Wigner function and the entanglement of a quantized Bessel–Gaussian vortex state of a quantized radiation field

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A new kind of quantum non-Gaussian state with a vortex structure, termed a Bessel–Gaussian vortex state, is constructed, which is an eigenstate of the sum of squared annihilation operators $a^2 + b^2$. The Wigner function of the quantum vortex state is derived and exhibits negativity which is an indication of nonclassicality. It is also found that a quantized vortex state is always in entanglement. And a scheme for generating such quantized vortex states is proposed.

Keywords: Wigner distribution function, quantized vortex state, quantum non-Gaussian state

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1. Introduction

In recent years there has been the considerable interest in constructing some new multimode quantum states of radiation fields because it has been widely recognized that quantum state entanglement represents a valuable resource for achieving efficient quantum communication protocols.[1,2] In the context of quantum information and communication, non-Gaussian states are equally as important as Gaussian states because theoretical investigations have shown that some quantum-information tasks are unlikely to be implemented only by using Gaussian states or Gaussian local operations.[3−5] Therefore, preparing non-Gaussian states has attracted special attention due to their remarkable usefulness. In fact, pair-coherent states[6−16] and quantized vortex states[17−22] are seen as important two-mode non-Gaussian states. A large number of studies demonstrate that pair-coherent states show remarkable nonclassical properties and may be realized with currently available technology.[17,22] However, there are only few papers treating the quantized vortex states. Since Agarwal *et al.*[17] first used a two-mode squeezed vacuum to generate a circularly symmetric quantum vortex, Agarwal and Biswas[21] found that such vortex states bear stronger nonclassical properties and are always in entanglement. Bandyopadhyay and Singh,[19] and Bandyopadhyay *et al.*[20] proposed that a quantum optical elliptic vortex can also be generated by coupling squeezed coherent states of two modes with a beam splitter or a dual channel directional coupler. Very recently, Agarwal[22] also pointed out that a two-mode quantized vortex state may be prepared through a squeezed vacuum by subtracting a photon. Indeed, optical vortex fields with a helical wave front can be produced in a controlled manner. Due to their specific spatial structure and associated orbital angular momentum, they find profound applications in the fields of quantum information and computation because they can be used to create higher-dimensional entangled states and have been demonstrated to be good carriers of quantum information.[23,24] Optical vortices have received a great deal of attention in the last few decades, and thereby prompted the start of a new branch in physical optics, known as singular optics.[25] In the present paper, a non-Gaussian state with a vortex structure, called a Bessel–Gaussian (BG) vortex state, is introduced and the vortex state is an eigenstate of the sum of squared annihilation operators $a^2 + b^2$. We focus our study on the Wigner function of a BG vortex state and its entanglement property. For such a BG vortex state, which is different from the quantum vortex states previously presented, its wavefunction has a BG function structure

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in quadrature space and to prepare a BG vortex state (a pair-coherent state) is employed through a parametric frequency conversion. The rest of this paper is organized as follows. In Section 2, a BG vortex state is introduced and its vortex structure of wavefunction in quadrature space is discussed. In Section 3 the Wigner function of such a vortex state is calculated and investigated in detail with numerical evaluations. And the entanglement properties of BG vortex states are analysed in Section 4. Finally, some conclusions are drawn from the present study in Section 5.

2. BG vortex state of radiation fields and its generation

In this paper the following two-mode quantized field state:

$$|\psi\rangle_{\rm BG} = N_q \int_0^{2\pi} {\rm e}^{{\rm i}q\theta} \left|{\rm i}\alpha\cos\theta\right\rangle_a \left|{\rm i}\alpha\sin\theta\right\rangle_b {\rm d}\theta \quad (1)$$

is named a BG vortex state, where $|\ \rangle_a$ and $|\ \rangle_b$ represent coherent states, respectively, in the modes a and b, α characterizes the amplitude of the coherent state and is assumed to be real throughout the paper for simplicity, and N_q is the normalized constant and expressed as

$$N_q^{-2} = 4\pi^2 \exp\left(-\alpha^2\right) {\rm I}_q\left(\alpha^2\right) \quad (2)$$

with ${\rm I}_q(\ )$ being the modified Bessel function of the q-th order. It is easily proved that the state is the eigenstate of sum of squared two-mode annihilation operator a^2+b^2 with an eigenvalue α^2 because

$$\left(a^2+b^2\right)|\psi\rangle_{\rm BG} = -\alpha^2 |\psi\rangle_{\rm BG}\,. \quad (3)$$

