A single adaptive controller with one variable for synchronization of fractional-order chaotic systems*

Zhang Ruo-Xun(张若洵)[a)b)] and Yang Shi-Ping(杨世平)[a)†]

[a)] *College of Physics Science and Information Engineering, Hebei Normal University, Shijiazhuang* 050016, *China*
[b)] *College of Elementary Education, Xingtai University, Xingtai* 054001, *China*

(Received 4 December 2011; revised manuscript received 28 January 2012)

In this paper we investigate the synchronization of a class of three-dimensional fractional-order chaotic systems. Based on the Lyapunov stability theory and adaptive control technique, a single adaptive–feedback controller is developed to synchronize a class of fractional-order chaotic systems. The presented controller which only contains a single driving variable is simple both in design and in implementation. Numerical simulation and circuit experimental results for fractional-order chaotic system are provided to illustrate the effectiveness of the proposed scheme.

Keywords: adaptive control, fractional-order chaotic system, a single controller

PACS: 05.45.Xt **DOI:** 10.1088/1674-1056/21/8/080505

1. Introduction

Even though fractional calculus as a mathematical topic has more than a 300 year history, its application to physics and engineering has attracted lots of attention only in recent years. It has been found that many systems in interdisciplinary fields can be described by fractional differential equations, such as viscoelastic systems,[1] dielectric polarization,[2] electrode–electrolyte polarization,[3] some finance systems, and electromagnetic waves.[4] Moreover, applications of fractional calculus have been reported in many areas such as signal processing,[5] image processing,[6] automatic control,[7] and robotics.[8,9] These examples and many other similar samples perfectly clarify the importance of consideration and analysis of dynamical systems with fractional-order models.

Recently, studying fractional-order chaotic systems has become an active research field. Synchronization of fractional-order chaotic systems started to attract increasing attention due to its potential applications in secure communication and control processing. Some approaches have been proposed to achieve chaos synchronization in fractional-order chaotic systems, such as PC control, a nonlinear state observer method, adaptive control, unidirectional linear error feedback coupling, sliding mode control, a scalar transmitted signal method,[10−22] etc. However, in the formulation of the chaos synchronization problem, the proposed controllers in previous work in most cases are too complex both in design and in implementation.

In this paper, we design a single adaptive controller with one driving variable to synchronize a class of three-dimensional fractional-order chaotic systems. The proposed scheme, based on Lyapunov stability theory and adaptive control technique, is simple, and globally and theoretically rigorous. To show the wider applicability of our method, we give illustrations by using three different fractional-order chaotic systems with numerical simulations to verify the effectiveness of the proposed synchronization scheme. The rest of the paper is organized as follows. In Section 2 the main results are given. In Section 3, three groups of examples are used to verify the effectiveness of the proposed scheme. The paper is concluded in Section 4.

2. Preliminaries and definitions

2.1. Definitions of fractional derivatives

Fractional calculus plays an important role in modern science. In this paper, we use both Reimann–

*Project supported by the Natural Science Foundation of Hebei Province, China (Grant No. A2010000343).
†Corresponding author. E-mail: yangship@mail.hebtu.edu.cn
© 2012 Chinese Physical Society and IOP Publishing Ltd

Liouville and Caputo fractional operators as our main tools. Caputo fractional derivative is defined as follows:[23]

$$D_t^\alpha x(t) = J^{m-\alpha} x^{(m)}(t) \quad \text{with} \quad \alpha \in (0,1), \tag{1}$$

where $m = [\alpha]$, i.e., m is the first integer which is not less than α, $x^{(m)}$ is the m-order derivative in the usual sense, and J^β $(\beta > 0)$ is the β-order Reimann–Liouville integral operator with expression:

$$J^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau) \mathrm{d}\tau, \tag{2}$$

where Γ denotes Gamma function.

2.2. Some properties and Lemma of fractional derivatives

One general property of the fractional-order derivative will be used, which is recalled below:

Property 1 Caputo fractional derivative operator is a linear operator

$$D_t^\alpha(\lambda x(t) + \mu y(t)) = \lambda D_t^\alpha x(t) + \mu D_t^\alpha y(t), \quad (\lambda, \mu \text{ are real constants}), \tag{3}$$

especially, $D_t^\alpha x(t) = D_t^\alpha(x(t) + 0) = D_t^\alpha x(t) + D_t^\alpha 0$, then, we have $D_t^\alpha 0 = 0$.

