Long-time limit behavior of the solution to an atom's master equation*

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We study the long-time limit behavior of the solution to an atom's master equation. For the first time we derive that the probability of the atom being in the α-th ($\alpha = j + 1 - j_z$, j is the angular momentum quantum number, j_z is the z-component of angular momentum) state is $\{(1 - K/G)/[1 - (K/G)^{2j+1}]\}(K/G)^{\alpha-1}$ as $t \to +\infty$, which coincides with the fact that when $K/G > 1$, the larger the α is, the larger the probability of the atom being in the α-th state (the lower excited state) is. We also consider the case for some possible generalizations of the atomic master equation.

Keywords: master equation, angular momentum, long-time limit behavior

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1. Introduction

In quantum optics theory one of the major topics is the atom interacting with all modes of the light radiation field in the electric dipole and rotating wave approximation. In the interacting process the evolution of pure photon states into mixed states occurs[1−6] and is described by a master equation. Solving master equations is one of the fundamental tasks in quantum optics. Usually people use the various quasi-probability representations, such as P-representation, Q-representation, complex P-representation, Wigner functions, and so on for the density operators describing photon states, to convert the master equations into the corresponding c-number equations. Recently, a new approach,[7] using the thermal entangled state representations of converting the operator master equations to c-number equations is presented which can simplify the procedures for solving master equations in many cases.

On the other hand, the master equation for the reduced density operator of the atom reads[1−3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = K\left(2J_-\rho J_+ - J_+J_-\rho - \rho J_+J_-\right) + G\left(2J_+\rho J_- - J_-J_+\rho - \rho J_-J_+\right), \quad (1)$$

where K and G are dissipation and gain coefficient, J_+ and J_- are ascending and lowering angular momentum operators describing atoms, satisfying the commutative relations

$$[J_+, J_-] = 2J_z, \ \ [J_\pm, J_z] = \mp J_\pm. \quad (2)$$

For example, for a two-level atom $j = 1/2$

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (3)$$

and the reduced density operator of the atom can be expressed as

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \quad (4)$$

The aim of this paper is to investigate all the possible generalization of Eq. (1) and the long-time limit behavior (stationary solution) of the atom governed by this master equation. As one can see shortly later, we find that the probability of the atom being in the α-th ($\alpha = j + 1 - j_z$, j is the angular momentum quantum number, j_z is the z-component of angular momentum) state is $\{(1 - K/G)/[1 - (K/G)^{2j+1}]\}(K/G)^{\alpha-1}$ as $t \to +\infty$, which coincides with the fact that when $K/G > 1$, the larger α is, the larger probability of the atom being in the α-th state (the lower excited state) is.

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2. Discussion on the generalization of master Eq. (1)

Let us consider the generalization of Eq. (1). The most general equation which contains all kinds of the terms in Eq. (1) reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\rho =& 2A_1 J_-\rho J_+ - B_1 J_+ J_-\rho - C_1\rho J_+ J_- \\ &+ 2A_2 J_+\rho J_- - B_2 J_- J_+\rho - C_2\rho J_- J_+, \quad (5)\end{aligned}$$

where the complex coefficients $A_{1,2}$, $B_{1,2}$, and $C_{1,2}$ cannot be arbitrary, there are some restrictions on them due to physics requirements, we want to find them by using the following procedures.

In the $(2j+1)$-dimensional representation of $SU(2)$, the matrix elements of the generators are

$$\begin{aligned}(J_+)_{\alpha\beta} &= \delta_{\alpha+1,\beta}\sqrt{\alpha\,(2j-\alpha+1)}, \\ (J_-)_{\alpha\beta} &= \delta_{\alpha,\beta+1}\sqrt{\beta\,(2j-\beta+1)}, \quad (6) \\ (J_z)_{\alpha\beta} &= (j+1-\alpha)\,\delta_{\alpha\beta}, \quad (7)\end{aligned}$$

so

$$(J_+J_-)_{\alpha\beta} = J_{+\alpha,\gamma}J_{-\gamma\beta} = \delta_{\alpha+1,\gamma}\sqrt{\alpha\,(2j-\alpha+1)}\delta_{\gamma,\beta+1}\sqrt{\beta\,(2j-\beta+1)} = \delta_{\alpha\beta}\alpha\,(2j-\alpha+1)\,, \quad (8)$$

where the repeated subscript indices implies summation from 1 to $2j+1$,

$$\begin{aligned}(J_-J_+)_{\alpha\beta} &= J_{-\alpha,\gamma}J_{+\gamma\beta} = \delta_{\alpha,\gamma+1}\sqrt{\gamma\,(2j-\gamma+1)}\delta_{\gamma+1,\beta}\sqrt{\gamma\,(2j-\gamma+1)} \\ &= \delta_{\alpha\beta}\sqrt{(\alpha-1)\,(2j-\alpha+2)\,(\beta-1)\,(2j-\beta+2)} = \delta_{\alpha\beta}\,(\alpha-1)\,(2j-\alpha+2)\,. \quad (9)\end{aligned}$$