Next, we evaluate the expression $\langle x,y|\psi\rangle$, where $|x,y\rangle$ is the eigenvector of $\left(a^\dagger+a\right)/\sqrt{2}$ and $\left(b^\dagger+b\right)/\sqrt{2}$ with the corresponding eigenvalues x and y. Based on the relation

$$\langle x|\alpha\rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{x^2}{2} + \sqrt{2}\alpha x - \frac{\alpha^2+|\alpha|^2}{2}\right), \quad (4)$$

under the Cartesian coordinates the state $|\psi\rangle_{\rm BG}$ becomes

$$\begin{aligned} &\psi_{\rm BG}(x,y) \\ &= \langle x,y|\psi\rangle_{\rm BG} \\ &= N_q \int_0^{2\pi} {\rm e}^{{\rm i}q\theta} \langle x|{\rm i}\alpha\cos\theta\rangle_a \langle y|{\rm i}\alpha\sin\theta\rangle_b \,{\rm d}\theta \\ &= \frac{N_q}{\sqrt{\pi}} \exp\left(-\frac{x^2+y^2}{2}\right) \\ &\quad \times \int_0^{2\pi} \exp\left[{\rm i}q\theta + \sqrt{2}{\rm i}\alpha\left(x\cos\theta + y\sin\theta\right)\right] {\rm d}\theta. \quad (5) \end{aligned}$$

Introducing the polar coordinates $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, one obtains

$$\begin{aligned} &\psi_{\rm BG}(x,y) \Rightarrow \psi_{\rm BG}(\rho,\varphi) \\ &= \frac{N_q}{\sqrt{\pi}} \exp\left(-\frac{\rho^2}{2}\right) \\ &\quad \times \int_0^{2\pi} \exp\left[{\rm i}q\theta + \sqrt{2}{\rm i}\alpha\rho\cos\left(\theta-\varphi\right)\right] {\rm d}\theta. \quad (6) \end{aligned}$$

Making use of the integral form of the modified Bessel function we finally have

$$\begin{aligned} &\psi_{\rm BG}(\rho,\varphi) \\ &= 2{\rm i}^q\sqrt{\pi}N_q \exp\left(-\frac{\rho^2}{2}\right) {\rm J}_q\left(\sqrt{2}\alpha\rho\right) \exp\left({\rm i}q\varphi\right), \quad (7) \end{aligned}$$

where ${\rm J}_q(\ )$ is the Bessel function of the q-th order. As is well known, Eq. (7) is just a standard BG wavepacket carrying vortex structure with a topological charge index q, which represents a new kind of non-Gaussian state. In addition, because the angular momentum operator in coordinate configuration space is

$$L_z^{x,y} = xp_y - yp_x = {\rm i}\hbar\partial/\partial\varphi \quad (8)$$

as expected, the wavefunction $\psi_{\rm BG}(x,y)$ is an eigenfunction of the angular momentum operator $L_z^{x,y}$ with the corresponding eigenvalue $q\hbar$.

State (1) may be prepared by making use of the parametric conversion process including pair-coherent states. The Hamiltonian describing a parametric conversion process reads[26–28]

$$H_{\rm pc} = \omega_a a^\dagger a + \omega_b b^\dagger b + \kappa\left(a^\dagger b\,{\rm e}^{{\rm i}\eta t} + a b^\dagger\,{\rm e}^{-{\rm i}\eta t}\right), \quad (9)$$

where $\eta=\omega_b-\omega_a$ (we constrain the discussion to the simplest case of exact resonance) and κ is the real coupling constant (the phase-matching condition is fulfilled). The well-known exact solutions of the Heisenberg equations of motion are

$$\begin{aligned} a_t &= {\rm e}^{-{\rm i}\omega_a t}\left(a\cos\tau - {\rm i}b\sin\tau\right) \\ b_t &= {\rm e}^{-{\rm i}\omega_b t}\left(b\cos\tau - {\rm i}a\sin\tau\right), \end{aligned} \quad (10)$$

where $\tau=\kappa t$ is the scaled time. And the corresponding evolution operator is represented as

$$U(t) = \exp\left(-{\rm i}H_{\rm pc}t/\hbar\right). \quad (11)$$

Due to $U(t)|0,0\rangle = |0,0\rangle$, an initial two-mode coherent state $|\alpha,\beta\rangle$ passing through such a parametric

conversion process will evolve into a new two-mode coherent state

$$U(t)|\alpha,\beta\rangle = D_{a,b}(\alpha_t,\beta_t)|0,0\rangle = |\alpha_t,\beta_t\rangle, \quad (12)$$

where the two-mode displacement operator is now transformed into

$$\begin{aligned} D^t_{a,b}(\alpha,\beta) &= U(t)\,D(\alpha,\beta)\,U^+(t) \\ &= D_{a,b}(\alpha_t,\beta_t), \end{aligned} \quad (13)$$

with

$$\begin{aligned} \alpha_t &= \alpha\,\mathrm{e}^{-\mathrm{i}\omega_a t}\cos\tau - \mathrm{i}\beta\,\mathrm{e}^{-\mathrm{i}\omega_b t}\sin\tau, \\ \beta_t &= \beta\,\mathrm{e}^{-\mathrm{i}\omega_b t}\cos\tau - \mathrm{i}\alpha\,\mathrm{e}^{-\mathrm{i}\omega_a t}\sin\tau. \end{aligned} \quad (14)$$

