Fractional differential equations of motion in terms of combined Riemann–Liouville derivatives*

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In this paper, we focus on studying the fractional variational principle and the differential equations of motion for a fractional mechanical system. A combined Riemann–Liouville fractional derivative operator is defined, and a fractional Hamilton principle under this definition is established. The fractional Lagrange equations and the fractional Hamilton canonical equations are derived from the fractional Hamilton principle. A number of special cases are given, showing the universality of our conclusions. At the end of the paper, an example is given to illustrate the application of the results.

Keywords: fractional Hamilton principle, fractional Lagrange equation, fractional Hamilton canonical equation, combined Riemann–Liouville fractional derivative

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1. Introduction

Fractional derivatives, or more precisely, derivatives of arbitrary orders, are a generalization of classical calculus and have been used successfully in various fields of science and engineering.[1−5] Nowadays physicists have used this powerful tool to deal with some problems, which are not solvable in the classical sense. Therefore, the fractional calculus becomes one of the most powerful and widely useful tools in describing and explaining some physical complex systems. As pointed out in Ref. [2], there has been hardly a field or science or engineering that has remained untouched by this field. For example, almost all systems contain internal damping, yet the traditional energy based approach cannot be used to obtain equations describing the behaviour of a non-conservative system. Riewe[6−7] used the fractional calculus to develop a new approach which can be used for both conservative and non-conservative systems. His approach allows fractional derivatives to appear in the Lagrangian and the Hamiltonian, whereas traditional Lagrangian mechanics often deals with first order derivatives. In a sequel to Riewe's work, Agrawal[8−10] further presented the fractional variational problems, and the resulting equations are found to be similar to those for variation problems containing integral order derivatives. Baleanu and Agrawal[11] gave the fractional Hamilton formalism within Caputo's derivative. Rabei *et al.*[12] obtained the Hamilton equations of motion in the same manner as those obtained by using the formulation of Euler–Lagrange equations from variational problems within the Riemann–Liouville fractional derivatives. In recent years, the study of fractional Lagrangian mechanics and Hamiltonian mechanics is a subject of current strong research of mathematics, physics, and mechanics, and has achieved a series of important results.[13−21]

In the present work, the fractional Hamilton principle, the fractional Lagrange equations, and the fractional Hamilton canonical equations are derived in terms of combined Riemann–Liouville derivatives.

2. Definitions and properties of fractional derivative

In this section we briefly review some basic definitions and properties of fractional derivatives used in the following sections. Detailed discussion and proof can be found in Refs. [1]–[5].

The left Riemann–Liouville fractional derivative is defined as[2]

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \times \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\mathrm{d}\tau. \quad (1) $$

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The right Riemann–Liouville fractional derivative is defined as

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \times \int_t^b (\tau - t)^{n-\alpha-1} f(\tau)\mathrm{d}\tau, \quad (2) $$

where $\Gamma(*)$ denotes the Gamma function, and α is the order of the derivative such that $n-1 \le \alpha < n$. If α is an integer, these derivatives are defined in the usual sense, i.e.

$$ {}_aD_t^\alpha f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t), $$
$$ {}_tD_b^\alpha f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t). \quad (3) $$

The formula of fractional integration by part is as follows:[6]

$$ \int_a^b g(t)\,{}_aD_t^\alpha f(t)\mathrm{d}t = \int_a^b f(t)\,{}_tD_b^\alpha g(t)\mathrm{d}t, \quad (4) $$

where we assume that $\mathrm{d}^k f/\mathrm{d}t^k = 0$ or $\mathrm{d}^k g/\mathrm{d}t^k = 0$ $(k = 0, 1, \ldots, n-1)$ at $t = a$ and $t = b$.

We define a combined Riemann–Liouville fractional derivative operator by

$$ D_\gamma^{\alpha,\beta} f(t) = \gamma\, {}_aD_t^\alpha f(t) + (1-\gamma)(-1)^n\, {}_tD_b^\beta f(t), \quad (5) $$

