Thermal properties of Lense–Thirring spacetime in tetrad theory of gravity

Gamal G. L. Nashed[a)b)c)†]

[a)] *Mathematics Department, Faculty of Science, King Faisal University, P. O. Box* 380, *Al-Ahsaa* 31982, *the Kingdom of Saudi Arabia*
[b)] *Mathematics Department, Faculty of Science, Ain Shams University, Cairo* 11566, *Egypt*
[c)] *Center for Theoretical Physics, British University of Egypt, Sherouk City* 11837, *P. O. Box* 43, *Egypt*

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Using the divergence term appearing in the Lagrangian of the teleparallel equivalent of general relativity (TEGR), we calculate the thermodynamic quantities of four tetrads' spacetime reproducing Lense–Thirring (LT) metric. We also investigate the first law of thermodynamics and the quantum statistical relation.

Keywords: Euclidean continuation method, thermodynamic quantities, first law of thermodynamic, quantum statistical relation

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1. Introduction

Hawking radiation, which is the most striking quantum effect arising from the quantum field in a curved spacetime background, attracts many efforts to investigate it. There are several kinds of derivations to obtain it. Hawking's original derivation is to calculate the Bogoliubov coefficients between the in and out states in a black hole background.[1−3] The subsequent Damour–Ruffini method is to calculate the particles' emitting rate by analytically extending the outgoing wave from the outside of horizon to the inside.[4] Parikh and Wilczek calculated WKB amplitude by considering the pair creation of particles and antiparticles near the horizon and supposing that particles tunnel across the classical forbidden path where the barrier is created by the outgoing particles themselves.[5−7] Another approach to calculate Hawking radiation is due to the anomaly through calculating the energy–momentum tensor in the black hole backgrounds.[8−12] The anomaly in field theory occurs if the symmetry of the action or the corresponding conservation law is valid in the classical theory but will violate in the quantized case. Anomalies can include a conformal anomaly (or trace anomaly), an anomaly in gauge symmetries, and a gravitation anomaly. The pioneer work of Christensen and Fulling[8] told us that the strength of Hawking radiation flux is determined by the trace of energy–momentum tensor. Thinking of the two-dimensional (2D) massless field, either the Hawking effect or conformal anomaly can be deduced from the other. In Robinson and Wilczek's work,[10] Hawking radiation can be understood as compensating flux to cancel the gravitation anomaly at the horizon.

The existence of symmetries in a physical system means that there exists one or more transformations that leave the physical system invariant. Since physical systems can be completely described by their Lagrangians/Hamiltonians, it is therefore natural to expect the symmetry transformations to be canonical ones. The number of symmetry transformations that can be generated from the system depends on the number of conserved quantities of the associated system. It has been well known that conserved quantities of a physical system relate closely to the generator of the associated symmetry transformations.[13−15]

Energy is a subtle issue in general relativity. Indeed, in accordance with the equivalence principle, the gravitational contribution to energy cannot be localized. Nevertheless, in the presence of asymptotic conditions, the total energy, as well as the angular momentum, of the system can be defined. However, even in such a case, some confusions used to exist in the literature: there are many different methods and in some cases they did not all give the same charges. For

†Corresponding author. E-mail: nashed@bue.edu.eg, Egyptian Relativity Group (ERG) URL: http://www.erg.eg.net
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example, as has already been emphasized,[16] not even in four dimensions did all authors obtain the same expression for the energy of Kerr–AdS black holes and some of these expressions were in disagreement with the first law of black hole thermodynamics.

Nowadays, a number of methods give reliable results. Gibbons *et al.*[16] computed the energy of Kerr–AdS black holes first indirectly by integrating the first law and then using the conformal definition of Ashtekar, Magnon, and Das,[17,18] and they showed that both of them agree. Besides the Regge–Teitelboim method[19] some of the various other definitions are the following: the approach of Abbott and Deser,[20−22] the spinor definition[23,24] based on the electric Weyl tensor, covariant phase space methods,[25−28] cohomological techniques,[29,30] the KBL approach,[31−33] Noether methods,[34−39] the counter term subtraction method,[40,41] (and more references on this in Ref. [27] for example), improved surface integrals,[42] and regularization of the Euclidean action.[43,44]

The aim of the present work is to apply the Euclidean continuation method to many tetrads' spacetime reproducing an LT black hole. In Section 2, a brief review of the derivation of the field equations of the gravitational field is given. In Section 3, we give a slowly rotating LT spacetime in a teleparallel equivalent with general relativity. The thermodynamic quantities of this tetrad are calculated also in Section 3. In Section 4, we give three different tetrads' reproduced LT spacetime and calculate their thermodynamic quantities. The final section is devoted to our main result. We also give a table of comparison to compare the result of energy calculated using the Euclidean continuation method with the calculation of energy using other methods like gravitational energy momentum tensor, translation momentum, conservation laws.

