New approach for obtaining the time-evolution law of a chaotic light field in a damping–gaining process*

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(Received 7 January 2012; revised manuscript received 29 February 2012)

A new approach for studying the time-evolution law of a chaotic light field in a damping–gaining coexisting process is presented. The new differential equation for determining the parameter of the density operator $\rho(t)$ is derived and the solution of f' for the damping and gaining processes are studied separately. Our approach is direct and the result is concise since it is not necessary for us to know the Kraus operators in advance.

Keywords: chaotic light field, damping–gaining process, time evolution law, quantum communication

PACS: 03.65.–w, 42.50.–p **DOI:** 10.1088/1674-1056/21/8/080302

1. Introduction

The chaotic light field[1] described by the density operator $\rho_{\rm c} = (1-{\rm e}^{f})\,{\rm e}^{fa^{\dagger}a}$, (a mixed state, or named thermo state when $f = -\beta\hbar\omega$, $\beta = 1/(kT)$ is the Boltzman constant), has been studied since the birth of quantum optics theory, where $a^{\dagger}a$ is the photon number operator, $\left[a, a^{\dagger}\right] = 1$. The thermo light is a characteristic of obeying Bose distribution in average photon numbers, i.e., ${\rm Tr}(a^{\dagger}a\rho_{\rm c}) = [{\rm e}^{\hbar\omega/(kT)}-1]^{-1}$.

As a common thermo source, the chaotic light field is closely related to quantum dissipation, which has been a hot topic of physicists.[2] Thus, one has to treat quantum dissipation as an essential ingredient in quantum optics theory since quantum dissipation always brings out quantum decoherence. In this work we explore how an initial chaotic light field gets damped or gained in a quantum channel governed by the master equation,[3]

$$\begin{aligned}\frac{{\rm d}\rho(t)}{{\rm d}t} &= g\left[2a^{\dagger}\rho(t)a - aa^{\dagger}\rho(t) - \rho(t)aa^{\dagger}\right]\\ &\quad + \kappa\left[2a\rho(t)a^{\dagger} - a^{\dagger}a\rho(t) - \rho(t)a^{\dagger}a\right], \quad (1)\end{aligned}$$

where $\rho(t)$ is the density operator of the system at time t, g and κ represent the gain and the loss of the cavity, respectively. Such a master equation can also describe a laser process. When $g = 0$, it reduces to the amplitude decay model (or amplitude dissipative channel)

$$\frac{{\rm d}\rho(t)}{{\rm d}t} = \kappa\left[2a\rho(t)a^{\dagger} - a^{\dagger}a\rho(t) - \rho(t)a^{\dagger}a\right]. \quad (2)$$

On the other hand, when $\kappa = 0$, equation (1) reduces to the gain model

$$\frac{{\rm d}\rho(t)}{{\rm d}t} = g\left[2a^{\dagger}\rho(t)a - aa^{\dagger}\rho(t) - \rho(t)aa^{\dagger}\right]. \quad (3)$$

The rest of this paper is arranged as follows: In Section 2 we make a reasonable ansatz that $\rho(t)$, into which the initial chaotic light field $\left(1-{\rm e}^{f}\right){\rm e}^{fa^{\dagger}a} = \rho(0)$ evolves, is also a chaotic state, and then we derive a new differential equation to determine the parameter involved in $\rho(t)$. In Section 3, we exactly solve this differential equation for damping and gaining cases, and know the time-evolution law of the chaotic light field in a damping–gaining coexisting process. In Section 4 we make a brief conclusion.

*Project supported by the National Natural Science Foundation of China (Grant Nos. 61141007, 11047133, and 11175113), the Natural Science Foundation of Jiangxi Province of China (Grant Nos. 2010GQS0080 and 2010GQW0027), the Research Foundation of the Education Department of Jiangxi Province of China (Grant No. GJJ11339), and the Sponsored Program for Cultivating Youths of Outstanding Ability in Jiangxi Normal University.