where $\gamma \in [0, 1]$. From Eq. (5), we have

$$ \gamma = 1 : D_1^{\alpha,\beta} f(t) = {}_aD_t^\alpha f(t), \quad (6) $$
$$ \gamma = 0 : D_0^{\alpha,\beta} f(t) = (-1)^n\, {}_tD_b^\beta f(t), \quad (7) $$
$$ \gamma = \frac{1}{2} : D_{1/2}^{\alpha,\alpha} f(t) = \frac{1}{2}\left({}_aD_t^\alpha f(t) + (-1)^n\, {}_tD_b^\alpha f(t)\right) = {}_a^RD_b^\alpha f(t), \quad (8) $$
$$ \gamma = \frac{1}{2}(1 + \mathrm{i}\mu), \quad \mathrm{i} = \sqrt{-1}, \mu \in C : $$
$$ D_{(1+\mathrm{i}\mu)/2}^{\alpha,\beta} f(t) = \frac{1}{2}\left({}_aD_t^\alpha f(t) + (-1)^n\, {}_tD_b^\beta f(t)\right) + \frac{1}{2}\mathrm{i}\mu\left({}_aD_t^\alpha f(t) - (-1)^n\, {}_tD_b^\beta f(t)\right) = \boldsymbol{D}_\mu^{\alpha,\beta}, \quad (9) $$

where ${}_a^RD_b^\alpha f(t)$ denotes the Riesz fractional derivative,[10] and $\boldsymbol{D}_\mu^{\alpha,\beta}$ denotes the fractional derivative given by Cresson.[16]

By utilizing Eqs. (4) and (5), the formula of fractional integration by part for the combined Riemann–Liouville fractional derivatives can be written as

$$ \int_a^b g(t)\, D_\gamma^{\alpha,\beta} f(t)\,\mathrm{d}t = (-1)^n \int_a^b f(t)\, D_{1-\gamma}^{\beta,\alpha} g(t)\,\mathrm{d}t, \quad (10) $$

where the function f or g and its derivatives up to order $n-1$ with respect to t are all zero at the ends $t = a$ and $t = b$. Especially, for $0 < \alpha, \beta < 1$, formulae (5) and (10) give

$$ D_\gamma^{\alpha,\beta} f(t) = \gamma\, {}_aD_t^\alpha f(t) - (1-\gamma)\, {}_tD_b^\beta f(t), \quad (11) $$
$$ \int_a^b g(t)\, D_\gamma^{\alpha,\beta} f(t)\,\mathrm{d}t = -\int_a^b f(t)\, D_{1-\gamma}^{\beta,\alpha} g(t)\,\mathrm{d}t. \quad (12) $$

3. Fractional Hamilton principle

Now we consider a mechanical system whose configuration is determined by N generalized coordinates q^k $(k = 1, 2, \ldots, N)$. The Lagrangian of the system is $L\left(t, q^k, D_\gamma^{\alpha,\beta} q^k\right)$, which is a function with continuous first and second (partial) derivatives with respect to all its arguments. A fractional Hamilton action is

$$ I = \int_a^b L\left(t, q^k, D_\gamma^{\alpha,\beta} q^k\right)\mathrm{d}t. \quad (13) $$

Under the combined Riemann–Liouville fractional derivatives, a fractional Hamilton principle can be expressed as follows:

$$ \delta I = \delta \int_a^b L\left(t, q^k, D_\gamma^{\alpha,\beta} q^k\right)\mathrm{d}t = 0 \quad (14) $$

with the commutation relation

$$ \delta D_\gamma^{\alpha,\beta} q^k = D_\gamma^{\alpha,\beta} \delta q^k \ (k = 1, 2, \ldots, N) \quad (15) $$

as well as the terminal conditions

$$ \delta q^k\,|_{t=a} = \delta q^k\,|_{t=b} = 0\,(k = 1, 2, \ldots, N). \quad (16) $$

The Hamilton principle shows that under the same interval, the same starting and ending positions, as well as under the same constraints, in all possible motions of a fractional mechanical system, the actual motion makes the value of the fractional Hamilton action stationary.

4. Fractional Lagrange equations

The fractional Lagrange equations will be derived from the fractional Hamilton principle as follows.

From formula (14), we obtain

$$ \delta I = \delta \int_a^b L\mathrm{d}t = \int_a^b \delta L\mathrm{d}t = 0. \quad (17) $$

Calculating δL, we obtain

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial q^k}\delta q^k + \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q^k}\delta D_\gamma^{\alpha,\beta} q^k \\ &= \frac{\partial L}{\partial q^k}\delta q^k + \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q^k} D_\gamma^{\alpha,\beta}\delta q^k. \end{aligned} \tag{18}$$

