is chosen to be large enough below, T_0 is a sufficiently large positive constant, $\zeta = \max(\xi^{M+1}\eta^m, \xi^m\eta^{m+1})$, and

$$\begin{aligned} Z_m =\ & g\ \exp\frac{(a-b)d}{R}t - \Big[\sum_{i=1}^{m}\int_0^t\Big[\frac{\mathrm{d}T_{i0}}{\mathrm{d}\tau} + F_{(i-1)0}\Big] \\ & \times\ \exp\frac{(a-b)d}{R}(t-\tau)\,\mathrm{d}\tau\Big]\xi^i\eta \\ & - \Big[\sum_{i=0,j=1}^{m}\int_0^t[G_{ij} + F_{(i-1)(j-1)}] \\ & \times\ \exp\frac{(a-b)d}{R}(t-\tau)\,\mathrm{d}\tau\Big]\xi^i\eta^j. \end{aligned}$$

Obviously, we have

$$\alpha \le \beta,\ 0 \le t \le T_0, \quad (14)$$

and

$$\alpha\,|_{t=-0} \le g \le \beta\,|_{t=0}\,. \quad (15)$$

Now we prove that

$$\begin{aligned} & \frac{\mathrm{d}\alpha}{\mathrm{d}t} - \frac{ad}{R}\alpha + \frac{bd}{R}\alpha(t-\eta) + \xi\eta\ f(\alpha) \le 0, \\ & \quad 0 \le t \le T_0, \end{aligned} \quad (16)$$

$$\begin{aligned} & \frac{\mathrm{d}\beta}{\mathrm{d}t} - \frac{ad}{R}\beta + \frac{bd}{R}\beta(t-\eta) + \xi\eta\ f(\beta) \ge 0, \\ & \quad 0 \le t \le T_0. \end{aligned} \quad (17)$$

From the hypothesis it follows that for ξ, η small enough there exists a positive constant M such that

$$\begin{aligned} & \frac{\mathrm{d}\alpha}{\mathrm{d}t} - \frac{ad}{R}\alpha + \frac{bd}{R}\alpha(t-\eta) + \xi\eta\ f(\alpha) \\ =\ & \frac{\mathrm{d}Z_m}{\mathrm{d}t} - \frac{ad}{R}Z_m + \frac{bd}{R}Z_m(t-\eta) + \xi\eta\ f(Z_m) \\ & - 2r\zeta - \xi\eta\{f(Z_m) - f(Z_m - r\zeta)] \\ \le\ & \sum_{i=0}^{m}\left[\frac{\mathrm{d}T_{i0}}{\mathrm{d}t} - \frac{(a-b)d}{R}T_{i0}\right]\xi^i \\ & + \sum_{i=0}^{m}\left[\frac{\mathrm{d}T_{i1}}{\mathrm{d}t} - \frac{(a-b)d}{R}T_{i1} + \frac{\mathrm{d}T_{i0}}{\mathrm{d}t} + F_{(i-1)0}\right]\xi^i\eta \\ & + \sum_{i,=0j=2}^{m}[\frac{\mathrm{d}T_{ij}}{\mathrm{d}t} - \frac{(a-b)d}{R}T_{ij} + G_{ij} \\ & + F_{(i-1)(j-1)}]\xi^i\eta^j + M\tau - r\delta\xi\eta\zeta - 2r\zeta \\ =\ & (M - (r\delta\xi\eta + 2r)\zeta. \end{aligned}$$

Selecting $r > M/2$, we prove Eq. (16). Analogously, we can prove Eq. (17). From Eqs. (14)–(17), α and β are upper and lower solutions of problems (1) and (2), respectively. Thus, there is a solution $T(t,\xi,\eta)$ of the problems (1) and (2), and $\alpha(t,\xi,\eta) \le y(t,\xi,\eta) \le \beta(t,\xi,\mu)$ holds. And from Eq. (13), we have the relation

$$\begin{aligned} T(t,\xi,\eta) =\ & g\ \exp\frac{(a-b)d}{R}t \\ & - \Big[\sum_{i=1}^{m}\int_0^t\Big[\frac{\mathrm{d}T_{i0}}{\mathrm{d}\tau} + F_{(i-1)0}\Big] \\ & \times\ \exp\frac{(a-b)d}{R}(t-\tau)\,\mathrm{d}\tau\Big]\xi^i\eta \\ & - \Big[\sum_{i=0,j=1}^{m}\int_0^t[G_{ij} + F_{(i-1)(j-1)}] \\ & \times\ \exp\frac{(a-b)d}{R}(t-\tau)\,\mathrm{d}\tau\Big]\xi^i\eta^j + O(\zeta), \\ & 0 \le t \le T_0,\ \ 0 < \xi = c/\eta,\ \eta \ll 1, \end{aligned} \quad (18)$$

where $\zeta = \max(\xi^{m+1}\eta^m,\ \xi^m\eta^{m+1}) = \max(c^{m+1}/\eta,\ c^m\eta)$. Therefore, Eq. (12) is a uniformly valid asymptotic expansion.

Then we know that the obtained expansions of sea surface temperature anomaly T for the time delay sea–air oscillator model (1) and (2) of the EI Niño Oscillation of equatorial Pacific using the perturbation method possess better accuracy

5. Examples

Example 1 For the nonlinear time delay model (1) and (2), we set the non-dimensional parameters to be $a = 1$, $b = 2$, $c = 0.5$, $d = 2$, $R = 1$, $g = 2$, $\eta = 0.2$, and the perturbation term is $0.5\sin T$. Then from Eq. (18), we can obtain the approximate solution

$T_{\rm a}(t)$ of time delay model (1), (2) as

$$T_{\rm a}(t) = 2\ \exp(-2t) - 0.5\int_0^t \sin\ 2\exp(-2\tau) \times\ \exp(-2(t-\tau))\,{\rm d}\tau - 0.4\,t\exp(-2\,t). \quad (19)$$

From Eq. (19) and model (1), (2), we can obtain the curves and data of approximate solution $T_{\rm a}$ and simulate solution $T_{\rm n}$ as shown in Fig. 1 and Table 1 respectively.