2. The TEGR for gravitation

In a spacetime with absolute parallelism the parallel vector field $h_a{}^\mu$ defines the nonsymmetric affine connection

$$\Gamma^\lambda{}_{\mu\nu} \overset{\text{def.}}{=\!=} h_a{}^\lambda h^a{}_{\mu,\nu}, \tag{1}$$

where $h_{a\mu,\nu} = \partial_\nu h_{a\mu}$.[1)] The curvature tensor defined by $\Gamma^\lambda{}_{\mu\nu}$ is identically vanishing, however. The metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu \tag{2}$$

with the following Minkowski metric as $\eta_{ab} = \text{diag}(+1\,,-1\,,-1\,,-1)$.[2)]

The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by[45][3)]

$$\begin{aligned}\mathscr{L}_{\rm G} &= \sqrt{-g} L_{\rm G} \\ &= -\frac{\sqrt{-g}}{16\pi}\left(\frac{T^{abc}T_{abc}}{4} + \frac{T^{abc}T_{bac}}{2} - T^a T_a\right) \\ &\quad - 2\partial_\mu\left(\sqrt{-g}T^\mu\right) - L_{\rm m} \\ &= -\frac{\sqrt{-g}}{16\pi}\Sigma^{abc}T_{abc} - 2\partial_\mu\left(\sqrt{-g}T^\mu\right) - L_{\rm m},\end{aligned} \tag{3}$$

where $g = \det(g_{\mu\nu})$. The tensor Σ^{abc} is defined by

$$\begin{aligned}\Sigma^{abc} \overset{\text{def.}}{=\!=} &\frac{1}{4}\left(T^{abc} + T^{bac} - T^{cab}\right) \\ &+ \frac{1}{2}\left(\eta^{ac}T^b - \eta^{ab}T^c\right).\end{aligned} \tag{4}$$

T^{abc} and T^a are the torsion tensor and the basic vector field defined by

$$T^a{}_{\mu\nu} \overset{\text{def.}}{=\!=} h^a{}_\lambda T^\lambda{}_{\mu\nu} = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu, \tag{5}$$

and

$$T^\mu \overset{\text{def.}}{=\!=} T^\nu{}_\nu{}^\mu, \qquad T^a \overset{\text{def.}}{=\!=} h^a{}_\mu T^\mu = T^b{}_b{}^a. \tag{6}$$

The quadratic combination $\Sigma^{abc}T_{abc}$ is proportional to the scalar curvature R and $\partial_\mu\left(\sqrt{-g}\,T^\mu\right)$ is a total divergence term that has no contribution to the field equation.[46] $L_{\rm m}$ represents the Lagrangian density for matter fields.

The gravitational field equations for the system described by $L_{\rm G}$ and $L_{\rm m}$ are the following

$$\begin{aligned}&h_{a\lambda}h_{b\mu}\partial_\nu\left(\sqrt{-g}\Sigma^{b\lambda\nu}\right) \\ &-\sqrt{-g}\left(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}h_{a\mu}T_{bcd}\Sigma^{bcd}\right) \\ &= \frac{1}{2}\kappa\sqrt{-g}\,T_{a\mu},\end{aligned} \tag{7}$$

where

$$\frac{\delta L_{\rm m}}{\delta h^{a\mu}} \equiv \sqrt{-g}T_{a\mu}.$$

[1)] spacetime indices $\mu,\ \nu,\ldots$ and SO(3,1) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated to $\mu = 0, i$, and $a = (0), (i)$.

[2)] Latin indices are raising and lowering with the aid of η_{ab} and η^{ab}.

[3)] Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$.