Using formula (12) of integration by part, and taking notice of condition (16), we have

$$\begin{aligned}&\int_a^b \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q^k} D_\gamma^{\alpha,\beta}\delta q^k \,\mathrm{d}t \\ &= -\int_a^b \delta q^k D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q^k}\,\mathrm{d}t. \end{aligned} \tag{19}$$

Substituting Eq. (18) into Eq. (17), and using relation (19), we have

$$\int_a^b \left\{ \left(\frac{\partial L}{\partial q^k} - D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q^k} \right) \delta q^k \right\} \mathrm{d}t = 0. \tag{20}$$

This identity holds for arbitrary interval $[a, b]$, and therefore the integrand should be zero, i.e.

$$\left(\frac{\partial L}{\partial q^k} - D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q^k} \right) \delta q^k = 0. \tag{21}$$

Formula (21) may be called a fractional d'Alembert–Lagrange principle. Furthermore, by the independence of δq^k, we obtain

$$\frac{\partial L}{\partial q^k} - D_{1-\gamma}^{\beta,\alpha} \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q^k} = 0 \ \ (k = 1, 2, \ldots N). \tag{22}$$

Equation (22) may be called the fractional Lagrange equations by means of the combined Riemann–Liouville fractional derivatives.

Now we come to study some important cases of Eq. (22).

Case 1 If we choose $\gamma = 1$, and the fractional Lagrangian is $L\left(t, q^k, {}_a D_t^\alpha q^k\right)$, then Eq. (22) gives

$$\frac{\partial L}{\partial q^k} + {}_t D_b^\alpha \frac{\partial L}{\partial {}_a D_t^\alpha q^k} = 0, \ \ (k = 1, 2, \ldots N). \tag{23}$$

This is a result given by Agrawal.[8]

Case 2 If we choose $\gamma = 0$, and the fractional Lagrangian is $L(t, q^k, {}_t D_b^\beta q^k)$, then Eq. (22) gives

$$\frac{\partial L}{\partial q^k} + {}_a D_t^\beta \frac{\partial L}{\partial {}_t D_b^\beta q^k} = 0, \ \ (k = 1, 2, \ldots N). \tag{24}$$

Equation (24) is also in accordance with Agrawal's result.[8]

Case 3 If we choose $\gamma = 1/2$, $\beta = \alpha$, and the fractional Lagrangian is $L\left(t, q^k, {}_a^R D_b^\alpha q^k\right)$, then Eq. (22) gives

$$\frac{\partial L}{\partial q^k} - {}_a^R D_b^\alpha \frac{\partial L}{\partial {}_a^R D_b^\alpha q^k} = 0 \, (k = 1, 2, \ldots, N). \tag{25}$$

Equation (25) is the Lagrange equations by means of Riesz fractional derivatives.

Case 4 If we choose $\gamma = (1 + \mathrm{i}\mu)/2$, $\mathrm{i} = \sqrt{-1}, \mu \in C$, and the fractional Lagrangian is $L\left(t, q^k, \boldsymbol{D}_\mu^{\alpha,\beta} q^k\right)$, then Eq. (22) gives

$$\frac{\partial L}{\partial q^k} - \boldsymbol{D}_{-\mu}^{\beta,\alpha} \frac{\partial L}{\partial \boldsymbol{D}_\mu^{\alpha,\beta} q^k} = 0, \ \ (k = 1, 2, \ldots, N). \tag{26}$$

Equation (26) is in accordance with Cresson's result.[16]

Case 5 In these cases mentioned above, if we take $\alpha, \beta \to 1$, then Eq. (22) becomes

$$\frac{\partial L}{\partial q^k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^k} = 0. \tag{27}$$

These are the classical Lagrange equations.

5. Fractional Hamilton canonical equations

The fractional Hamilton canonical equations will be derived from the fractional Hamilton principle as follows.

Under the combined Riemann–Liouville fractional derivatives, the generalized momentum can be defined as

$$\begin{aligned}&p_{\gamma,k}^{\alpha,\beta} = \frac{\partial L}{\partial D_\gamma^{\alpha,\beta} q^k} \\ &(k = 1, 2, \ldots, N; \ 0 < \alpha, \beta < 1; \ 0 \le \gamma \le 1). \end{aligned} \tag{28}$$