Lemma 1 (See Ref. [24]) For the system

$$D_t^\alpha x = f(t, x), \tag{4}$$

where $\alpha \in (0,1)$, we assume that $f(t,x)$ satisfies the Lipschitz condition with Lipschitz constant $l > 0$, i.e.

$$|f(t, x_2) - f(t, x_1)| \le l|x_2 - x_1|, \tag{5}$$

where l is a positive constant.

3. Main results

Let a three-dimensional fractional-order chaotic error system be

$$\begin{aligned} D_t^\alpha x &= f_1(x, z), \\ D_t^\alpha z &= f_2(x, z), \end{aligned} \tag{6}$$

where $D_t^\alpha = d^\alpha / \mathrm{d}t^\alpha$, $0 < \alpha < 1$, $x \in R^2$, $z \in R^1$ are the state error vectors, f_1 and f_2 are continuous differential nonlinear functions with $f_1(0,0) = f_2(0,0) = 0$.

To describe the new design and analysis, the following assumption is needed.

Assumption 1 The function $f_1(x,z)$ is smooth in a neighbourhood of $z = 0$, and the subsystem $D_t^\alpha x = f_1(x, 0)$ is asymptotically stable about the origin $x = 0$ for all x.

Remark 1 System (6) is very general, which contains almost all fractional-order chaotic error systems.

Remark 2 The vector function $f_1(x,z)$ is smooth in a neighbourhood of $z = 0$, i.e., there are two positive constants λ_0, λ_1 such that $||f_1(x,z) - f_1(x,0)|| \le \lambda_0|z|$, and then $||x||_1 \le \lambda_1|z|$. Thus, for chaos error system (6), there exist two positive numbers p_1, p_2 and a large enough positive constant λ_2 such that

$$||f_2(x,z)|| \le p_1||x||_1 + p_2|z| \le p_1\lambda_1|z| + p_2|z| \le \lambda_2|z|.$$

In order to stabilize error system (6) to its equilibrium point $x = 0$, $z = 0$, we add the following adaptive controller u into error system (6), so that controlled error system (6) is given by

$$\begin{aligned} D_t^\alpha x &= f_1(x, z), \\ D_t^\alpha z &= f_2(x, z) + u. \end{aligned} \tag{7}$$

The adaptive controller u is modified into

$$u = -ks, \tag{8}$$

where s is defined as

$$D_t^{1-\beta} s = z, \tag{9}$$

and k is adapted according to the following update law:

$$\dot{k} = \theta s^2, \tag{10}$$

where θ is an arbitrary positive real number.

Remark 3 According to the property of Caputo's fractional derivative operator, we have $D_t^\beta z = \dot{s}$.

Remark 4 For Eq. (9), according to Property 1 and Lemma 1, there exists a positive number λ_3, such that $||z|| \le \lambda_3||s||$, and if $s = 0$, then $z = 0$.

Theorem 1 Starting from any initial value, controlled error system (7) is asymptotically stable under the controller (8) and updating law (10).

Proof Consider the following Lyapunov function candidate

$$V(t) = \frac{1}{2}s^2 + \frac{1}{2\theta}(k - k^*)^2,$$

where $k^* > \lambda_2\lambda_3$. Taking the derivative of $V(t)$ with respect to time, one has

$$\begin{aligned}\dot{V}(t) &= s\dot{s} + (k-k^*)\dot{k}/\theta = sD_t^\alpha z + (k-k^*)\dot{k}/\theta \\ &= s(f_2(x,z) - ks) + (k-k^*)s^2 \\ &\le \lambda_2|s||z| - ks^2 + (k-k^*)s^2 \\ &\le \lambda_2\lambda_3 s^2 - k^* s^2 \\ &= -(k^* - \lambda_2\lambda_3)s^2 \le 0.\end{aligned}$$

Using Barbalat's lemma, it can be concluded that s, $\dot{s} \in L_\infty$. In a finite time, s approaches zero. Since s is driven to zero in a finite time, then the error z is driven to zero in a finite time. Moreover, since z is driven to zero in a finite time, according to Assumption 1, then after such a finite time, x asymptotically converges to zero as t tends to infinity. Therefore, the errors x, z asymptotically converge to zero as t tends to infinity.

