Does the eigenratio λ_2/λ_N represent the synchronizability of a complex network?*

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In the study of complex networks, it is commonly believed that the eigenratio λ_2/λ_N of the Laplacian matrix of a network represents the network synchronizability, especially for symmetric networks. This paper gives two counterexamples to show that this is not true for the case where the network has a disconnected synchronized region. Consequently, a simple answer is presented to the question of when the eigenratio λ_2/λ_N does represent the network synchronizability.

Keywords: synchronizability, synchronized region, Laplacian eigenratio, complex network

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1. Introduction

The subject of network synchronization has attracted increasing attention from various fields of science of engineering in recent years (see Refs. [1]–[12] and references therein). Of particular interest is how the synchronization ability depends on the underlying network topology and characteristics. Some important results have been established by introducing the notions of master stability function and synchronized region.[13,14] According to the current literature,[14−16] the analysis of network synchronizability can be divided into two concerns: one is related to the eigenvalues of the network coupling matrix, and the other is related to the synchronized region determined by both the network node dynamics and the inner-linking function. It is known today that the easier the products of the network coupling strength and the eigenvalues of the network Laplacian matrix fall into the synchronized region, the more efficiently the network achieves synchronization, hence the stronger the synchronizability. More precisely, the larger the synchronized region, or the larger the eigenratio λ_2/λ_N, the easier the synchronization. Therefore, the eigenratio λ_2/λ_N is referred to as the network synchronizability index, and the relationships between the synchronizability index and various network structural characteristics have been studied.[6,17−20] This viewpoint also inspired the study using the spectrum of the Laplacian matrix to characterize the network synchronizability.[21−26]

For linearized networks, the synchronized region is actually reduced to the stable region of a matrix pencil with respect to a parameter.[15] The existence of disconnected synchronized regions is demonstrated in Refs. [13], [15], and [16], which brings up the following concern: if part of the eigenvalues falls into one synchronized sub-region, and part of the eigenvalues falls into the other synchronized sub-region, then how can the eigenratio represent the synchronizability? One extreme case is that each eigenvalue falls into one independent synchronized sub-region. In this situation, the synchronizability index is actually very complicated and unlikely to be reflected by the eigenratio λ_2/λ_N. This paper gives two specific examples to illustrate this problem.

2. Preliminaries

Consider a dynamical network consisting of N coupled identical nodes, with each node being an n-dimensional dynamical system, described by

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}H(x_j), \quad i = 1, 2, \ldots, N, \quad (1)$$

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where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ is the state vector of node i, $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector-valued function, constant $c > 0$ represents the coupling strength, $H(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is called the inner-linking function, and $A = (a_{ij})_{N\times N}$ is called the outer coupling matrix which represents the coupling configuration of the entire network. If the network is connected, then A is an irreducible matrix, and if the entries of A satisfy

$$a_{ij} = a_{ji}, \quad a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, \quad i = 1, 2, \ldots, N,$$

then the network is called a diffusively and symmetrically coupled network. In this case, zero is an eigenvalue of A with multiplicity 1 and all the other eigenvalues of A are strictly negative, which are denoted by

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N. \tag{2}$$

The dynamical network (1) is said to achieve (asymptotical) synchronization if

$$\|x_i(t) - x_j(t)\| \to 0, \quad \text{as } t \to \infty,\ i, j = 1, \ldots, N. \tag{3}$$

As shown in Ref. [14], the stability of the synchronized manifold $x_1 = x_2 = \cdots = x_N$ can be determined by analyzing the following equation, known as the master stability equation:

$$\dot{\omega} = [Df(s) + \alpha DH(s)]\omega, \tag{4}$$

where $\alpha \in \mathbb{R}$, $Df(s)$ and $DH(s)$ are the Jacobian matrices of functions f and H at s, respectively, and $s \in \mathbb{R}^n$ is the constant equilibrium of $f(\cdot)$. In this case, system (4) is simply written as

$$\dot{\omega} = [F + \alpha H]\omega. \tag{5}$$

Hence, the synchronized region S becomes the stability region of the matrix pencil $F + \alpha H$ with respect to the parameter α.