Respectively, a fractional Hamiltonian $H(t, q^k, p_{\gamma,k}^{\alpha,\beta})$ is

$$H = p_{\gamma,k}^{\alpha,\beta} D_\gamma^{\alpha,\beta} q^k - L. \tag{29}$$

Therefore the fractional Lagrangian is

$$L = p_{\gamma,k}^{\alpha,\beta} D_\gamma^{\alpha,\beta} q^k - H. \tag{30}$$

Substituting expression (30) into Eq. (14), we have

$$\delta I = \delta \int_a^b \left(p_{\gamma,k}^{\alpha,\beta} D_\gamma^{\alpha,\beta} q^k - H \right) \mathrm{d}t = 0. \tag{31}$$

After a variational calculation, we have

$$\begin{aligned}&\int_a^b \left(D_\gamma^{\alpha,\beta} q^k \delta p_{\gamma,k}^{\alpha,\beta} + p_{\gamma,k}^{\alpha,\beta} \delta D_\gamma^{\alpha,\beta} q^k - \frac{\partial H}{\partial q^k}\delta q^k \right. \\ &\left. - \frac{\partial H}{\partial p_{\gamma,k}^{\alpha,\beta}} \delta p_{\gamma,k}^{\alpha,\beta} \right) \mathrm{d}t = 0. \end{aligned} \tag{32}$$

Using property (12) and condition (16), the second term in the bracket of Eq. (32) can be written as

$$\begin{aligned}&\int_a^b p_{\gamma,k}^{\alpha,\beta}\delta D_\gamma^{\alpha,\beta}q^k\,\mathrm{d}t\\&=\int_a^b p_{\gamma,k}^{\alpha,\beta}D_\gamma^{\alpha,\beta}\delta q^k\,\mathrm{d}t\\&=-\int_a^b \delta q^k D_{1-\gamma}^{\beta,\alpha}p_{\gamma,k}^{\alpha,\beta}\,\mathrm{d}t.\end{aligned}\tag{33}$$

Substituting Eq. (33) into Eq. (32), we have

$$\begin{aligned}&\int_a^b\left\{\left(D_\gamma^{\alpha,\beta}q^k-\frac{\partial H}{\partial p_{\gamma,k}^{\alpha,\beta}}\right)\delta p_{\gamma,k}^{\alpha,\beta}\right.\\&\left.+\left(-D_{1-\gamma}^{\beta,\alpha}p_{\gamma,k}^{\alpha,\beta}-\frac{\partial H}{\partial q^k}\right)\delta q^k\right\}\mathrm{d}t=0.\end{aligned}\tag{34}$$

Taking partial derivation of Eq. (29) with respect to $p_{\gamma,k}^{\alpha,\beta}$, we obtain

$$\frac{\partial H}{\partial p_{\gamma,k}^{\alpha,\beta}}=D_\gamma^{\alpha,\beta}q^k,\ (k=1,2,\ldots,N).\tag{35}$$

Taking partial derivation of Eq. (29) with respect to t, we obtain

$$\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}.\tag{36}$$

Substituting Eq. (35) into Eq. (34), and considering the arbitrariness of the integral interval and the independence of δq^k, we obtain

$$\frac{\partial H}{\partial q^k}=-D_{1-\gamma}^{\beta,\alpha}p_{\gamma,k}^{\alpha,\beta},\ (k=1,2,\ldots,N).\tag{37}$$

Combining Eq. (37) with Eqs. (35) and (36), we obtain the fractional Hamilton canonical equations within the combined Riemann–Liouville derivatives for the fractional mechanical system under consideration.

The following are some important cases of the fractional Hamilton canonical equations (35)–(37).

Case 1 If we choose $\gamma=1$, then the fractional Hamilton canonical equations (35)–(37) give

$$\begin{aligned}&\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t},\ \frac{\partial H}{\partial p_k^\alpha}={}_aD_t^\alpha q^k,\\&\frac{\partial H}{\partial q^k}={}_tD_b^\alpha p_k^\alpha,\end{aligned}\tag{38}$$

where $p_k^\alpha=\partial L/\partial_a D_t^\alpha q^k$. Equation (38) is in accordance with the results obtained by Rabei *et al.*[12]

Case 2 If we choose $\gamma=0$, then the fractional Hamilton canonical equations (35)–(37) give

$$\begin{aligned}&\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t},\ \frac{\partial H}{\partial p_k^\beta}={}_tD_b^\beta q^k,\\&\frac{\partial H}{\partial q^k}={}_aD_t^\beta p_k^\beta,\end{aligned}\tag{39}$$

where $p_k^\beta=\partial L/\partial_t D_b^\beta q^k$. Equation (39) is also in accordance with the results given by Rabei.[12]

Case 3 If we choose $\gamma=1/2$, $\beta=\alpha$, then the fractional Hamilton canonical equations (35)–(37) give

$$\begin{aligned}&\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t},\ \frac{\partial H}{\partial p_k^\alpha}={}_a^RD_b^\alpha q^k,\\&\frac{\partial H}{\partial q^k}=-{}_a^RD_b^\alpha p_k^\alpha,\end{aligned}\tag{40}$$

where $p_k^\alpha=\partial L/\partial_a^R D_b^\alpha q^k$. Equation (40) is the fractional Hamilton equations under Riesz fractional derivatives.