4. Applications

In this section of the paper, we present three illustrative examples to demonstrate the effectiveness of the proposed synchronization scheme. MATLAB software or circuit experiment is used in this simulation.

Example 1 Consider the fractional-order Chua's system[25]

$$\begin{aligned}D_t^\alpha x_1 &= a\left(x_2 + \frac{x_1 - 2x_1^3}{7}\right), \\ D_t^\alpha x_2 &= x_1 - x_2 + x_3, \\ D_t^\alpha x_3 &= -\frac{100}{7}x_2. \end{aligned} \tag{11}$$

When $\alpha = 0.95$ and $a = 12$, the fractional-order Chua's system is chaotic. Let system (11) be the master system. Then, the slave system with variable y is given as

$$\begin{aligned}D_t^\alpha y_1 &= a\left(y_2 + \frac{y_1 - 2y_1^3}{7}\right), \\ D_t^\alpha y_2 &= y_1 - y_2 + y_3, \\ D_t^\alpha y_3 &= -\frac{100}{7}y_2, \end{aligned} \tag{12}$$

and the error system ($e = y - x$) is as follows:

$$\begin{aligned}D_t^\alpha e_1 &= a\left(e_2 + \frac{1 - 2(x_1^2 + x_1y_1 + y_1^2)}{7}e_1\right), \\ D_t^\alpha e_2 &= e_1 - e_2 + e_3, \\ D_t^\alpha e_3 &= -\frac{100}{7}e_2. \end{aligned} \tag{13}$$

It is easy to see that if $e_1 = 0$, the following two-dimensional subsystem of the system (13):

$$\begin{aligned}D_t^\alpha e_2 &= -e_2 + e_3, \\ D_t^\alpha e_3 &= -\frac{100}{7}e_2,\end{aligned}$$

is uniformly exponentially stable about the origin $e_2 = 0$, $e_3 = 0$ for all e_2, e_3. Therefore, if we let $x = (e_2, e_3)^{\mathrm{T}}$, $z = e_1$, system (13) satisfies Assumption 1, thus, the controlled error system (13) is

$$\begin{aligned}D_t^\alpha e_1 &= a\left(e_2 + \frac{1 - 2(x_1^2 + x_1y_1 + y_1^2)}{7}e_1\right) - ks, \\ D_t^\alpha e_2 &= e_1 - e_2 + e_3, \\ D_t^\alpha e_3 &= -\frac{100}{7}e_2, \end{aligned} \tag{14}$$

i.e. the controlled slave system (12) is

$$\begin{aligned}D_t^\alpha y_1 &= a\left(y_2 + \frac{y_1 - 2y_1^3}{7}\right) - ks, \\ D_t^\alpha y_2 &= y_1 - y_2 + y_3, \\ D_t^\alpha y_3 &= -\frac{100}{7}y_2, \end{aligned} \tag{15}$$

where $D_t^{1-\alpha}s = e_1$, and $\dot{k} = \theta s^2$ ($\theta > 0$).

According to Theorem 1, the controlled error system (14) can be stabilized, i.e., the master system (11) can synchronize the slave system (15) with the single controller.

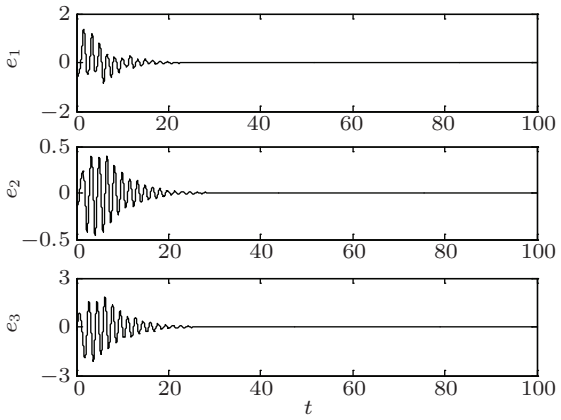


Fig. 1. Synchronization errors e_1, e_2, e_3 of the fractional-order Chua chaotic system.