The largest Lyapunov exponent $L_{\max}$ of network (1), which can be calculated from system (4) and is a function of α, is referred to as the master stability function. In addition, the region S of negative real α where $L_{\max}$ is also negative is called the synchronized region of network (1). Based on the results of Ref. [14], the synchronized solution of network (1) is asymptotically stable only if

$$c\lambda_k \in S, \quad k = 2, 3, \ldots, N. \tag{6}$$

The synchronized region S can be an unbounded region, a bounded region, an empty set, or a union of several sub-regions.[15,27]

3. Two examples

Example 1 Consider network (1), now consisting of the fourth-order generalized chaotic Chua's circuits[28] described by

$$\begin{aligned}
\dot{x}_{i1} &= -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha(ax_{i1}^3 + bx_{i1}),\\
\dot{x}_{i2} &= kx_{i1} - kx_{i2} + kx_{i3},\\
\dot{x}_{i3} &= k\beta x_{i2} + k\gamma x_{i4},\\
\dot{x}_{i4} &= -0.1x_{i2},
\end{aligned} \tag{7}$$

where $\alpha, \beta, \gamma, a, b$, and k are parameters. The vector x_i in Eq. (1) is $(x_{i1}, x_{i2}, x_{i3}, x_{i4})^{\mathrm{T}}$ here. Linearizing system (7) at its zero equilibrium yields

$$\dot{x}_i = Fx_i, \quad F = \begin{pmatrix} -k\alpha - k\alpha b & k\alpha & 0 & 0\\ k & -k & k & 0\\ 0 & k\beta & 0 & k\gamma\\ 0 & -0.1 & 0 & 0 \end{pmatrix}. \tag{8}$$

Take parameters $k = 3$, $\alpha = -0.1$, $\beta = -0.2$, $\gamma = 0.2$, $a = 1$, and $b = -25$. Then F in Eq. (8) is stable, i.e., the node system (7) is locally stable about zero. One can easily choose a parameter $\beta_0 = -0.2$ such that all roots of $\det(sI - F) - \beta_0$ are real. Following Theorem 2 in Ref. [15], one can find $\alpha_1 = 0.000345356$, $\alpha_2 = 10$, $\alpha_3 = 15.005$, $\alpha_4 = 25.005$, and

$$H = \begin{pmatrix} 34.3780902138340 & 18.3313854199063 & 4.9588330372823 & 6.3542213067607\\ 11.6296319128101 & 6.0084351727926 & 2.2119654981287 & 2.2045002656647\\ 155.1293157037622 & 82.1429320159836 & 23.2147692977359 & 28.7654072889066\\ -344.1645925125970 & -182.2909833003194 & -51.3251468678921 & -63.6012946843625 \end{pmatrix}.$$

Then, by simple computation, one can show that $F + \alpha H$ has four disconnected stable sub-regions: $S_1 = [-0.00012, 0]$, $S_2 = [-10.0072, -10.000008]$, $S_3 = [-19.99999, -19.996]$, and $S_4 = [-49.9989, -49.9985]$, so the complete synchronized region is $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Further, consider the two graphs G_1 and G_2 shown in Figs. 1 and 2, both having the same number of nodes $N = 6$.

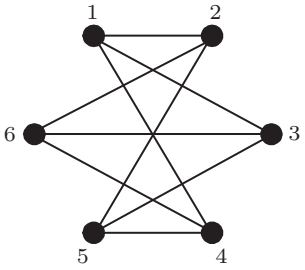


Fig. 1. Graph G_1.

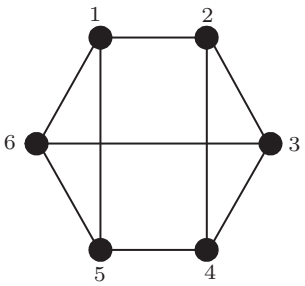


Fig. 2. Graph G_2.