In this representation, the matrix elements of terms in Eq. (5) are

$$\begin{aligned}(J_-\rho J_+)_{\alpha\beta} &= 2\delta_{\alpha,\gamma+1}\sqrt{\gamma\,(2j-\gamma+1)}\rho_{\gamma\epsilon}\delta_{\epsilon+1,\beta}\sqrt{\epsilon\,(2j-\epsilon+1)} \\ &= 2\sqrt{(\alpha-1)\,(2j-\alpha+2)}\sqrt{(\beta-1)\,(2j-\beta+2)}\rho_{\alpha-1,\beta-1}, \\ (J_+J_-\rho)_{\alpha\beta} &= \alpha\,(2j-\alpha+1)\,\rho_{\alpha\beta}, \quad (\rho J_+J_-)_{\alpha\beta} = \beta\,(2j-\beta+1)\,\rho_{\alpha\beta}, \\ (J_+\rho J_-)_{\alpha\beta} &= 2\delta_{\alpha+1,\gamma}\sqrt{\alpha\,(2j-\alpha+1)}\rho_{\gamma\epsilon}\delta_{\epsilon,\beta+1}\sqrt{\beta\,(2j-\beta+1)} = 2\sqrt{\alpha\,(2j-\alpha+1)}\sqrt{\beta\,(2j-\beta+1)}\rho_{\alpha+1,\beta+1}, \\ (J_-J_+\rho)_{\alpha\beta} &= (\alpha-1)\,(2j-\alpha+2)\,\rho_{\alpha\beta}, \quad (\rho J_-J_+)_{\alpha\beta} = (\beta-1)\,(2j-\beta+2)\,\rho_{\alpha\beta}. \quad (10)\end{aligned}$$

Firstly, we must have $\mathrm{Tr}\,\rho(t) \equiv 1$, so

$$\begin{aligned}0 \equiv \mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho\right) &= (2A_1 - B_1 - C_1)\,\mathrm{Tr}\,(J_+J_-\rho) \\ &+ (2A_2 - B_2 - C_2)\,Tr\,(J_-J_+\rho) \\ &= [(2A_1 - B_1 - C_1) + (2A_2 - B_2 - C_2)] \\ &\times \mathrm{Tr}\,(J_+J_-\rho) - 2\,(2A_2 - B_2 - C_2)\,\mathrm{Tr}\,(J_z\rho)\,. \quad (11)\end{aligned}$$

Equation (11) must hold for all possible initial values $\rho\,(0)$. Choosing the initial density matrix

$$\rho\,(0)_{\alpha\beta} = \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} = \delta_{\alpha i}\delta_{\beta i} \quad (12)$$

with $i = 1, 2, \ldots, 2j+1$, then

$$\begin{aligned}\mathrm{Tr}\,(J_+J_-\rho) &= \sum_{\alpha\beta}(J_+J_-)_{\alpha\beta}\,\rho_{\beta\alpha} \\ &= \sum_{\alpha\beta}\alpha\,(2j-\alpha+1)\,\delta_{\alpha\beta}\delta_{\alpha i}\delta_{\beta i} \\ &= i\,(2j-i+1)\,, \\ \mathrm{Tr}\,(J_z\rho) &= \sum_{\alpha\beta}(J_z)_{\alpha\beta}\,\rho_{\beta\alpha} \\ &= \sum_{\alpha\beta}(j-\alpha+1)\,\delta_{\alpha\beta}\delta_{\alpha i}\delta_{\beta i} \\ &= j-i+1. \quad (13)\end{aligned}$$