In the simulation, we choose the initial conditions

$$x(0) = (0.2, -0.2, 0.3)^{\mathrm{T}},$$

$$y(0) = (-0.1, 0.1, -1)^{\mathrm{T}},$$

and $\alpha = 0.95$, $c = 1$, $\theta = 1$. The numerical results are illustrated in Figs. 1 and 2. Figure 1 displays that the error system $e = y - x$ is stabilized to the origin asymptotically as $t \to \infty$, that is to say, master system (11) and slave system (12) are synchronized by the above controller in Fig. 2.

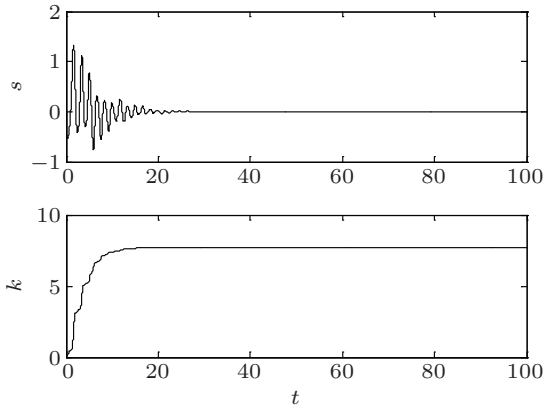


Fig. 2. Variables s and feedback gain k versus time t.

Example 2 The fractional-order unified chaotic system is[26]

$$\begin{aligned} D_t^\alpha x_1 &= (25a + 10)(x_2 - x_1), \\ D_t^\alpha x_2 &= (28 - 35a)x_1 + (29a - 1)x_2 - x_1 x_3, \\ D_t^\alpha x_3 &= x_1 x_2 - (a + 8)x_3/3, \end{aligned} \tag{16}$$

where $a \in [0, 1]$. When $a = 0.8$, system (16) becomes the fractional-order Lü system; when $a = 1$, it is the fractional-order Chen system. Let system (16) be the master system, then, the slave system with variable y is given as

$$\begin{aligned} D_t^\alpha y_1 &= (25a + 10)(y_2 - y_1), \\ D_t^\alpha y_2 &= (28 - 35a)y_1 \\ &\quad + (29a - 1)y_2 - y_1 y_3, \\ D_t^\alpha y_3 &= y_1 y_2 - (a + 8)y_3/3, \end{aligned} \tag{17}$$

and the error system ($e = y - x$) is as follows:

$$\begin{aligned} D_t^\alpha e_1 &= (25a + 10)(e_2 - e_1), \\ D_t^\alpha e_2 &= (28 - 35a)e_1 \\ &\quad + (29a - 1)e_2 - x_1 e_3 - y_3 e_1, \\ D_t^\alpha e_3 &= x_1 e_2 + y_2 e_1 - (a + 8)e_3/3. \end{aligned} \tag{18}$$

It is easy to see that if $e_2 = 0$, the following two-dimensional subsystem of system (18):

$$D_t^\alpha e_1 = -(25a + 10)e_1,$$

$$D_t^\alpha e_3 = y_2 e_1 - (a + 8)e_3/3, \tag{19}$$

is asymptotically stable about the origin $e_1 = 0$, $e_3 = 0$ for all e_1, e_3. Therefore, if we let $x = (e_1, e_3)^{\mathrm{T}}$, $z = e_2$, system (18) satisfies Assumption 1, thus, the controlled error system (18) is

$$\begin{aligned} D_t^\alpha e_1 &= (25a + 10)(e_2 - e_1), \\ D_t^\alpha e_2 &= (28 - 35a)e_1 + (29a - 1)e_2 \\ &\quad - x_1 e_3 - y_3 e_1 - ks, \\ D_t^\alpha e_3 &= x_1 e_2 + y_2 e_1 - (a + 8)e_3/3, \end{aligned} \tag{20}$$

i.e. controlled slave system (17) is

$$\begin{aligned} D_t^\alpha y_1 &= (25a + 10)(y_2 - y_1), \\ D_t^\alpha y_2 &= (28 - 35a)y_1 \\ &\quad + (29a - 1)y_2 - y_1 y_3 - ks, \\ D_t^\alpha y_3 &= y_1 y_2 - (a + 8)y_3/3, \end{aligned} \tag{21}$$

where $D_t^{1-\alpha} s = e_2$, and $\dot{k} = \theta s^2$ ($\theta > 0$).