It is possible to prove by explicit calculations that the left-hand side of the symmetric part of the field equations (7) is exactly given by[45]

$$\frac{\sqrt{-g}}{2}\left[R_{a\mu}-\frac{1}{2}h_{a\mu}R\right].$$

The axial-vector part of the torsion tensor A_μ is defined by

$$A_\mu \overset{\text{def.}}{=\!=} \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma}=\frac{1}{3}\epsilon_{\mu\nu\rho\sigma}\gamma^{\nu\rho\sigma}, \tag{8}$$

where

$$\epsilon_{\mu\nu\rho\sigma}\overset{\text{def.}}{=\!=}\sqrt{-g}\delta_{\mu\nu\rho\sigma}$$

with $\gamma_{\nu\rho\sigma}=\eta^{ab}h_{a\nu}h_{b\rho\,;\,\sigma}$ being the contorsion tensor and $\delta_{\mu\nu\rho\sigma}$ is completely antisymmetric and normalized as $\delta_{0123}=-1$.

The definition of the gravitational energy–momentum P^a four-vector has the form[46]

$$P^a=-\int_V \mathrm{d}^3x\partial_i\Pi^{ai}, \tag{9}$$

where V is an arbitrary volume of the three-dimensional (3D) space. In the configuration space we have

$$\Pi^{ai}=-\frac{2}{\kappa}\sqrt{-g}\Sigma^{a0i} \tag{10}$$

with

$$\partial_\nu(\sqrt{-g}\Sigma^{a\lambda\nu})=\frac{\kappa}{2}\sqrt{-g}\,h^a{}_\mu(t^{\lambda\mu}+T^{\lambda\mu}),$$

where

$$t^{\lambda\mu}=\frac{1}{2\kappa}\left(4\Sigma^{bc\lambda}T_{bc}{}^{\mu}-g^{\lambda\mu}\Sigma^{bcd}T_{bcd}\right).$$

Maluf *et al.*[45,46] defined

$$L^{ab}=2\int_V \mathrm{d}^3xM^{[ab]}, \tag{11}$$

as the four-angular-momentum of the gravitational field for an arbitrary volume V of the three-dimensional space.

3. Thermal properties of the spherically symmetric solution

Hawking and Gibbons[47] discussed the thermal properties of the Schwarzschild solution, for which the line-element squared takes the positive-definite standard form

$$\begin{aligned}\mathrm{d}s^2=&+\left(1-\frac{2M}{r}\right)\mathrm{d}\tau^2\\&+\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2,\end{aligned} \tag{12}$$

after the Euclidean continuation of the time variable, $t=-\mathrm{i}\tau$. By the transformation $x=4M(1-2M/r)^{1/2}$, the line-element squared becomes

$$\mathrm{d}s^2=+\left(\frac{x}{4M}\right)^2\mathrm{d}\tau^2+\left(\frac{r^2}{4M^2}\right)^2\mathrm{d}x^2+r^2\mathrm{d}\Omega^2, \tag{13}$$

which shows that τ can be regarded as an angular variable with period $8\pi M$. Now the Euclidean section of the Schwarzschild solution is the region defined by $8\pi M\geq\tau\geq 0$ and $x>0$, where the metric is positive-definite, asymptotically flat, and non-singular. They calculated the Euclidean action, $\hat{I}$, of general relativity from the surface term as follows:

$$\hat{I}=4\pi M^2=\frac{\beta^2}{16\pi}, \tag{14}$$

where $\beta=8\pi M=T^{-1}$ with T being interpreted as the absolute temperature of the Schwarzschild black hole.

For a canonical ensemble the energy is given by[47–49]

$$E=\frac{\sum_n E_n\,\mathrm{e}^{-\beta E_n}}{\sum_n\,\mathrm{e}^{-\beta E_n}}=-\frac{\partial}{\partial\beta}\log Z, \tag{15}$$

where E_n is the energy in the n-th state, and Z is the partition function, which is in the tree approximation related to the Euclidean action of the classical solution by

$$\hat{I}=-\log Z. \tag{16}$$

Using of Eqs. (16) and (14) in Eq. (13) gives

$$E=\frac{\beta}{8\pi}=M. \tag{17}$$

They also calculated the entropy of the Schwarzschild black hole to obtain

$$S=-\sum_n P_n\log P_n=\beta E+\log Z=4\pi M^2=\frac{1}{4}A, \tag{18}$$

where $P_n=Z^{-1}\,\mathrm{e}^{-\beta E_n}$, and A is the area of the event horizon of the Schwarzschild black hole.