Considering that the magnitude of the optical frequency is about 10^{15} but the typical magnitude of the coupling constant κ is 10^7 and adequately selecting the interaction time t, it is possible to let $\tau = \pi/4$, $\omega_a T = \pi\omega_a/4\kappa = 2n\pi$, and $\omega_b T = 2m\pi - \pi/2$ (n and m being integers), and Eq. (14) turns into

$$\alpha_T = \frac{\alpha+\beta}{\sqrt{2}} \quad \text{and} \quad \beta_T = \frac{\alpha-\beta}{\sqrt{2}i}. \quad (15)$$

Therefore, due to the expressions $(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})/2 = \cos\theta$ and $(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})/2\mathrm{i} = \sin\theta$, a pair-coherent state[7,8]

$$|\psi\rangle_{\mathrm{pair}} = N_q \int_0^{2\pi} \mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}q\theta}|\mathrm{i}\beta\,\mathrm{e}^{\mathrm{i}\theta}, \mathrm{i}\beta\,\mathrm{e}^{-\mathrm{i}\theta}\rangle$$

through such a parametric conversion process will conditionally evolve into

$$\begin{aligned} &|\psi(T)\rangle \\ &= U(T)|\psi\rangle_{\mathrm{pair}} \\ &= N_q \int_0^{2\pi} \mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}q\theta}\left|\sqrt{2}\mathrm{i}\beta\cos\theta, \sqrt{2}\mathrm{i}\beta\sin\theta\right\rangle, \end{aligned} \quad (16)$$

which is just the BG vortex state (1). While a pair-coherent state can be conditionally generated with current available technology.[7,9,16]

3. Wigner function of BG vortex states

Based on the displaced parity operator on two modes, defined as

$$\begin{aligned} &\hat{\Pi}(u,w) \\ &= D_a(u)\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a} D_a^+(u)\,D_b(w)\,\mathrm{e}^{\mathrm{i}\pi b^\dagger b} D_b^+(w), \end{aligned} \quad (17)$$

where $u, w \in \mathbb{C}$, a, and b are the mode operators, and $D_z(\eta) = \exp(\eta z^+ - \eta^* z)$ ($z = a$, b and $\eta = \alpha, \beta$ are single-mode displacement operators), the two-mode Wigner function $W(u,w)$ can be expressed as

$$W(u,w) = 4\Pi(u,w)/\pi^2, \quad (18)$$

while $\Pi(u,w) = \langle\hat{\Pi}(u,w)\rangle$ is the expectation value of the corresponding operator.

For a given state $|\psi\rangle$ of the quantized field, we have

$$\begin{aligned} \Pi(u,w) &= \langle\psi|\hat{\Pi}(u,w)|\psi\rangle = \langle\psi|D_a(u)\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a} D_a^+(u)\,D_b(w)\,\mathrm{e}^{\mathrm{i}\pi b^\dagger b} D_b^+(w)|\psi\rangle \\ &= N_q^2 \int_0^{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}q(\theta-\theta')}\,\mathrm{d}\theta\mathrm{d}\theta'\,\langle\mathrm{i}\alpha\cos\theta'|D_a(u)\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a} D_a^+(u)|\mathrm{i}\alpha\cos\theta\rangle \\ &\quad \times \langle\mathrm{i}\alpha\sin\theta'|D_b(w)\,\mathrm{e}^{\mathrm{i}\pi b^\dagger b} D_b^+(w)|\mathrm{i}\alpha\sin\theta\rangle. \end{aligned} \quad (19)$$

By virtue of the relations

$$\begin{aligned} D(\alpha)|\beta\rangle &= \exp\left(\frac{\alpha\beta^* - \alpha^*\beta}{2}\right)|\alpha+\beta\rangle, \\ &\quad \times \exp\left(\mathrm{i}\pi a^\dagger a\right)|\alpha\rangle = |-\alpha\rangle, \end{aligned} \quad (20)$$

we have

$$\begin{aligned} &\langle\beta|D_a(u)\,\mathrm{e}^{\mathrm{i}\pi a^\dagger a} D_a^+(u)|\alpha\rangle \\ &= \langle\beta|2u-\alpha\rangle\exp(\alpha u^* - \alpha^* u) \\ &= \exp\Bigg(2\alpha u^* + 2\beta^* u - \alpha\beta^* \\ &\quad - \frac{|\alpha|^2 + |\beta|^2 + 4|u|^2}{2}\Bigg). \end{aligned} \quad (21)$$