According to Theorem 1, controlled error system (20) can be stabilized, i.e., master system (16) can synchronize slave system (21) with a single controller.

According to the circuit of $1/s^\alpha$,[27,28] using the circuitry theory and the circuitry design method in Refs. [27] and [28], we design a circuit to realize the synchronization of two identical fractional-order unified chaotic systems with $a = 1$ and $\alpha = 0.9$ in Fig. 2, where unit (I) is the circuitry of the drive system, unit (II) is the circuitry of the response system, and unit (III) is the circuitry of the controller $u = ks$, where $D_t^{0.1} s = e_2$, and $\dot{k} = \theta s^2$. AD633 is a multiplier with an output coefficient of 0.1, TL084CN is a high speed J-FET input quad operational amplifier, $R_{01} = 63.10$ MΩ, $R_{02} = 1.598$ MΩ, $R_{03} = 15.81$ kΩ, $C_{01} = 0.4410$ μF, $C_{02} = 0.4870$ μF, $C_{03} = 0.2937$ μF, $R_{04} = 0.5849$ MΩ, $R_{05} = 2.382$ MΩ, $R_{03} = 1.432$ MΩ, $C_{04} = 631$ pF, $C_{05} = 2.517$ μF, $C_{06} = 0.02509$ μF, and the values of other resistances and capacitances are shown in Fig. 3. When we set delay-time $t = 5$ s, the circuit experiment results are depicted in Fig. 4 from which one can find that the two fractional systems (21) and (16) can achieve synchronisation quickly by the adaptive controller, as switch S is turned on.