Setting $i = 1$ and $i = 2j+1$ respectively in Eq. (13) and then substituting the result into Eq. (11), we respectively have

$$\begin{aligned}&2j\,[(2A_1 - B_1 - C_1) + (2A_2 - B_2 - C_2)] \\ &\quad - 2j\,(2A_2 - B_2 - C_2) = 0, \\ &2j\,(2A_2 - B_2 - C_2) = 0. \quad (14)\end{aligned}$$

So

$$2A_1 = B_1 + C_1, \quad 2A_2 = B_2 + C_2. \tag{15}$$

Besides, $\rho(t)$ must be Hermitian, so

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^\dagger - \rho\right) \\
&= 2\left(A_1^* - A_1\right) J_-\rho J_+ - \left(B_1^* - C_1\right)\rho J_+ J_- \\
&\quad - \left(C_1^* - B_1\right) J_+ J_-\rho \\
&\quad + 2\left(A_2^* - A_2\right) J_+\rho J_- - \left(B_2^* - C_2\right)\rho J_- J_+ \\
&\quad - \left(C_2^* - B_2\right) J_- J_+\rho \equiv 0.
\end{aligned} \tag{16}$$

Again, equation (16) must hold for all possible initial values $\rho(0)$. Choosing $\rho(0)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i}$, where $i = 1, 2, \ldots, 2j+1$, then $\rho(0)_{i+1,i+1} = 0$, $\rho(0)_{i-1,i-1} = 0$, plugging these and Eqs. (6) and (7) into the right-hand side of Eq. (16), we see that only the terms from $(J_-\rho J_+)_{\alpha\beta}$ and $(J_+\rho J_-)_{\alpha\beta}$ make contribution, i.e.,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^\dagger - \rho\right)_{i+1,i+1}\big|_{t=0} \\
&= 2i\left(2j - i + 1\right)\left(A_1^* - A_1\right) = 0, \\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^\dagger - \rho\right)_{i-1,i-1}\big|_{t=0} \\
&= 2\left(A_2^* - A_2\right)(i-1)\,\alpha\,(2j - i + 2) = 0.
\end{aligned} \tag{17}$$

So

$$A_1^* = A_1, \; A_2^* = A_2, \tag{18}$$

i.e., A_1 and A_2 are both real numbers.

Now in Eq. (16) we choose the initial density matrix

$$\rho(0) = \begin{pmatrix} x & y \\ y^* & 1-x \end{pmatrix}, \tag{19}$$

i.e.,

$$\rho(0)_{\alpha\beta} = \begin{cases} x, & \text{if } \alpha = \beta = i, \\ 1-x, & \text{if } \alpha = \beta = k, \\ y, & \text{if } \alpha = i,\ \beta = k, \\ y^*, & \text{if } \alpha = k,\ \beta = i, \\ 0, & \text{else}, \end{cases} \tag{20}$$

where $x > 0$ and $x(1-x) - |y|^2 > 0$ are demanded for quarantining $\rho(0)$ being semi-positive and $\mathrm{Tr}\,\rho(0) = 1$. In this case $(J_-\rho J_+)_{i-1,k-1} = (J_+\rho J_-)_{i+1,k+1} = 0$. Then we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^\dagger - \rho\right)_{ik}\big|_{t=0} \\
&= -\left(B_1^* - C_1\right) i\left(2j - i + 1\right) y \\
&\quad - \left(C_1^* - B_1\right) k\left(2j - k + 1\right) y \\
&\quad - \left(B_2^* - C_2\right)(i-1)\left(2j - i + 2\right) y \\
&\quad - \left(C_2^* - B_2\right)(k-1)\left(2j - k + 2\right) y = 0.
\end{aligned} \tag{21}$$

Choosing $i = 1, k = 2j+1$, equation (21) becomes

$$-2jy\left(B_1^* - C_1\right) - 2jy\left(C_2^* - B_2\right) = 0. \tag{22}$$

Choosing $i = 1, k = 2j \neq 1$, equation (21) becomes

$$\begin{aligned}
&-2jy\left(B_1^* - C_1\right) - 2j\left(C_1^* - B_1\right) y \\
&\qquad - 2\left(2j - 1\right) y\left(C_2^* - B_2\right) = 0.
\end{aligned} \tag{23}$$

Subtracting Eq. (22) from Eq. (23), we have

$$-2j\left(C_1^* - B_1\right) - \left(2j - 2\right)\left(C_2^* - B_2\right) \equiv 0. \tag{24}$$