The Laplacian matrices of G_1 and G_2 are

$$G_1 = \begin{pmatrix} -3 & 1 & 1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 & 1 & 1 \\ 1 & 0 & -3 & 0 & 1 & 1 \\ 1 & 0 & 0 & -3 & 1 & 1 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 1 & 1 & 1 & 0 & -3 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} -3 & 1 & 0 & 0 & 1 & 1 \\ 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & 1 & -3 & 1 & 0 \\ 1 & 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 0 & 1 & -3 \end{pmatrix},$$

respectively. It can be easily verified that the eigenvalues of G_1 are $0, -3, -3, -3, -3$, and -6; the eigenvalues of G_2 are $0, -2, -3, -3, -5$, and -5. Obviously, $\lambda_2(G_1) = 3 > \lambda_2(G_2) = 2$, and $\lambda_2(G_1)/\lambda_6(G_1) = 0.5 > \lambda_2(G_2)/\lambda_6(G_2) = 0.4$. Therefore, according to the general understanding in the existing literature, the synchronizability of network G_1 is better than that of network G_2 based on the eigenratio comparison $\lambda_2(G_1)/\lambda_6(G_1) > \lambda_2(G_2)/\lambda_6(G_2)$.[29]

However, for the above node system (7) and the inner-linking matrix H, network Graph G_2 synchronizes with the coupling strength $c = 5.0015$ as shown by Fig. 3. At this time, every different nonzero eigenvalue of G_2 each falls into one synchronized sub-region S_i. But network G_1 does not synchronize for any coupling strength $c > 0.00003$ as shown by Fig. 4. This synchronization behavior is very different from the general understanding that the synchronizability of G_1 is better than G_2.

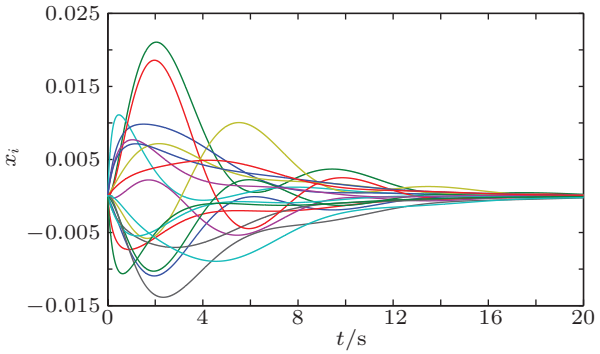


Fig. 3. (colour online) Synchronization of network (1) on Graph G_2 with $c = 5.0015$.

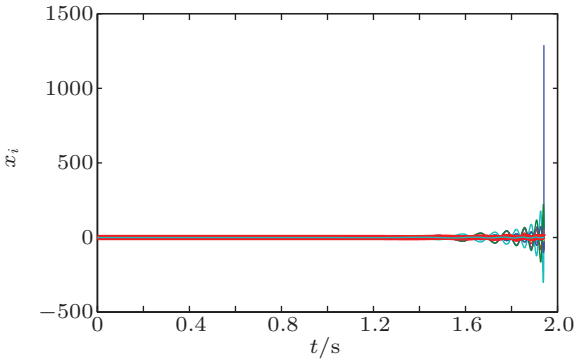


Fig. 4. (colour online) Non-synchronization of network (1) on Graph G_1 with $c = 5.0015$.

In this example, the node equation is a fourth-order chaotic Chua's circuit which may be too complex. However, one can actually construct a simple linear node equation to show the same argument from another point of view.

Example 2 According to the method designed in Ref. [15], take a stable polynomial

$$p(\alpha) = (\alpha + 0.002)(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5)(\alpha + 6)$$

and a scalar $\beta = 1$ such that $p(s) - \beta^6$ has six real roots, $-\beta_1, -\beta_2, \ldots, -\beta_6$. Then, F and H are ob-

tained and chosen respectively as follows:

$$F = \begin{pmatrix} -\beta & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & -\beta & \beta_2 & 0 & 0 & 0 \\ 0 & 0 & -\beta & \beta_3 & 0 & 0 \\ 0 & 0 & 0 & -\beta & \beta_4 & 0 \\ 0 & 0 & 0 & 0 & -\beta & \beta_5 \\ -\beta_6 & 0 & 0 & 0 & 0 & -\beta \end{pmatrix},$$

and

$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} & \det(sI - F - \alpha H) \\ = \; & (s+\beta)^6 + (\alpha+\beta_1)(\alpha+\beta_2) \\ & \times (\alpha+\beta_3)(\alpha+\beta_4)(\alpha+\beta_5)(\alpha+\beta_6). \end{aligned}$$

Obviously, the constant term in $\det(sI - F - \alpha H)$ is $(\alpha + 0.002)(\alpha + 2)(\alpha + 3)(\alpha + 4)(\alpha + 5)(\alpha + 6)$, which is larger than zero in four regions: $(-\infty, -6)$, $(-5, -4)$, $(-3, -2)$, and $(-0.002, +\infty)$. By simple computation, one finds that $F + \alpha H$ has six disconnected stable sub-regions with respect to α: $[-6.022, -6.0001]$, $[-4.999, -4.87]$, $[-4.21, -4.001]$, $[-2.999, -2.83]$, $[-2.08, -2.001]$, and $[-0.00199, 0]$. In the two regions $(-5, -4)$ and $(-3, -2)$, $F + \alpha H$ has two disconnected stable sub-regions.