After substituting Eq. (21) into Eq. (19) and performing some mathematical manipulations, Eq. (19) becomes

$$\begin{aligned} &W(u,w) \\ &= \frac{4}{\pi^2} N_q^2 \exp\left(-2|u|^2 - 2|w|^2 - \alpha^2\right)\Delta(u,w), \end{aligned} \quad (22)$$

with

$$\begin{aligned} \Delta(u,w) &= \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\theta\mathrm{d}\theta'\exp[\mathrm{i}q(\theta-\theta') \\ &\quad - \alpha^2\cos(\theta-\theta')] \\ &\quad \times \exp[2\mathrm{i}\alpha(u^*\cos\theta + w^*\sin\theta) \\ &\quad - 2\mathrm{i}\alpha(u\cos\theta' + w\sin\theta')]. \end{aligned} \quad (23)$$

Due to[29]

$$\begin{aligned}
&\exp\left[\pm \mathrm{i}\left(A\cos\theta + B\sin\theta\right)\right] \\
&= \exp\left[\pm\left(\frac{B+\mathrm{i}A}{2}\,\mathrm{e}^{\mathrm{i}\theta} - \frac{B-\mathrm{i}A}{2}\,\mathrm{e}^{-\mathrm{i}\theta}\right)\right] \\
&= \exp\left[\pm\frac{\sqrt{A^2+B^2}}{2}\left(\sqrt{\frac{B+\mathrm{i}A}{B-\mathrm{i}A}}\,\mathrm{e}^{\mathrm{i}\theta}\right.\right. \\
&\quad \left.\left. - \sqrt{\frac{B-\mathrm{i}A}{B+\mathrm{i}A}}\,\mathrm{e}^{-\mathrm{i}\theta}\right)\right] \\
&= \sum_{n=-\infty}^{\infty}\left(\frac{B+\mathrm{i}A}{B-\mathrm{i}A}\right)^{n/2}\mathrm{J}_n\left(\pm\sqrt{A^2+B^2}\right)\mathrm{e}^{\mathrm{i}n\theta}, \quad (24)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{J}_{-n}(x) &= (-1)^n\,\mathrm{J}_n(x), \\
\mathrm{J}_n(-x) &= (-1)^n\,\mathrm{J}_n(x), \quad (25)
\end{aligned}$$

where the generating function of the Bessel function

$$\exp\left[x\left(t-t^{-1}\right)/2\right] = \sum_{n=-\infty}^{\infty}\mathrm{J}_n(x)\,t^n$$

has been applied, therefore, the integration of Eq. (23) can be performed and the results can be expressed as

$$\begin{aligned}
\Delta(u,w) &= \sum_{n,m,k=-\infty}^{\infty}\left(\frac{w^*-\mathrm{i}u^*}{w^*+\mathrm{i}u^*}\right)^{n/2}\left(\frac{w-\mathrm{i}u}{w+\mathrm{i}u}\right)^{m/2}\mathrm{J}_n\left(2\alpha\sqrt{u^{*2}+w^{*2}}\right)\mathrm{J}_m\left(-2\alpha\sqrt{u^2+w^2}\right) \\
&\quad\times \mathrm{I}_k\left(-\alpha^2\right)\int_0^{2\pi}\mathrm{d}\theta\mathrm{d}\theta'\exp\left[\mathrm{i}\left(q+k\right)\left(\theta-\theta'\right)+\mathrm{i}n\theta+\mathrm{i}m\theta'\right] \\
&= 4\pi^2\sum_{k=-\infty}^{\infty}\mathrm{J}_{q+k}\left(2\alpha\sqrt{u^{*2}+w^{*2}}\right)\mathrm{J}_{q+k}\left(2\alpha\sqrt{u^2+w^2}\right)\mathrm{I}_k\left(-\alpha^2\right), \quad (26)
\end{aligned}$$

because of $\int_0^{2\pi}\exp\left(\mathrm{i}q\theta\right)\mathrm{d}\theta = 2\pi$ for $q=0$, but 0 for other values of q.