Equations (22) and (24) lead to

$$C_1^* = B_1, \; C_2^* = B_2. \tag{25}$$

Therefore, equations (15), (18), and (25) allow us to write

$$B_1 = C_1^* = A_1 + \mathrm{i}D_1, \quad B_2 = C_2^* = A_2 + \mathrm{i}D_2, \tag{26}$$

where A_1, D_1, A_2, D_2 are real numbers. i.e., only four independent real parameters are allowed in Eq. (5), so the general equation (5) takes the form

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\rho = {} & A_1\left(2J_-\rho J_+ - J_+J_-\rho - \rho J_+J_-\right) \\
& + \mathrm{i}D_1\left(\rho J_+J_- - J_+J_-\rho\right) \\
& + A_2\left(2J_+\rho J_- - J_-J_+\rho - \rho J_-J_+\right) \\
& + \mathrm{i}D_2\left(\rho J_-J_+ - J_-J_+\rho\right).
\end{aligned} \tag{27}$$

Equation (27) is more complicated than the standard Eq. (1), in the next section we shall simplify it.

3. Reduction of the generalized master equation

In this section we will prove that equation (27) can be simplified by using a unitary transformation

$$U = \mathrm{e}^{\mathrm{i}Ht}, \qquad H = \lambda J_+J_- + \mu J_z \tag{28}$$

with the result being an equation in exactly the same form as Eq. (1), where λ, μ are real parameters.

In fact, from the following well-known commutation relationship

$$\begin{aligned}
&[J_-, J_z] = J_-, \quad [J_+, J_z] = -J_+, \\
&[J_+, J_-] = 2J_z, \quad [J_+J_-, J_z] = 0,
\end{aligned} \tag{29}$$

then $[H, J_z] = [H, J_+J_-] = 0$, and

$$\begin{aligned}[H, J_-] &= \lambda [J_+, J_-] J_- + \mu [J_z, J_-] \\ &= (2\lambda J_z - \mu) J_-, \qquad (30)\\ [H, [H, J_-]] &= [H, (2\lambda J_z - \mu) J_-] \\ &= (2\lambda J_z - \mu) [H, J_-] = (2\lambda J_z - \mu)^2 J_-. \qquad (31)\end{aligned}$$

It follows then

$$\underbrace{[H, \ldots [H, [H}_{n}, J_-], \ldots] = (2\lambda J_z - \mu)^n J_-. \qquad (32)$$

Further, using the Baker–Hausdorff formula,

$$\begin{aligned}\mathrm{e}^{\mathrm{i}Ht} J_- \mathrm{e}^{-\mathrm{i}Ht} &= \exp [\mathrm{i} (2\lambda J_z - \mu) t] J_-, \\ \mathrm{e}^{\mathrm{i}Ht} J_+ \mathrm{e}^{-\mathrm{i}Ht} &= \left(\mathrm{e}^{\mathrm{i}Ht} J_- \mathrm{e}^{-\mathrm{i}Ht} \right)^\dagger \\ &= J_+ \exp [-\mathrm{i} (2\lambda J_z - \mu) t]. \qquad (33)\end{aligned}$$

Define

$$\tilde{\rho} \equiv \mathrm{e}^{\mathrm{i}Ht} \rho \mathrm{e}^{-\mathrm{i}Ht}, \qquad (34)$$

then by Eq. (27),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho} &= \mathrm{i} [H, \tilde{\rho}] + \mathrm{e}^{\mathrm{i}Ht} \left(\frac{\mathrm{d}}{\mathrm{d}t} \rho \right) \mathrm{e}^{-\mathrm{i}Ht} \\ &= \mathrm{i} [H, \tilde{\rho}] + 2A_1 \mathrm{e}^{\mathrm{i}Ht} J_- \mathrm{e}^{-\mathrm{i}Ht} \tilde{\rho} \mathrm{e}^{\mathrm{i}Ht} J_+ \mathrm{e}^{-\mathrm{i}Ht} \\ &\quad - A_1 (J_+ J_- \tilde{\rho} + \tilde{\rho} J_+ J_-) \\ &\quad + \mathrm{i} D_1 [\tilde{\rho}, J_+ J_-] + 2A_2 \mathrm{e}^{\mathrm{i}Ht} \\ &\quad \times J_+ \mathrm{e}^{-\mathrm{i}Ht} \tilde{\rho} \mathrm{e}^{\mathrm{i}Ht} J_- \mathrm{e}^{-\mathrm{i}Ht} \\ &\quad - A_2 (J_- J_+ \tilde{\rho} + \tilde{\rho} J_- J_+) + \mathrm{i} D_2 [\tilde{\rho}, J_- J_+] \\ &= \mathrm{i} [H - D_1 J_+ J_- - D_2 J_- J_+, \tilde{\rho}] \\ &\quad + A_1 \big(2\tilde{J}_- (t) \tilde{\rho} (t) \tilde{J}_+ (t) - \tilde{J}_+ (t) \\ &\quad \times \tilde{J}_- (t) \tilde{\rho} (t) - \tilde{\rho} (t) \tilde{J}_+ (t) \tilde{J}_- (t) \big) \\ &\quad + A_2 \big(2\tilde{J}_+ (t) \tilde{\rho} (t) \tilde{J}_- (t) - \tilde{J}_- (t) \\ &\quad \times \tilde{J}_+ (t) \tilde{\rho} (t) - \tilde{\rho} (t) \tilde{J}_- (t) \tilde{J}_+ (t) \big), \qquad (35)\end{aligned}$$