Now let us use the Hawking and Gibbons procedure to different axially symmetric tetrads that reproduce LT spacetime.

4. Thermodynamic quantities of the first tetrad

The covariant form of the first slowly rotating tetrad field having axial symmetry in spherical coordinates (t,r,θ,ϕ), can be written as

$$(h^a{}_\mu)_{\rm I} = \begin{pmatrix} -\dfrac{r-M}{r} & \dfrac{-M}{r-2M} & 0 & 0 \\ \dfrac{M(\sin\theta\cos\phi+\alpha\sin\phi)}{r} & \dfrac{\alpha_1\sin\theta}{r-2M} & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \dfrac{M(\sin\theta\sin\phi-\alpha\cos\phi)}{r} & \dfrac{\alpha_2\sin\theta}{r-2M} & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \dfrac{M\cos\theta}{r} & \dfrac{r-M}{r-2M}\cos\theta & -r\sin\theta & 0 \end{pmatrix}, \tag{19}$$

where α, α_1, and α_2 have the form

$$\alpha = \frac{-2a\sin\theta}{r}, \qquad \alpha_1 = (r-M)\cos\phi, \qquad \alpha_2 = (r-M)\sin\phi, \tag{20}$$

where M and a are the mass and the slowly rotation parameter respectively.[50] We consider an asymptotically flat spacetime in this paper, and impose the boundary condition that for $r\to\infty$ the tetrad (19) approaches the tetrad of Minkowski spacetime, in Cartesian coordinate, i.e., $\eta_{ab} = {\rm diag}(+1,-1,-1,-1)$. The metric tensor $g_{\mu\nu} \overset{\rm def.}{=\!=} \eta_{ab}\,(h^a{}_\mu)_{\rm I}\left(h^b{}_\nu\right)_{\rm I}$ associated with the tetrad field (19) has the form

$${\rm d}s^2 = \left(1-\frac{2Mr}{r}\right){\rm d}t^2 - \left(1-\frac{2Mr}{r}\right)^{-1}{\rm d}r^2 - r^2({\rm d}\theta^2+\sin^2\theta\,{\rm d}\phi^2) - \frac{4Ma\sin^2\theta}{r}{\rm d}t{\rm d}\phi, \tag{21}$$

which is the LT spacetime written in Boyer–Lindquist coordinates.

Since the solution gives an axially symmetric LT black hole solution, the variable τ can be regarded as an angular variable with period $4\pi r_+$ after the Euclidean continuation, $t=-{\rm i}\tau$. We use the index 4 instead of 0 in the Euclidean section. The Euclidean continuation of the tetrad (19) is given by

$$(h^a{}_\mu)_{\rm IE} = \begin{pmatrix} \dfrac{\alpha_1\sin\theta}{r-2M} & r\cos\theta\cos\phi & -r\sin\theta\sin\phi & \dfrac{M\sin\theta\cos\phi+\alpha\sin\phi}{r} \\ \dfrac{\alpha_2\sin\theta}{r-2M} & r\cos\theta\sin\phi & r\sin\theta\cos\phi & \dfrac{M\sin\theta\sin\phi-\alpha\cos\phi}{r} \\ \dfrac{r-M}{r-2M}\cos\theta & -r\sin\theta & 0 & \dfrac{M\cos\theta}{r} \\ \dfrac{-M}{r-2M} & 0 & 0 & \dfrac{r-M}{r} \end{pmatrix}, \tag{22}$$

the line-element squared of tetrad (22) takes the positive-definite form

$$\begin{aligned} {\rm d}s^2 &= +\left(1-\frac{2Mr}{r}\right){\rm d}\tau^2 \\ &\quad + \left(1-\frac{2Mr}{r}\right)^{-1}{\rm d}r^2 + r^2({\rm d}\theta^2+\sin^2\theta\,{\rm d}\phi^2) \\ &\quad + \frac{4Ma\sin^2\theta}{r}{\rm d}\tau{\rm d}\phi. \end{aligned} \tag{23}$$

The Euclidean action is given by[53]

$$\begin{aligned} \hat{I} &= -\frac{1}{2\kappa}\int\sqrt{g}\,(R-2T^\mu{}_{;\mu})\,{\rm d}^4x \\ &= \frac{1}{\kappa}\int\sqrt{g}\;T^\mu{}_{;\mu}{\rm d}^4x. \end{aligned} \tag{24}$$

where R is the Riemann–Christoffel scalar curvature, which is vanishing for LT spacetime, and T^μ is the basic vector field defined by Eq. (6).