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2. New differential equation for determining the parameter involved in $\rho(t)$

Let the initial thermo state be $\rho(0) = (1 - \mathrm{e}^{f})\,\mathrm{e}^{fa^{\dagger}a}$. In fact, the generation mechanism of the initial thermo state is the process with both damping and gaining, thus the density operator $\rho(t)$ should keep its essentiality and we can think that $\rho(t)$ keeps a thermo state (Bose distribution in average photon numbers still holds) during this damping–gaining process. There is no squeezing or other nonlinear effects in this process, so we make an ansatz for the final state

$$\rho(t) = (1 - \mathrm{e}^{f'})\,\mathrm{e}^{f'a^{\dagger}a}, \tag{4}$$

where $f' = f'(t)$ is to be determined. Substituting Eq. (4) into the left-hand side of Eq. (1), one has

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(1 - \mathrm{e}^{f'})\,\mathrm{e}^{f'a^{\dagger}a} \\
&= \mathrm{e}^{f'a^{\dagger}a}\frac{\mathrm{d}}{\mathrm{d}t}(1 - \mathrm{e}^{f'}) + (1 - \mathrm{e}^{f'})\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{f'a^{\dagger}a} \\
&= \left[(1 - \mathrm{e}^{f'})a^{\dagger}a - \mathrm{e}^{f'}\right]\mathrm{e}^{f'a^{\dagger}a}\frac{\mathrm{d}f'}{\mathrm{d}t}.
\end{aligned} \tag{5}$$

Substituting Eq. (4) into the right-hand side (RS) of Eq. (1) yields

$$\begin{aligned}
&\text{Right-hand side of Eq. (1)} \\
&= 2(1 - \mathrm{e}^{f'})\left\{g\left[a^{\dagger}a\,\mathrm{e}^{-f'} - aa^{\dagger}\right]\right. \\
&\quad \left. + \kappa\left[aa^{\dagger}\,\mathrm{e}^{f'} - a^{\dagger}a\right]\right\}\mathrm{e}^{f'a^{\dagger}a} \\
&= 2(1 - \mathrm{e}^{f'})\left\{a^{\dagger}a\left[g(\mathrm{e}^{-f'} - 1)\right.\right. \\
&\quad \left.\left. + \kappa(\mathrm{e}^{f'} - 1)\right] + \kappa\,\mathrm{e}^{f'} - g\right\}\mathrm{e}^{f'a^{\dagger}a}.
\end{aligned} \tag{6}$$

Combining Eq. (5) with Eq. (6) and comparing the terms involving $a^{\dagger}a\,\mathrm{e}^{f'a^{\dagger}a}$, one obtains the new differential equation

$$\frac{\mathrm{d}f'}{\mathrm{d}t} = 2\left[g(\mathrm{e}^{-f'} - 1) + \kappa(\mathrm{e}^{f'} - 1)\right], \tag{7}$$

while in Eq. (6), comparing the terms involving $\mathrm{e}^{f'a^{\dagger}a}$, one obtains

$$2(1 - \mathrm{e}^{f'})(\kappa\,\mathrm{e}^{f'} - g) = -\,\mathrm{e}^{f'}\frac{\mathrm{d}f'}{\mathrm{d}t}, \tag{8}$$

these two equations are the same, so one only needs to solve Eq. (7). It then follows

$$\int \frac{\mathrm{d}f'}{g(\mathrm{e}^{-f'} - 1) + \kappa(\mathrm{e}^{f'} - 1)} = 2\int \mathrm{d}t + c, \tag{9}$$

where c is the integration constant.