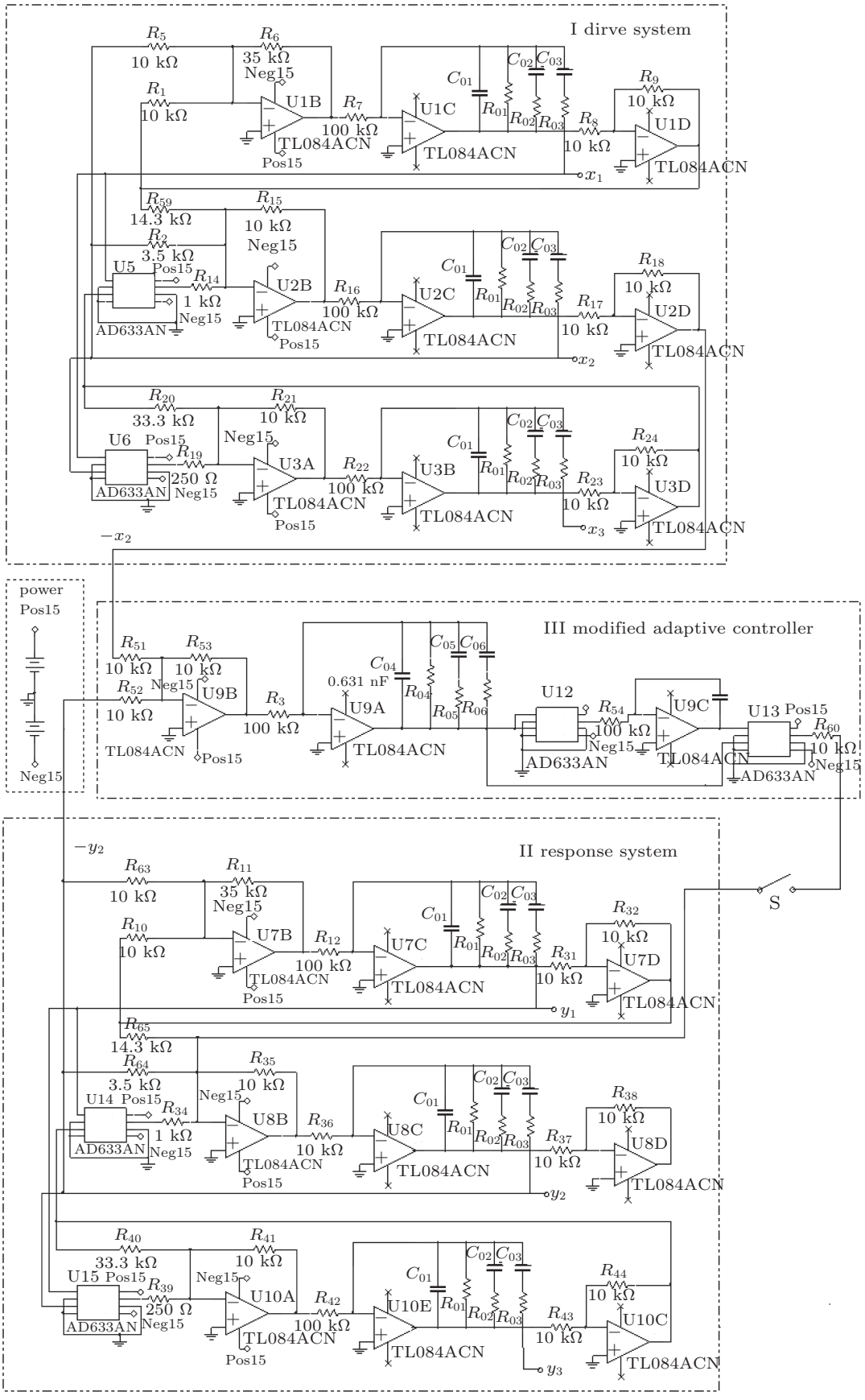


Fig. 3. Synchronisation circuit between fractional-order Chen chaotic systems (16) and (21): (I) circuitry of drive system, (II) circuitry of response system, (III) circuitry of controller.

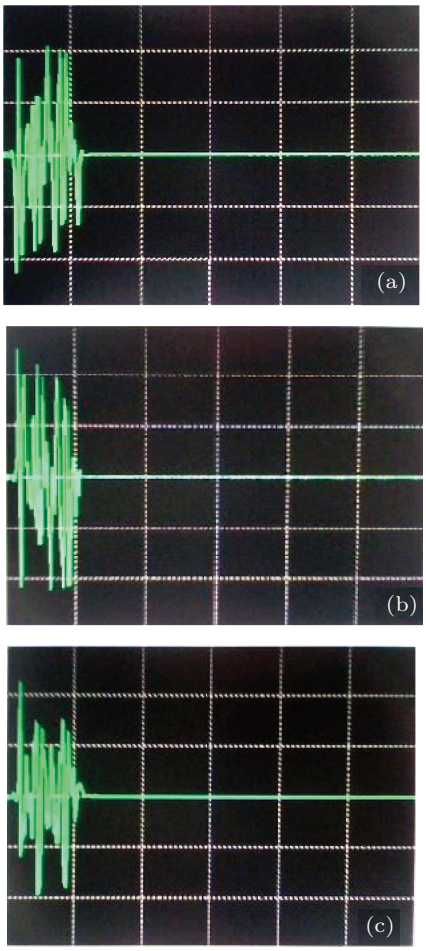


Fig. 4. (colour online) Observed synchronization error evolutions with time on the oscilloscope (a) e_1, (b) e_2, (c) e_3.

Example 3 The fractional-order financial system is[29]

$$\begin{aligned} D_t^\alpha x_1 &= x_3 + (x_2 - 1)x_1, \\ D_t^\alpha x_2 &= 1 - 0.1x_2 - x_1^2, \\ D_t^\alpha x_3 &= -x_1 - x_3. \end{aligned} \tag{22}$$

When $\alpha = 0.95$, the fractional-order financial system is chaotic. Let system (22) be the master system, then, the slave system with variable y is given as

$$\begin{aligned} D_t^\alpha y_1 &= y_3 + (y_2 - 1)y_1, \\ D_t^\alpha y_2 &= 1 - 0.1y_2 - y_1^2, \\ D_t^\alpha y_3 &= -y_1 - y_3, \end{aligned} \tag{23}$$

and the error system ($e = y - x$) is as follows:

$$\begin{aligned} D_t^\alpha e_1 &= e_3 - e_1 + x_1 e_2 + y_2 e_1, \\ D_t^\alpha e_2 &= -0.1e_2 - (x_1 + y_1)e_1, \\ D_t^\alpha e_3 &= -e_1 - e_3. \end{aligned} \tag{24}$$