Now, consider a weighted graph with its Laplacian matrix being

$$L = \begin{pmatrix} -1.6722 & -1.6707 & 0.3836 & 0.3836 & 0.8586 & 0.8586 & 0.8586 \\ -1.6707 & -1.6722 & 0.3836 & 0.3836 & 0.8586 & 0.8586 & 0.8586 \\ 0.3836 & 0.3836 & -2.6964 & -0.6464 & 0.8586 & 0.8586 & 0.8586 \\ 0.3836 & 0.3836 & -0.6464 & -2.6964 & 0.8586 & 0.8586 & 0.8586 \\ 0.8586 & 0.8586 & 0.8586 & 0.8586 & -3.4323 & -0.4838 & 0.4819 \\ 0.8586 & 0.8586 & 0.8586 & 0.8586 & -0.4838 & -3.4323 & 0.4819 \\ 0.8586 & 0.8586 & 0.8586 & 0.8586 & 0.4819 & 0.4819 & -4.3981 \end{pmatrix}.$$

The eigenvalues of L are 0, -0.0015, -2.05, -2.9485, -4.11, -4.88, and -6.01. Obviously, every nonzero eigenvalue each falls into one stable sub-region of $F + \alpha H$. Therefore, the network (1) with the coupling strength $c = 1$ and with node equation $\dot{x}_i = Fx_i$ and inner-linking matrix H, specified above, achieves synchronization as shown by Fig. 5.

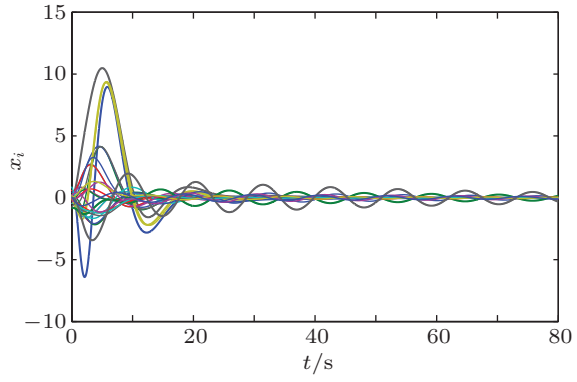


Fig. 5. (colour online) Synchronization of network (1) with linear dynamics.

In this example, the eigenratio λ_2/λ_N does not play the role as the synchronizability index, simply because λ_2 and λ_N belong to different synchronized sub-regions respectively.

Remark Clearly, in the above two examples, the eigenratio λ_2/λ_N does not represent anything regarding the network synchronization behavior. In fact, only if all the nonzero eigenvalues of the network Laplacian matrix fall into the same synchronized sub-region, can the eigenratio λ_2/λ_N be used as a synchronizability measure. Otherwise, the synchronizability index needs to be studied on a case-by-case basis and very carefully.

4. Conclusion

Two examples have been given to show that the eigenratio λ_2/λ_N cannot represent the synchronizability of those complex networks whose synchronized regions are disconnected. This demonstrates the in-

trinsic complexity of network synchronization behaviors. When the network node dynamics and the inner-linking function are given, it depends on the characteristics of the network's synchronized regions to determine whether or not the eigenratio λ_2/λ_N represents the synchronizability. Nevertheless, even if the synchronized region is disconnected, if all the eigenvalues of the network Laplacian matrix fall into the same synchronized sub-region, then the eigenratio λ_2/λ_N can still be used as a valid synchronizability index. Therefore, the network synchronizability should be studied by combining the Laplacian eigenvalues with the network synchronized regions, not just by the eigenratio λ_2/λ_N.

Considering complex synchronized regions in weighted networks[13] and consensus regions in multi-agent systems,[30] there are many important and interesting problems remaining for further studies in this direction.

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