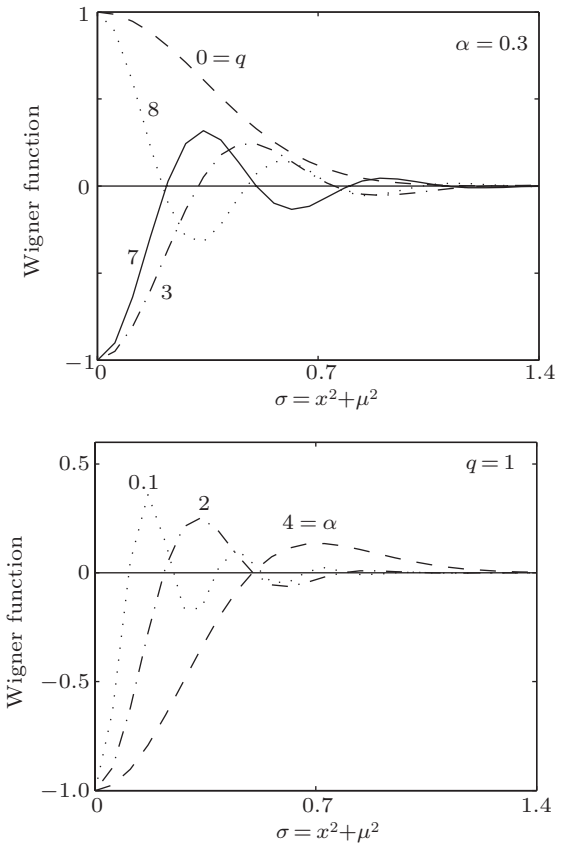


Fig. 1. Wigner function $W(u = x + \mathrm{i}y, w = \mu + \mathrm{i}\nu)$ for BG state versus ρ at fixed values of q and α in the case of $y = \nu = 0$. The parameters are denoted on the corresponding curves.

From Eqs. (22) and (26), the behaviour of the four-dimensional $W(u = x + \mathrm{i}y, w = \mu + \mathrm{i}\nu)$ function depends essentially on the topological charge index q and the displacement parameter α. It can be reduced to two variables by choosing the other two to be specified values. Especially, when $x = \mu = 0$ or $y = \nu = 0$, correspondingly, u and w become purely imaginary or purely real and $W(u,w)$ reduces to depend only on single variable $\rho_r = x^2 + \mu^2$ or $\rho_i = y^2 + \nu^2$. In Fig. 1 the Wigner functions are shown respectively corresponding to the different state parameters q and α.

Obviously, the Wigner functions may be negative, which considerably deviates from the Gaussian distribution and may be a signature of occurrence of nonclassical properties.

In addition, we examine the marginal Wigner function defined as

$$\begin{aligned}
&W_m(y,\nu) \\
&= \frac{4}{\pi^2}\int \mathrm{d}x\,\mathrm{d}\mu\,\langle\Pi\left(u = x+\mathrm{i}y, w = \mu+\mathrm{i}\nu\right)\rangle\,. \quad (27)
\end{aligned}$$

Inserting Eq. (22) into this expression and performing the integration, we obtain

$$\begin{aligned}
&W_m(y,\nu) \\
&= \frac{4N_q^2}{\pi^2}\exp\left(-2y^2-2\nu^2-\alpha^2\right)\int_0^{2\pi}\mathrm{d}\theta\mathrm{d}\theta'\,\mathrm{e}^{\mathrm{i}q\left(\theta-\theta'\right)} \\
&\quad\times\exp[2\alpha\left(y\cos\theta'+\nu\sin\theta'\right) \\
&\quad+2\alpha\left(y\cos\theta+\nu\sin\theta\right)-\alpha^2\cos\left(\theta-\theta'\right)]
\end{aligned}$$

$$
\begin{aligned}
&\times \int \mathrm{d}x\,\mathrm{d}\mu \exp[-2\left(x^2+\mu^2\right) \\
&+2\mathrm{i}\alpha x\left(\cos\theta-\cos\theta'\right)+2\mathrm{i}\alpha\mu\left(\sin\theta-\sin\theta'\right)] \\
=\ & \frac{2N_q^2}{\pi}\exp\left(-2y^2-2\nu^2-2\alpha^2\right) \\
&\times \int_0^{2\pi}\mathrm{d}\theta\,\mathrm{d}\theta'\,\mathrm{e}^{\mathrm{i}q\left(\theta-\theta'\right)} \\
&\times \exp[2\alpha\left(y\cos\theta'+\nu\sin\theta'\right) \\
&+2\alpha\left(y\cos\theta+\nu\sin\theta\right)]. \qquad (28)
\end{aligned}
$$