The volume integral (24) of tetrad (22) is calculated to give

$$\hat{I} = \frac{1}{\kappa}\int\sqrt{g}\,(T^\mu{}_{;\mu})\,{\rm d}^4x = \pi {r_+}^2, \tag{25}$$

where r_+ is the outer event horizon located at the largest positive root of the relation $g^{rr}=0$, i.e.,

$(1-2M/r)=0$.[4] Using Eq. (25) in Eq. (15) we obtain the energy of tetrad (22) using the Euclidean continuation method to have the form

$$E=M, \tag{26}$$

which is an acceptable result.[51−54] The area of Eq. (22) is given by

$$A=\int\sqrt{g_{\theta\theta}g_{\phi\phi}}\,\mathrm{d}\theta\,\mathrm{d}\phi=4\pi {r_+}^2, \tag{27}$$

where r_+ is the event horizon. Substituting Eqs. (25) and (26) into Eq. (18) we obtain the entropy of tetrad (22) to have the form

$$S=4\pi M^2=\frac{A}{4}. \tag{28}$$

The Hawking temperature of tetrad (22) has the form

$$T=\frac{1}{8\pi M}. \tag{29}$$

Using Eqs. (26), (28), and (29) we obtain

$$\mathrm{d}M=T\mathrm{d}S. \tag{30}$$

Equation (30) shows that the first law of thermodynamics is satisfied. From Eqs. (25), (26), (28), and (29) we find the following relation

$$E-TS\cong T\hat{I} \tag{31}$$

is satisfied. Equation (31) shows that quantum statistical relation holds.

5. Thermodynamic quantities of equivalent tetrads which reproduce LT spacetime

5.1. The second tetrad

The covariant form of the second rotating tetrad field having axial symmetry in spherical coordinates (t,r,θ,ϕ), can be written as

$$(h^a{}_\mu)_{\mathrm{II}}=\begin{pmatrix}\sqrt{1-2M/r} & 0 & 0 & 0\\ 0 & -1/\sqrt{1-2M/r} & 0 & 0\\ 0 & 0 & r & 0\\ 2Ma\sin\theta/r^2 & 0 & 0 & r\sin\theta\end{pmatrix}. \tag{32}$$

The metric tensor $g_{\mu\nu}\overset{\mathrm{def.}}{=\!=}\eta_{ab}\,(h^a{}_\mu)_{\mathrm{II}}\left(h^b{}_\nu\right)_{\mathrm{II}}$ associated with tetrad field (32) has the form (21).

We are going to apply the Gibbons and Hawking procedure to the axially symmetric LT black hole solution, namely, to the tetrad (32). Following the same procedure done for the first tetrad, the Euclidean continuation of the tetrad (32) is given by

$$(h^a{}_\mu)_{\mathrm{IIE}}=\begin{pmatrix}-1/\sqrt{1-2M/r} & 0 & 0 & 0\\ 0 & r & 0 & 0\\ 0 & 0 & r\sin\theta & 2Ma\sin\theta/r^2\\ 0 & 0 & 0 & \sqrt{1-2M/r}\end{pmatrix}. \tag{33}$$

[4] The singularity at $r=2M$ can be physical or merely the result of a bad choice of coordinates. If the singularity is physical, then the region $r\leq 2M$ has nothing to do with the region $r>2M$ that describes the exterior of massive bodies. However, if the singularity at $r=2M$ is just a coordinate singularity, we can use another coordinate system that is related to Schwarzschild's in the region $r>2M$ by a standard coordinate change but such that the metric is regular at $r=2M$. The analytic extension of the Schwarzschild metric obtained in this way will also cover the region $r<2M$. To find the nature of the singularities it is necessary to perform an analysis of the curvature and torsion invariants. If one carries out the calculations of invariants, i.e., $T^{\mu\nu\rho}T_{\mu\nu\rho}$, $T^{\mu}T_{\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ ... one can conclude that $r=2M$ is not a true singularity, but the singularity at $r=0$ is certainly a physical singularity that will be present in any coordinate system.