where

$$\begin{aligned}\tilde{J}_- (t) &= \mathrm{e}^{\mathrm{i}Ht} J_- \mathrm{e}^{-\mathrm{i}Ht} = \exp [\mathrm{i} (2\lambda J_z - \mu) t] J_-, \\ \tilde{J}_+ (t) &= \mathrm{e}^{\mathrm{i}Ht} J_+ \mathrm{e}^{-\mathrm{i}Ht} \\ &= J_+ \exp [-\mathrm{i} (2\lambda J_z - \mu) t]. \qquad (36)\end{aligned}$$

If we set

$$\lambda = D_1 + D_2, \qquad \mu = -2D_2 \qquad (37)$$

in Eq. (28), then

$$H = \lambda J_+ J_- + \mu J_z = D_1 J_+ J_- + D_2 J_- J_+, \qquad (38)$$

equation (35) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho} &= A_1 \big(2\tilde{J}_- (t) \tilde{\rho} (t) \tilde{J}_+ (t) - \tilde{J}_+ (t) \\ &\quad \times \tilde{J}_- (t) \tilde{\rho} (t) - \tilde{\rho} (t) \tilde{J}_+ (t) \tilde{J}_- (t) \big) \\ &\quad + A_2 \big(2\tilde{J}_+ (t) \tilde{\rho} (t) \tilde{J}_- (t) - \tilde{J}_- (t) \\ &\quad \times \tilde{J}_+ (t) \tilde{\rho} (t) - \tilde{\rho} (t) \tilde{J}_- (t) \tilde{J}_+ (t) \big), \qquad (39)\end{aligned}$$

which takes exactly the same form of Eq. (1), except that here $\tilde{J}_- (t)$, $\tilde{J}_+ (t)$ are time-dependent. An extra term with the form $\mathrm{i} [H', \rho]$ is often included in Eq. (1) in many literatures, we can see straightforwardly that this term can also be annihilated by setting an appropriate H. An example can be found in Ref. [9], in this work the authors discussed the Jaynes–Cumming model of a two-level atom system interacting with a single-mode radiation field. Thus Eq. (1) is "rigid" in form if we allow the operators (J_+, J_-, etc.) to be time-dependent.

4. Long-time limit behavior of $j = 1/2$ system

Now let us discuss the long-time behavior of the system described by Eq. (1). For example, for a two-level atom $j = 1/2$

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \qquad (40)$$

Writing

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \qquad (41)$$

then

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= K (2J_- \rho J_+ - J_+ J_- \rho - \rho J_+ J_-) \\ &\quad + G (2J_+ \rho J_- - J_- J_+ \rho - \rho J_- J_+) \\ &= \begin{pmatrix} 2G\rho_{22} - 2K\rho_{11} & -(K+G)\rho_{12} \\ -(K+G)\rho_{21} & 2K\rho_{11} - 2G\rho_{22} \end{pmatrix}, \qquad (42)\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \rho_{12} \\ \rho_{21} \end{pmatrix} = -(K+G) \begin{pmatrix} \rho_{12} \\ \rho_{21} \end{pmatrix}, \qquad (43)$$

we can solve

$$\begin{aligned}\rho_{12} (t) &= \exp [-(K+G) t] \rho_{12} (0), \\ \rho_{21} (t) &= \exp [-(K+G) t] \rho_{21} (0). \qquad (44)\end{aligned}$$