Similarly, using

$$
\begin{aligned}
&\exp\left(A\cos\theta+B\sin\theta\right) \\
=\ & \exp\left[\frac{A}{2}\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)+\frac{B}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}\right)\right] \\
=\ & \exp\left(\frac{A-\mathrm{i}B}{2}\mathrm{e}^{\mathrm{i}\theta}+\frac{A+\mathrm{i}B}{2}\mathrm{e}^{-\mathrm{i}\theta}\right) \\
=\ & \exp\left[\frac{\sqrt{A^2+B^2}}{2}\left(\sqrt{\frac{A+\mathrm{i}B}{A-\mathrm{i}B}}\mathrm{e}^{\mathrm{i}\theta}\right.\right. \\
&\left.\left.+\sqrt{\frac{A-\mathrm{i}B}{A+\mathrm{i}B}}\mathrm{e}^{-\mathrm{i}\theta}\right)\right] \\
=\ & \sum_{n=-\infty}^{\infty}\left(\frac{A+\mathrm{i}B}{A-\mathrm{i}B}\right)^{n/2}\mathrm{I}_n\left(\sqrt{A^2+B^2}\right)\mathrm{e}^{\mathrm{i}n\theta} \qquad (29)
\end{aligned}
$$

and the generating function of the modified Bessel function $\exp\left[x\left(t+t^{-1}\right)/2\right]=\sum\limits_{n=-\infty}^{\infty}\mathrm{I}_n\left(x\right)t^n$, we finally have

$$
\begin{aligned}
W_m\left(y,\nu\right) =\ & 8\pi N_q^2\exp\left(-2y^2-2\nu^2-2\alpha^2\right) \\
&\times \mathrm{I}_q^2\left(2\alpha\sqrt{y^2+\nu^2}\right), \qquad (30)
\end{aligned}
$$

which is just a modified BG function form.

4. Entanglement of BG vortex states

Based on high correlation moment functions of continuous-variable states, Agarwal and Biswas[21] recently developed a criterion to detect the inseparability of the quantum vortex states. The condition is established by the uncertainty relations of $su(1,1)$ algebra, whose three generators are respectively defined as

$$
\begin{aligned}
&J_x=\frac{1}{2}\left(a^\dagger b+ab^\dagger\right), \quad J_y=\frac{\mathrm{i}}{2}\left(a^\dagger b-ab^\dagger\right), \\
&J_z=\frac{1}{2}\left(a^\dagger a-b^\dagger b\right), \qquad (31)
\end{aligned}
$$

which have the canonical commutation relations $[J_x,J_y]=\mathrm{i}J_z$ (and cyclic permutations). It is found that any separable states must satisfy the following inequality

$$
\begin{aligned}
&[1+4\left(\Delta J_x\right)^2][1+4\left(\Delta J_y\right)^2] \\
\geq\ & \left(1+\left\langle a^\dagger a+b^\dagger b\right\rangle\right)^2. \qquad (32)
\end{aligned}
$$

In what follows we will use an indication function of inseparability as

$$
A=\frac{[1+4\left(\Delta J_x\right)^2][1+4\left(\Delta J_y\right)^2]}{\left(1+\left\langle a^\dagger a+b^\dagger b\right\rangle\right)^2}-1. \qquad (33)
$$

According to Eq. (33), if $A<0$, the state must be entangled.

To detect inseparability of state (1), we need to calculate the following expectation values associated with state (1)

$$
\begin{aligned}
\left\langle a^\dagger a\right\rangle =\ & N_q^2\int_0^{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}q\left(\theta-\theta'\right)}\left\langle \mathrm{i}\alpha\cos\theta',\mathrm{i}\alpha\sin\theta'\right|a^\dagger a\left|\mathrm{i}\alpha\cos\theta,\mathrm{i}\alpha\sin\theta\right\rangle\mathrm{d}\theta\mathrm{d}\theta' \\
=\ & N_q^2\alpha^2\,\mathrm{e}^{-\alpha^2}\int_0^{2\pi}\int_0^{2\pi}\exp\left[\mathrm{i}q\left(\theta-\theta'\right)+\alpha^2\cos\left(\theta-\theta'\right)\right]\cos\theta\cos\theta'\,\mathrm{d}\theta\mathrm{d}\theta' \\
=\ & \frac{1}{2}N_q^2\alpha^2\,\mathrm{e}^{-\alpha^2}\sum_l\mathrm{I}_l\left(\alpha^2\right)\int_0^{2\pi}\int_0^{2\pi}\exp\left[\mathrm{i}\left(q+l\right)\left(\theta-\theta'\right)\right]\cos\left(\theta-\theta'\right)\mathrm{d}\theta\mathrm{d}\theta' \\
=\ & \frac{1}{4}\alpha^2\left[\mathrm{I}_{q+1}\left(\alpha^2\right)+\mathrm{I}_{q-1}\left(\alpha^2\right)\right]\cdot 4\pi^2N_q^2\,\mathrm{e}^{-\alpha^2}=\frac{1}{4}\alpha^2T_{1,q}\left(\alpha^2\right)=\left\langle b^\dagger b\right\rangle, \qquad (34)
\end{aligned}
$$