It is easy to see that if $e_1 = 0$, the following two-dimensional subsystem of system (24):

$$\begin{aligned} D_t^\alpha e_2 &= -0.1e_2, \\ D_t^\alpha e_3 &= -e_3, \end{aligned} \tag{25}$$

is asymptotically stable about the origin $e_2 = 0$, $e_3 = 0$ for all e_2, e_3. Therefore, if we let $x = (e_2, e_3)^{\mathrm{T}}$, $z = e_1$, system (24) satisfies Assumption 1, thus, the controlled error system (24) is

$$\begin{aligned} D_t^\alpha e_1 &= e_3 - e_1 + x_1 e_2 + y_2 e_1 - ks, \\ D_t^\alpha e_2 &= -0.1e_2 - (x_1 + y_1)e_1, \\ D_t^\alpha e_3 &= -e_1 - e_3, \end{aligned} \tag{26}$$

i.e. the controlled slave system (23) is

$$\begin{aligned} D_t^\alpha y_1 &= y_3 + (y_2 - 1)y_1 - ks, \\ D_t^\alpha y_2 &= 1 - 0.1y_2 - y_1^2, \\ D_t^\alpha y_3 &= -y_1 - y_3, \end{aligned} \tag{27}$$

where $D_t^{1-\alpha} s = e_1$, and $\dot{k} = \theta s^2$ ($\theta > 0$).

According to Theorem 1, controlled error system (26) can be stabilized, i.e., master system (22) can synchronize slave system (27) with a single controller.

In the simulation, we choose the initial conditions $x(0) = (1, 2, 1)^{\mathrm{T}}$, $y(0) = (-1, -1, 2)^{\mathrm{T}}$, and $\alpha = 0.96$, $\theta = 3$. The numerical results are illustrated in Figs. 5 and 6. Figure 5 displays that the error system $e = y - x$ is stabilized to the origin asymptotically as $t \to \infty$, that is to say, master system (22) and slave system (27) are synchronized by the above controller in Fig. 6.

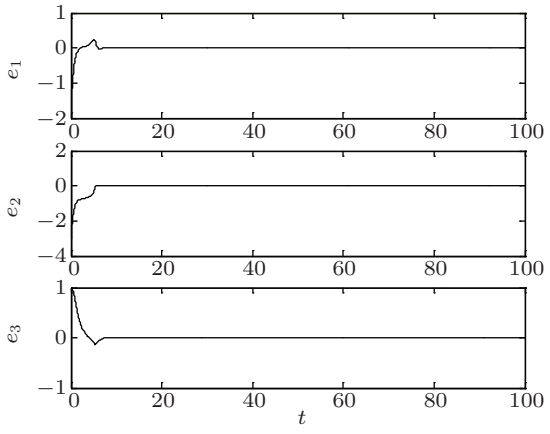


Fig. 5. Synchronization errors e_1, e_2, e_3 of the fractional-order financial system.

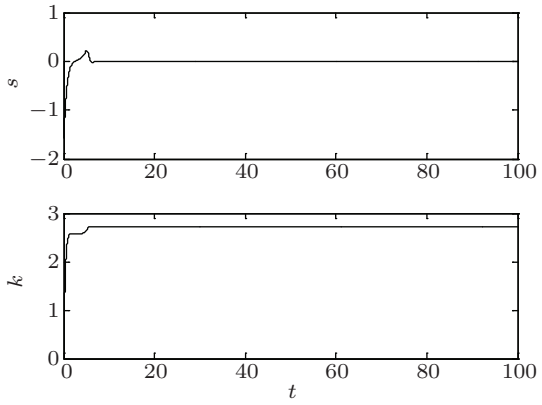


Fig. 6. Variables s and feedback gain k versus time t.

We have implemented and verified our method for other fractional-order chaotic systems, namely the Liu chaotic systems,[30] Genesio Tesi system,[31] Qi system,[32,33] and Arneodo system.[34] Therefore, our proposed control approach has wider applicability, it is very simple and could be easily realized experimentally compared with previous techniques in which complex control functions are used.