Thus we see that the non-diagonal elements decay exponentially, when $t \to +\infty$, they go to zero. From Eq. (42) we also see

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\rho_{11}\\ \rho_{22}\end{pmatrix} = 2\begin{pmatrix}-K & G\\ K & -G\end{pmatrix}\begin{pmatrix}\rho_{11}\\ \rho_{22}\end{pmatrix}, \tag{45}$$

its solution is

$$\begin{aligned}\rho_{11}(t) &= \frac{G}{K+G} + \left[\rho_{11}(0) - \frac{G}{K+G}\right] \\ &\quad \times \exp\left[-(K+G)t\right], \\ \rho_{22}(t) &= \frac{K}{K+G} - \left[\rho_{11}(0) - \frac{G}{K+G}\right] \\ &\quad \times \exp\left[-(K+G)t\right],\end{aligned} \tag{46}$$

which preserves $\mathrm{Tr}\,\rho(t) \equiv 1$. The above exact time-dependent solution for $j = 1/2$ case can be found in Ref. [10]. In this work the authors applied an interesting method developed by Puri and Agarwal.[9] When $t \to \infty$, $\rho_{11}(t)$, $\rho_{22}(t)$ approach $G/(K+G)$, $K/(K+G)$, respectively, and we see the limit behavior of $\rho_{11}(t)$, $\rho_{22}(t)$,

$$\lim_{t\to+\infty} \rho_{11}(t)/\rho_{22}(t) = G/K. \tag{47}$$

It is remarkable that

(i)

$$\begin{aligned}\lim_{t\to+\infty}\rho(t) &= \begin{pmatrix}G/(K+G) & 0\\ 0 & K/(K+G)\end{pmatrix} \\ &= \rho(+\infty)\end{aligned} \tag{48}$$

is a diagonal matrix.

(ii) $\rho(+\infty)$ is completely determined by Eq. (1). It does not depend on the initial density matrix $\rho(0)$. Investigating the first few values of j, $j = 1/2$, 1, 3/2, we reach the same conclusion, i.e., (i) and (ii). Enlightened by these facts, we investigate long-term behavior of ρ for general j value in the next section.

5. Long-time limit behavior for general j case

In general cases for $(2j+1)$-dimensional representation of $SU(2)$, we have the following equations by Eqs. (6) and (7)

$$\begin{aligned}&\left(2J_-\rho J_+ - J_+J_-\rho - \rho J_+J_-\right)_{\alpha\beta} \\ &= 2\sqrt{(\alpha-1)(2j-\alpha+2)} \\ &\quad \times \sqrt{(\beta-1)(2j-\beta+2)}\rho_{\alpha-1,\beta-1} \\ &\quad - \alpha(2j-\alpha+1)\rho_{\alpha\beta} - \beta(2j-\beta+1)\rho_{\alpha\beta}, \\ &\quad \left(2J_+\rho J_- - J_-J_+\rho - \rho J_-J_+\right)_{\alpha\beta} \\ &= 2\sqrt{\alpha(2j-\alpha+1)}\sqrt{\beta(2j-\beta+1)}\rho_{\alpha+1,\beta+1} \\ &\quad - (\alpha-1)(2j-\alpha+2)\rho_{\alpha\beta} \\ &\quad - (\beta-1)(2j-\beta+2)\rho_{\alpha\beta}.\end{aligned} \tag{49}$$

The evolution equation of the diagonal elements $\rho_{\alpha\alpha}$ reads

$$\begin{aligned}\frac{\mathrm{d}\rho_{\alpha\alpha}}{\mathrm{d}t} &= 2(\alpha-1)(2j-\alpha+2)\left(K\rho_{\alpha-1,\alpha-1} - G\rho_{\alpha\alpha}\right) \\ &\quad -2\alpha(2j-\alpha+1)\left(K\rho_{\alpha\alpha} - G\rho_{\alpha+1,\alpha+1}\right).\end{aligned} \tag{50}$$