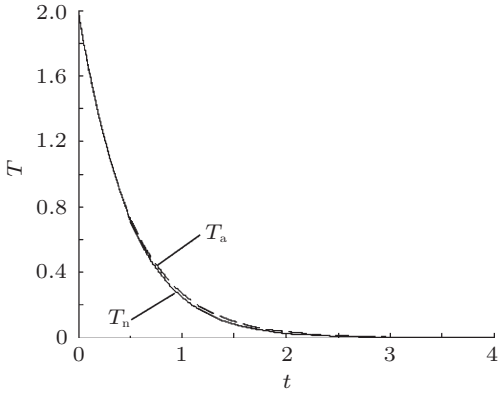


Fig. 1. Comparison between curves of approximate solution $T_{\rm a}$ and simulate solution $T_{\rm n}$ with $a = 1$, $b = 2$, $c = 0.5$, $d = 2$, $R = 1$, $g = 2$, and $\eta = 0.2$.

Table 1. Comparison between the data of approximate solution $T_{\rm a}$ and simulate solution $T_{\rm n}$ with $a = 1$, $b = 2$, $c = 0.5$, $d = 2$, $R = 1$, $g = 2$, and $\eta = 0.2$.

t	$T_{\rm n}$	$T_{\rm a}$	t	$T_{\rm n}$	$T_{\rm a}$
0	2.000	2.000	2.5	0.011	0.015
0.5	1.333	1.324	3.0	0.003	0.005
1.0	0.375	0.399	3.5	0.001	0.002
1.5	0.111	0.129	4.0	0.000	0.000
2.0	0.034	0.043			

From Fig. 1, Table 1 and $b > a$, we can see that the solution SST of the nonlinear time delay model (1), (2) is stable, then the ENSO oscillation well vanish generally.

Example 2 For the nonlinear time delay model (1), (2), we set the non-dimensional parameters to be $a = 2$, $b = 1$, $c = 0.02$, $d = 2$, $g = 2$, and $\eta = 0.05$, and the disturbed term to be $0.02\sin T$, then from Eq. (18), we can obtain the approximate solution $T_{\rm a}(t)$ of time delay model (1), (2) as follows:

$$T_{\rm a}(t) = 2\exp(2t) - 0.02\int_0^t \sin(\exp(2\exp(2\tau))) \times\ \exp(2(t-\tau))\,{\rm d}\tau - 0.1\,t\exp(2t). \quad (20)$$

From Eq. (20) and model (1), (2), we can obtain the curves and data of approximate solution $T_{\rm a}$ and simulate solution $T_{\rm n}$ as shown in Fig. 2 and Table 2 respectively.

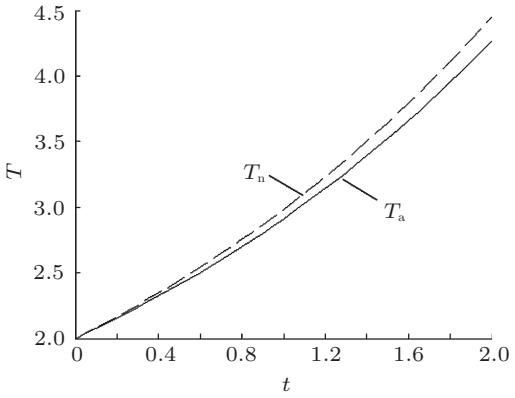


Fig. 2. Comparison between the curves of approximate solution $T_{\rm a}$ and simulate solution $T_{\rm n}$ with $a = 2$, $b = 1$, $c = 0.02$, $d = 2$, $g = 2$, and $\eta = 0.05$.

Table 2. Comparison between the data of approximate solution $T_{\rm a}$ and simulate solution $T_{\rm n}$ with $a = 2$, $b = 1$, $c = 0.02$, $d = 2$, $g = 2$, and $\eta = 0.05$.

t	$T_{\rm n}$	$T_{\rm a}$	t	$T_{\rm n}$	$T_{\rm a}$
0	2.000	2.000	1.2	3.141	2.984
0.2	2.155	2.167	1.4	3.391	3.501
0.4	2.322	2.347	1.6	3.661	3.793
0.6	2.503	2.543	1.8	3.953	4.109
0.8	2.699	2.754	2.0	4.269	4.451
1.0	2.911	2.984			

From Fig. 2, Table 2 and $b < a$, we can see that the solution SST of the nonlinear time delay model (1), (2) is unstable, then the ENSO oscillation well increases generally.

From Figs. 1 and 2 and data 1 and 2 of examples, we can see that the obtained approximate solutions using approximate method possess better degree of accuracy.

6. Conclusions

1) The ENSO is a complicated natural phenomenon. Hence, we need to reduce basic models for the sea–air oscillator and solve them using the approximate method. The perturbation method is a simple and valid method.

2) The perturbation method is an asymptotic analytic method, which differs from the general numerical method. The expansions of solution obtained

through the perturbation method can be used analytically. Thus, from Eq. (6), we can further study the fixed qualitative and quantitative behaviours of the temperature anomaly in the equatorial Pacific.

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