Case 4 If we choose $\gamma=(1+\mathrm{i}\mu)/2$, $\mathrm{i}=\sqrt{-1}$, and $\mu\in C$, then the fractional Hamilton canonical equations (35)–(37) give

$$\begin{aligned}&\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t},\ \frac{\partial H}{\partial p_k}=\boldsymbol{D}_\mu^{\alpha,\beta}q^k,\\&\frac{\partial H}{\partial q^k}=-\boldsymbol{D}_{-\mu}^{\beta,\alpha}p_k,\end{aligned}\tag{41}$$

where $p_k=\partial L/\partial\boldsymbol{D}_\mu^{\alpha,\beta}q^k$. Equation (41) is in accordance with the results obtained by Cresson.[16]

Case 5 In these cases listed above, if we take $\alpha,\beta\to1$, then the equations give

$$\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t},\ \frac{\partial H}{\partial p_k}=\dot{q}^k,\ \frac{\partial H}{\partial q^k}=-\dot{p}_k.\tag{42}$$

This is the classical Hamilton canonical equations.

6. An illustrative example

The Lagrangian of a fractional mechanical system is[17]

$$L=\frac{1}{2}\left(D_\gamma^{\alpha,\beta}q\right)^2+cD_\gamma^{\alpha,\beta}q,\tag{43}$$

where c is a real constant of nonzero.

From Eq. (43), fractional Lagrange equation (22) gives

$$-D_{1-\gamma}^{\beta,\alpha}\left(D_\gamma^{\alpha,\beta}q+c\right)=0.\tag{44}$$

Generalized momentum (28) gives

$$p_\gamma^{\alpha,\beta}=\frac{\partial L}{\partial D_\mu^{\alpha,\beta}q}=D_\gamma^{\alpha,\beta}q+c.\tag{45}$$

Fractional Hamiltonian (29) gives

$$H=p_\gamma^{\alpha,\beta}D_\gamma^{\alpha,\beta}q-L=\frac{1}{2}\left(p_\gamma^{\alpha,\beta}-c\right)^2.\tag{46}$$

Fractional Hamilton canonical equations (35)–(37) give

$$\frac{\partial H}{\partial p_\gamma^{\alpha,\beta}} = D_\gamma^{\alpha,\beta} q, \quad \frac{\partial H}{\partial q} = -D_{1-\gamma}^{\beta,\alpha} p_\gamma^{\alpha,\beta},$$

that is,

$$p_\gamma^{\alpha,\beta} - c = D_\gamma^{\alpha,\beta} q, -D_{1-\gamma}^{\beta,\alpha} p_\gamma^{\alpha,\beta} = 0. \tag{47}$$

Comparing Eq. (44) with Eq. (47), and taking Eq. (45) into account, we come to the conclusion that the fractional Lagrange equations and the fractional Hamilton canonical equations give the same differential equations of motion.

Here, if we choose $\gamma = 1$, then the above conclusions are in accordance with the results obtained by Baleanu *et al.*[17]

If take $\alpha,\ \beta \to 1$, then we obtain the corresponding equations of classical case of integer order.

7. Conclusion

Fractional calculus is now a powerful tool widely used in various fields of science and engineering. Fractional calculus is also used to establish dynamical models in non-conservative or dissipative systems, and enables us to solve some problems which are too difficult to solve by using classical methods under integer order derivatives. In this paper, we defined a combined Riemann–Liouville fractional derivative operator, and within this definition, we established a fractional Hamilton principle (14)–(16) of a fractional mechanical system, and further derived the fractional Lagrange equation (22) and the fractional Hamilton canonical equations (35)–(37). The method and results in this paper are of universal significance, and several fractional Euler–Lagrange equations and fractional Hamilton canonical equations given before are special cases of this paper. The classical Lagrange equations and Hamilton canonical equations are naturally a special case of our results.

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