3. Solutions of $\rho(t)$ for damping–gaining process

Using the integration formula:

$$\begin{aligned}
&\int \frac{\mathrm{d}x}{g(\mathrm{e}^{-x} - 1) + \kappa(\mathrm{e}^{x} - 1)} \\
&= \frac{\ln\left[(\kappa\,\mathrm{e}^{x} - g)/(1 - \mathrm{e}^{x})\right]}{g - \kappa} + c',
\end{aligned} \tag{10}$$

thus we have

$$\ln\left[(\kappa\,\mathrm{e}^{f'} - g)/(1 - \mathrm{e}^{f'})\right] = 2(g - \kappa)t + c'', \tag{11}$$

where c'' is the integration constant. When $t = 0$, $f' \to f$, so one can determine

$$c'' = \ln\left(\frac{\kappa\,\mathrm{e}^{f} - g}{1 - \mathrm{e}^{f}}\right), \tag{12}$$

and

$$\frac{\kappa\,\mathrm{e}^{f'} - g}{1 - \mathrm{e}^{f'}} = \frac{\kappa\,\mathrm{e}^{f} - g}{1 - \mathrm{e}^{f}}\,\mathrm{e}^{2(g-\kappa)t}. \tag{13}$$

Then we obtain

$$f' = \ln\left[\frac{g(1 - \mathrm{e}^{f}) + (\kappa\,\mathrm{e}^{f} - g)\,\mathrm{e}^{2(g-\kappa)t}}{\kappa(1 - \mathrm{e}^{f}) + (\kappa\,\mathrm{e}^{f} - g)\,\mathrm{e}^{2(g-\kappa)t}}\right], \tag{14}$$

and the time-evolution law of the chaotic light field in a damping–gaining coexisting process as follows:

$$\begin{aligned}
\rho(t) &= \frac{(\kappa - g)(1 - \mathrm{e}^{f})}{\kappa(1 - \mathrm{e}^{f}) + (\kappa\,\mathrm{e}^{f} - g)\,\mathrm{e}^{2(g-\kappa)t}} \\
&\quad \times \left[\frac{g(1 - \mathrm{e}^{f}) + (\kappa\,\mathrm{e}^{f} - g)\,\mathrm{e}^{2(g-\kappa)t}}{\kappa(1 - \mathrm{e}^{f}) + (\kappa\,\mathrm{e}^{f} - g)\,\mathrm{e}^{2(g-\kappa)t}}\right]^{a^{\dagger}a}.
\end{aligned} \tag{15}$$

In particular, when $\kappa = 0$, i.e., no loss, equation (14) reduces to

$$f' = \ln\left[(\mathrm{e}^{f} - 1)\,\mathrm{e}^{-2gt} + 1\right], \tag{16}$$

the density operator at t for the gaining process is

$$\begin{aligned}
\rho(t) &= (1 - \mathrm{e}^{f'})\,\mathrm{e}^{f'a^{\dagger}a} \\
&= \mathrm{e}^{-2gt}(1 - \mathrm{e}^{f})\,\mathrm{e}^{a^{\dagger}a\ln[1 - \mathrm{e}^{-2gt}(1 - \mathrm{e}^{f})]} \\
&= \mathrm{e}^{-2gt}(1 - \mathrm{e}^{f})\left[1 - \mathrm{e}^{-2gt}(1 - \mathrm{e}^{f})\right]^{a^{\dagger}a}.
\end{aligned} \tag{17}$$

While for $g = 0$, i.e., no gain, equation (14) reduces to

$$f' = \ln\frac{1}{1 + (\mathrm{e}^{-f} - 1)\,\mathrm{e}^{2t\kappa}}, \tag{18}$$

and the density operator at t for the damping process is

$$\rho(t) = \frac{(\mathrm{e}^{-f} - 1)\,\mathrm{e}^{2\kappa t}}{(\mathrm{e}^{-f} - 1)\,\mathrm{e}^{2\kappa t} + 1}\left[\frac{1}{(\mathrm{e}^{-f} - 1)\,\mathrm{e}^{2\kappa t} + 1}\right]^{a^{\dagger}a}. \tag{19}$$

This result can be checked by substituting Eq. (14) into the left-hand side and the right-hand side of Eq. (7), respectively.

4. Conclusion

In this paper, we have exactly solved the master equation for the damping–gaining coexisting process when the initial field is a chaotic one in a direct and concise way. The approach in this work is different from that which we reported in our previous review papers,[4−6] where we must know the Kraus operators for Eq. (1) in advance.

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