The line-element squared of the above tetrad takes the positive-definite form as given by Eq. (23).

The volume integral (24) for tetrad (33) is calculated to give

$$\begin{aligned}\hat{I} &= \frac{1}{\kappa}\int \sqrt{g}\,(T^{\mu}{}_{;\mu})\,\mathrm{d}^4x \\ &= 8\pi r_+ M - 12M^2 = 4\pi M^2, \end{aligned} \tag{34}$$

where r_+ is the outer event horizon located at the largest positive root of $g^{rr}=0$. Substituting Eq. (34) into Eq. (15) we obtain the energy of tetrad (33) using the Euclidean continuation method to have the form

$$E = M, \tag{35}$$

which is an acceptable result.[51−54] Following the same procedure done for the first tetrad to calculate the thermodynamic quantities associated to tetrad (33) we can obtain the same result of thermodynamic quantities of the first tetrad.

5.2. The third tetrad

The covariant form of the third rotating tetrad field having axial symmetry in spherical coordinates (t, r, θ, ϕ), can be written as

$$(h^a{}_\mu)_{\mathrm{III}} = \begin{pmatrix} \sqrt{1-2M/r} & 0 & 0 & 2Ma\sin^2\theta/\sqrt{r^2-2Mr} \\ 0 & -1/\sqrt{1-2M/r} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{pmatrix}. \tag{36}$$

The metric tensor $g_{\mu\nu} \overset{\text{def.}}{=\!=} \eta_{ab}\,(h^a{}_\mu)_{\mathrm{III}}\,(h^b{}_\nu)_{\mathrm{III}}$ associated with tetrad field (36) has the form (21).

Following the same procedure done for the first and second tetrads, the Euclidean continuation of the tetrad (36) is given by

$$(h^a{}_\mu)_{\mathrm{IIIE}} = \begin{pmatrix} -1/\sqrt{1-2M/r} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r\sin\theta & 0 \\ 0 & 0 & -2Ma\sin^2\theta/\sqrt{r^2-2Mr} & \sqrt{1-2M/r} \end{pmatrix}. \tag{37}$$

The line-element squared of the above tetrad, i.e., equation (37) takes the positive-definite form given by Eq. (23).

The volume integral (24) for tetrad (37) is calculated to give the same value as Eq. (34). Therefore, the energy is the same as Eq. (35). Following the same procedure done for the first tetrad to calculate the thermodynamic quantities associated to tetrad (37) we obtain the same result of thermodynamic quantities.

5.3. The fourth tetrad

The covariant form of the fourth rotating tetrad field having axial symmetry in spherical coordinates (t, r, θ, ϕ), can be written as

$$(h^a{}_\mu)_{\mathrm{IV}} = \begin{pmatrix} \sqrt{1-2M/r} & 0 & 0 & a\sin^2\theta\sqrt{1-2M/r} \\ 0 & 1/\sqrt{1-2M/r} & 0 & 0 \\ 0 & 0 & r & 0 \\ a\sin\theta/r & 0 & 0 & r\sin\theta \end{pmatrix}. \tag{38}$$

The metric tensor $g_{\mu\nu} \overset{\text{def.}}{=\!=} \eta_{ab}\,(h^a{}_\mu)_{\rm IV}\,\big(h^b{}_\nu\big)_{\rm IV}$ associated with tetrad field (38) has the form (21).

Following the same procedure done for the previous tetrads the Euclidean continuation of the tetrad (38) is given by

$$(h^a{}_\mu)_{\rm IVE} = \begin{pmatrix} 1/\sqrt{1-2M/r} & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r\sin\theta & a\sin\theta/r \\ 0 & 0 & -a\sin^2\theta\sqrt{1-2M/r} & \sqrt{1-2M/r} \end{pmatrix}. \tag{39}$$

The line-element squared of the above tetrad takes the positive-definite form given by Eq. (23).

The volume integral (24) for the tetrad (39) is calculated to give the same value as Eq. (34). Therefore, the energy associated to tetrad (39) is the same as given by Eq. (35). Following the same procedure done for the first tetrad to calculate the thermodynamic quantities associated to tetrad (39) we obtain the same result of thermodynamic quantities.