5. Conclusion

In this paper, we investigated the synchronization of three-dimensional fractional-order chaotic systems based on Lyapunov stability theory, and proposed a single state adaptive–feedback control law to realize a chaos synchronization goal. Our proposed controller which only contains one variable is very simple and could be easily realized experimentally. Furthermore, the presented method is universal, that is, for a class of fractional-order chaotic systems, the controllers obtained are similar in form. We demonstrated the performance of our technique using several examples of three-dimensional fractional-order chaotic systems with different structures. Numerical simulations and circuit experiments are provided in all the cases to illustrate the effectiveness and validity of this method.

References

[1] Bagley R L and Calico R A 1991 *J. Guid. Control Dyn.* **14** 304
[2] Sun H H, Abdelwahad A A and Onaral B 1984 *IEEE Trans. Autom. Control* **29** 441
[3] Ichise M, Nagayanagi Y and Kojima T 1971 *J. Electroanal. Chem. Interfacial Electrochem* **33** 253
[4] Heaviside O 1971 *Electromagnetic Theory* (New York: Chelsea)
[5] Mandelbrot B and Van Ness J W 1968 *SIAM Rev.* **10** 422
[6] Oustaloup A and La Derivation 1995 *Non Entiere: Theorie, Synthase et Applications* Editions Hermes, Paris, France
[7] Podlubny I 1999 *IEEE Trans. Automat. Control* **44** 208
[8] Linares H, Baillot Ch, Oustaloup A and Ceyral Ch *Generation of a Fractal Ground: Application in Robotics*, in: *International Congress in IEEE-SMC CESA'96 IMACS Multiconf.*, Lille, July 1996
[9] Duarte F B M and Macado J A T 2002 *Nonlinear Dyn.* **29** 315
[10] Li C P and Deng W H 2006 *Int. J. Mod. Phys.* B **20** 791
[11] Lu J G 2006 *Physica* A **359** 107
[12] Wu X J, Lu H T and Shen S L 2009 *Phys. Lett.* A **373** 2329
[13] Zhang R X and Yang S P 2010 *Chin. Phys.* B **19** 020510
[14] Odibat Z M 2010 *Nonlinear Dyn.* **60** 479
[15] Wang X Y, Zhang Y L, Lin D and Zhang N 2011 *Chin. Phys.* B **20** 030506
[16] Tavazoei M S and Haeri M 2008 *Physica* A **387** 57
[17] Zhang R X and Yang S P 2011 *Nonlinear Dyn.* **66** 831
[18] Lu J G 2006 *Chaos, Solitions and Fractals* **27** 519
[19] Wu C J, Zhang Y B and Yang N N 2011 *Chin. Phys.* B **20** 060505
[20] Wang M J, Wang X Y and Niu Y J 2011 *Chin. Phys.* B **20** 010508
[21] Qi D L,Yang J and Zhang J L 2010 *Chin. Phys.* B **19** 100506
[22] Zhang R X and Yang S P 2011 *Chin. Phys.* B **20** 090512
[23] Podlubny I 1999 *Fractional Differential Equations* (New York: Academic Press)
[24] Li Y, Chen Y Q and Podlubny I 2010 *Comput. Math. Appl.* **59** 1810
[25] Hartley T T, Lorenzo C F and Qammer H K 1995 *IEEE Trans. CAS* I **42** 485
[26] Wang X J, Li J and Chen G R 2008 *J. Franklin Institute* **345** 392
[27] Zhang R X and Yang S P 2009 *Acta Phys. Sin.* **58** 2957 (in Chinese)
[28] Zhang R X and Yang S P 2009 *Chin. Phys.* B **18** 3295
[29] Chen W C 2008 *Chaos, Solitons and Fractals* **36** 1305
[30] Xu Z and Liu C X 2008 *Chin. Phys.* B **17** 4033
[31] Lu J G 2005 *Chin. Phys.* **14** 1517
[32] Qi G Y, Chen G R, Du S Z and Chen Z Q 2005 *Physica* A **352** 295
[33] Song L, Yang J Y and Xu S Y 2010 *Nonlinear Analysis: Theory Methods & Applications* **72** 2326
[34] Lu J G 2005 *Chaos, Solitons and Fractals* **26** 1125