where

$$
\int_0^{2\pi}\mathrm{e}^{\mathrm{i}m\left(\theta-\theta'\right)}\cos\left(\theta+\theta'\right)\mathrm{d}\theta\mathrm{d}\theta'=0
$$

has been used.

Similarly, we easily have

$$
\left\langle a^\dagger b\right\rangle=-\left\langle ab^\dagger\right\rangle=\frac{1}{2\mathrm{i}}q, \qquad (35)
$$

$$
\left\langle a^{\dagger 2}a^2\right\rangle=\left\langle b^{\dagger 2}b^2\right\rangle=\frac{\alpha^4}{4}\left[1+\frac{T_{2,q}\left(\alpha^2\right)}{4}\right], \qquad (36)
$$

$$\left\langle a^{+2}b^2\right\rangle = \left\langle a^2b^{+2}\right\rangle = \frac{\alpha^4}{4}\left[1-\frac{T_{2,q}\left(\alpha^2\right)}{4}\right], \quad (37)$$

$$\left\langle a^{\dagger}ab^{\dagger}b\right\rangle = \frac{\alpha^4}{16}T_{2,q}\left(\alpha^2\right), \quad (38)$$

where

$$T_{k,q}\left(x\right) = \frac{\mathrm{I}_{q+k}\left(x\right)+\mathrm{I}_{q-k}\left(x\right)}{\mathrm{I}_q\left(x\right)}. \quad (39)$$

Therefore, it can be directly found that in such a two-mode vortex state both the individual a- and b-mode field have the exactly identical photon number and photon number fluctuation.

Obviously, the results show that $\langle J_x\rangle = \langle J_z\rangle = 0$ and $\langle J_y\rangle = -n/2$. And making use of these above-mentioned facts, the numerical results are displayed in Fig. 2 and the values of parameter A are all negative, therefore, such a BG vortex state is always in entanglement.

Moreover, for sufficiently large values of x, with the help of the approximate expansion of modified Bessel function[29]

$$\mathrm{I}_n\left(x\right) \simeq \frac{\mathrm{e}^x}{\sqrt{2\pi x}}\left[1-\frac{4n^2-1}{8x} + \frac{\left(4n^2-1\right)\left(4n^2-9\right)}{2\left(8x\right)^2}\right], \quad (40)$$

we have

$$T_{1,q}\left(x\right) \simeq 2-\frac{1}{x} \quad \text{and}$$
$$T_{2,q}\left(x\right) \simeq 2-\frac{4}{x}+\frac{4q^2+2}{x^2}. \quad (41)$$

In the expression of $T_{2,q}\left(x\right)$ we retain the higher-order term. For large values of α, it is easy to prove that

$$1+4\langle\left(\Delta J_x\right)^2\rangle = \frac{1}{2}\alpha^2\left[T_{1,q}\left(\alpha^2\right)+\alpha^2\right]+1 \simeq \frac{\left(1+\alpha^2\right)^2}{2}, \quad (42)$$

$$1+4\langle\left(\Delta J_y\right)^2\rangle = \frac{1}{2}\alpha^2\left[\frac{\alpha^2}{2}T_{2,q}\left(\alpha^2\right)+T_{1,q}\left(\alpha^2\right)-\alpha^2\right]+1-q^2 \simeq \frac{1}{2}\alpha^2\left[\frac{\alpha^2}{2}\left(2-\frac{4}{\alpha^2}+\frac{4q^2+2}{\alpha^4}\right) + 2-\frac{1}{\alpha^2}-\alpha^2\right]+1-q^2 = 1, \quad (43)$$

so that

$$A = \frac{\left[1+4\left(\Delta J_x\right)^2\right]\left[1+4\left(\Delta J_y\right)^2\right]}{\left(1+\langle a^{\dagger}a+b^{\dagger}b\rangle\right)^2}-1 \to -1/2. \quad (44)$$

And, because of

$$\mathrm{I}_{\nu}\left(x\right) \simeq \frac{1}{\Gamma\left(\nu+1\right)}\left(\frac{x}{2}\right)^{\nu}$$

for very small values of α, we can prove that

$$A \to -\frac{|q|}{|q|+1}.$$

All those are completely consistent with the numerical results drawn in Fig. 2.