We also give a table (Table 1) of comparison to compare the results of energy calculated using the Euclidean continuation method with the calculation of energy using other methods like gravitational energy momentum tensor, translation momentum, and conservation laws.

Table 1. The comparison between the results of energy using the Euclidean continuation method and the other definitions.[5)]

Spacetime	Euclidean continuation method	Gravitational energy tensor	Riemannian connection 1-form	Superpotential of Mϕller 1978
	$\hat{I} = (1/\kappa)\int \sqrt{g}\,T^\mu{}_{;\mu}\,\mathrm{d}^4x$	$E = -\int_V \mathrm{d}^3x\partial_i \Pi^{0i}$	$\tilde{E} = \int_{\partial S}\tilde{H}_{\hat{0}}$ $\tilde{H}_\alpha = (1/2\kappa)\tilde{\Gamma}^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma}$	$E(R) = \int_{r=R}\iint \tau_0{}^0\,\mathrm{d}^3x$ $\tau_\mu{}^\nu = \mathscr{U}_\mu{}^{\nu\lambda}{}_{,\lambda}$
First coframe	$E = M$	$E = M$	$E = M$	$E = M$
Second coframe	$E = M$	divergent	divergent, Ref. [70], Eq. (21)	divergent
Third coframe	$E = M$	divergent	divergent, Ref. [70], Eq. (46)	divergent
Fourth coframe	$E = M$	divergent	divergent, Ref. [70], Eq. (35)	divergent

6. Discussion and conclusion

Most of the gravitational models are invariant under both spacetime diffeomorphisms and local Lorentz transformations. However, the Lagrangian may be shifted by a total derivative term (equivalently, by a boundary term). This actually happens in many gravitational field models, for example: i) when one adds a non-invariant boundary term to the original Lagrangian; ii) when the gravitational dynamics are described in the purely tetrad framework;[55] iii) when the Lagrangian includes topological terms (e.g., the Chern–Simons type[56−65]).

The main results of this paper can be summarized in the following items.

(i) We use four tetrads to whom their associated metrics reproduce LT spacetime.

(ii) We calculate the energy of the first tetrad using the Euclidean continuation method. We show that the result of energy is acceptable and consistent with the previous result.[51−54] The thermodynamic quantities associated with this tetrad are calculated and the results are consistent with the previous ones.[66,67] We also show that the first law of thermodynamics is satisfied and the quantum statistical relation holds.

(iii) Repeating the same calculations done for the first tetrad for three different tetrad fields reproduces the same metric, i.e., LT, we obtain the same results of energy, thermodynamical quantities. We also show that the first law of thermodynamics is satisfied and

[5)] $\mathscr{U}_\mu{}^{\nu\lambda} = [(-g)^{1/2}/2\kappa]P_{\chi\rho\sigma}{}^{\tau\nu\lambda}[\Phi^\rho g^{\sigma\chi}g_{\mu\tau} - \lambda g_{\tau\mu}\gamma^{\chi\rho\sigma} - (1-2\lambda)g_{\tau\mu}\gamma^{\sigma\rho\chi}]$, where $P_{\chi\rho\sigma}{}^{\tau\nu\lambda}$ is defined as $P_{\chi\rho\sigma}{}^{\tau\nu\lambda} \overset{\text{def.}}{=\!=} \delta_\chi{}^\tau g_{\rho\sigma}{}^{\nu\lambda} + \delta_\rho{}^\tau g_{\sigma\chi}{}^{\nu\lambda} - \delta_\sigma{}^\tau g_{\chi\rho}{}^{\nu\lambda}$ with $g_{\rho\sigma}{}^{\nu\lambda}$ being a tensor defined by $g_{\rho\sigma}{}^{\nu\lambda} \overset{\text{def.}}{=\!=} \delta_\rho{}^\nu\delta_\sigma{}^\lambda - \delta_\sigma{}^\nu\delta_\rho{}^\lambda$.

the quantum statistical relation holds for these tetrad fields.

(iv) Using the Euclidean continuation method to calculate the energy associated with the four tetrad fields presented in this paper we obtain the ADM[51−54] value without any regularization[68] or even without introducing any local Lorentz transformation that remove the inertia, i.e., the divergence or the unphysics quantity.[69]

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