It is very reasonable to make the assumption that $\mathrm{d}\rho/\mathrm{d}t \to 0$ and $\rho_{\alpha\beta} \to 0$ $(\alpha \neq \beta)$ when $t \to +\infty$, since the system that ρ describes is coupled to a reservoir, when $t \to +\infty$, the system will reach thermal equilibrium, in this case ρ is a constant diagonal matrix. With this assumption, we can solve $\rho_{\alpha\alpha}$ recursively, first when $\alpha = 1$,

$$\frac{\mathrm{d}\rho_{11}}{\mathrm{d}t} = -4j\left(K\rho_{11} - G\rho_{2,2}\right) = 0 \tag{51}$$

tells us

$$\rho_{2,2} = \frac{K}{G}\rho_{11}. \tag{52}$$

By the induction method, assuming $\rho_{\alpha\alpha} = (K/G)\rho_{\alpha-1,\alpha-1}$ for some α, $1 < \alpha < 2j+1$, then $\alpha(2j-\alpha+1) \neq 0$ and we have

$$\begin{aligned}0 = \frac{\mathrm{d}\rho_{\alpha\alpha}}{\mathrm{d}t} &= -2\alpha(2j-\alpha+1) \\ &\quad \times \left(K\rho_{\alpha\alpha} - G\rho_{\alpha+1,\alpha+1}\right).\end{aligned} \tag{53}$$

So

$$\rho_{\alpha+1,\alpha+1} = \frac{K}{G}\rho_{\alpha,\alpha} = \cdots = c\left(\frac{K}{G}\right)^{\alpha} \tag{54}$$

for all $0 \leqslant \alpha \leqslant 2j$.

Since

$$\mathrm{Tr}\,\rho = \sum_{\alpha=1}^{2j+1}\rho_{\alpha\alpha} = c\left(1 + \frac{K}{G} + \cdots\left(\frac{K}{G}\right)^{2j}\right) = 1, \tag{55}$$

the constant c is

$$c = \frac{1}{1 + K/G + \cdots (K/G)^{2j}} = \frac{1 - K/G}{1 - (K/G)^{2j+1}}. \tag{56}$$

Finally we obtain the limit behavior of $\rho_{\alpha\alpha}$ when $t \to +\infty$

$$\rho_{\alpha\alpha} \to \frac{1 - K/G}{1 - (K/G)^{2j+1}}\left(\frac{K}{G}\right)^{\alpha-1}. \tag{57}$$

This result agrees with that in Ref. [8], where Hassant *et al.* dealt with a similar problem under atom

coherent state representation. In order to see the intrinsic consistency, we notice that $\rho_{\alpha\alpha}$ is the probability of the atom being in the α-th state, in our case, $|\alpha\rangle = |j, j_z = j + 1 - \alpha\rangle$, by statistical mechanics, we have

$$\rho_{\alpha\alpha} \propto \exp\left(\frac{-E_\alpha}{k_{\rm B}T}\right) \tag{58}$$

for system in thermal equilibrium. So the eigen-energy of $|\alpha\rangle$ is

$$E_\alpha = E_0 + j_z \Delta E, \tag{59}$$

which is evenly spaced. In particular, when

$$K = \frac{\gamma}{2}(1 + \bar{n}), \qquad G = \frac{\gamma}{2}\bar{n}, \tag{60}$$

this is the case as given in Refs. [1] and [2], $\bar{n}$ is the average photon number, we have

$$\frac{1 + \bar{n}}{\bar{n}} = \frac{K}{G} = \exp\left(\frac{\Delta E}{k_B T}\right), \tag{61}$$

so the average photon number

$$\bar{n} = \frac{1}{\exp(\Delta E/k_{\rm B}T) - 1}, \tag{62}$$

which is exactly photon's Bose–Einstein distribution.

It is noticeable that although equation (27) is different from Eq. (1), the equation of evolution for the diagonal elements $\rho_{\alpha\alpha}$ still takes the same form of Eq. (50)

$$\frac{{\rm d}\rho_{\alpha\alpha}}{{\rm d}t} = 2(\alpha-1)(2j-\alpha+2)(A_1\rho_{\alpha-1,\alpha-1} - A_2\rho_{\alpha\alpha}) - 2\alpha(2j-\alpha+1)(A_1\rho_{\alpha\alpha} - A_2\rho_{\alpha+1,\alpha+1}), \tag{63}$$

which gives a similar conclusion

$$\lim_{t\to+\infty}\rho_{\alpha\alpha}(t) = \frac{1 - A_1/A_2}{1 - (A_1/A_2)^{2j+1}}\left(\frac{A_1}{A_2}\right)^{\alpha-1}, \tag{64}$$

i.e., the new parameters D_1 and D_2 do not affect the long-time behavior of $\rho_{\alpha\alpha}$.

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