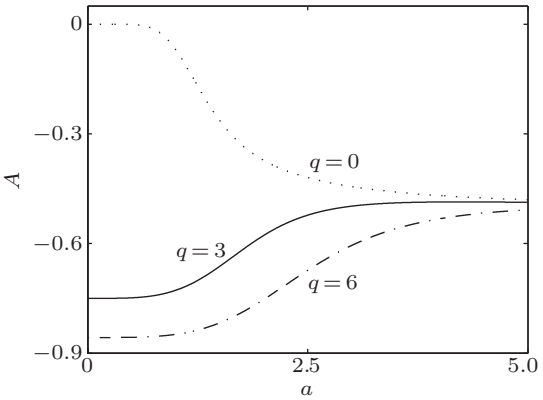


Fig. 2. Variations of A-parameter with α for different topological charge indices. The values are all negative.

5. Conclusion

In this paper, a new non-Gaussian state, termed a BG vortex state, is introduced, which is an eigenstate of the sum of squared two-mode annihilation operators a^2+b^2. In quadrature space its wavefunction has a BG vortex structure with topological charge index q and is an eigenfunction of the z-component L_z of the angular momentum operator. Its Wigner function is calculated and bears negative values, which is a good indication of the possibility of the occurrence of non-classical properties. The entanglement properties of a BG vortex state are also analysed in detail and it is found that the BG vortex state is always in entanglement. Theoretically, such BG vortex states may be prepared by making use of a pair-coherent state through a parametric conversion process.

References

[1] Braunstein S L and Pati A K 2003 *Quantum Information with Continuous Variables* (Dordrecht: Kluwer Academic)

[2] Braunstein S L and von Loock P 2005 *Rev. Mod. Phys.* **77** 513

[3] Eisert J, Scheel S and Plenio M B 2002 *Phys. Rev. Lett.* **89** 137903

[4] Fiurasek J 2002 *Phys. Rev. Lett.* **89** 137904
[5] Giedke G and Cirac J I 2002 *Phys. Rev.* A **66** 032316
[6] Bhaumik D, Bhaumik K and Dutta-Roy B 1976 *J. Phys.* A: *Math. Gen.* **9** 1507
[7] Agarwal G S 1986 *Phys. Rev. Lett.* **57** 827
[8] Agarwal G S 1988 *J. Opt. Soc. Am.* B **5** 1940
[9] Prakash G S and Agarwal G S 1995 *Phys. Rev.* A **52** 2335
[10] Gou S C, Steinbach J and Knight P L 1996 *Phys. Rev.* A **54** R1014
[11] Gilchrist A and Munro W J 2000 *J. Opt.* B: *Quantum Semiclass. Opt.* **2** 47
[12] Meng X G, Wang J S and Fan H Y 2007 *Phys. Lett.* A **363** 12
[13] Meng X G, Wang J S, Liang B L and Li H Q 2008 *Chin. Phys.* B **17** 1791
[14] Gerry C C and Mimih J 2010 *Phys. Rev.* A **82** 013831
[15] Gabris A and Agarwal G S 2007 *Int. J. Quantum Infor.* **5** 17
[16] Gerry C C, Mimih J and Birrittella R 2011 *Phys. Rev.* A **84** 023810
[17] Agarwal G S, Puri R R and Singh R P 1997 *Phys. Rev.* A **56** 4207
[18] Agarwal G S and Banerji J 2006 *J. Phys.* A: *Math. Gen.* **39** 11503
[19] Bandyopadhyay A and Singh R P 2011 *Opt. Commun.* **284** 256
[20] Bandyopadhyay A, Prabhakar S and Singh R P 2011 *Phys. Lett.* A **375** 1926
[21] Agarwal G S and Biswas A 2005 *New J. Phys.* **7** 211
[22] Agarwal G S 2011 *New J. Phys.* **13** 073008
[23] Molina-Terriza G, Torres J P and Torner L 2001 *Phys. Rev. Lett.* **88** 013601
[24] Leach J, Padgett M J, Barnett S M, Franke-Arnold S and Courtial J 2002 *Phys. Rev. Lett.* **88** 257901
[25] Allen L, Barnett S M and Padgett M J 2003 *Optical Angular Momentum* (Bristol: Institute of Physics)
[26] Louisell W H, Yariv A and Semjonov A E 1961 *Phys. Rev.* **124** 1646
[27] Tucker J and Walls D F 1969 *Ann. Phys.* NY **52** 1
[28] Dodonov A V, Dodonov V V and Mizrahi S S 2005 *J. Phys.* A: *Math. Gen.* **38** 683
[29] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)