A type of conserved quantity of Mei symmetry of Nielsen equations for a holonomic system*

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A type of structural equation and conserved quantity which are directly induced by Mei symmetry of Nielsen equations for a holonomic system are studied. Under the infinitesimal transformation of the groups, from the definition and the criterion of Mei symmetry, a type of structural equation and conserved quantity for the system by proposition 2 are obtained, and the inferences in two special cases are given. Finally, an example is given to illustrate the application of the results.

Keywords: Nielsen equation, Mei symmetry, structural equation, conserved quantity

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1. Introduction

In recent years, analytical mechanics has been widely applied in the fields of engineering and technology, and facilitated the development of mathematical methods such as calculus of variations and the geometric method.[1−4] Nielsen style is one of the three important mechanical systems and occupies an important position in the theories of analytical mechanics.[5] At present, scholars study questions about Lie symmetry and Mei symmetry mainly through the classical method, and many achievements have been obtained.[6−18] The relation between the form invariance of Nielsen equations and the Noether symmetry in Ref. [19], the form invariance of the Nielsen equation of a non-conservative mechanical system in Ref. [20], the conformal invariance and conserved quantities of Mei symmetry for generality in Ref. [21], and Lie symmetry and Hojman conserved quantity of the Nielsen equation in a dynamical system of the relative motion in Ref. [22] have been studied. Recently, a kind of conserved quantity which is directly induced by Mei symmetry of Nielsen equations for a Lagrangian system in Ref. [23], and a non-Noether conserved quantity constructed directly by using the form invariance of Nielsen equations for a non-conservative mechanical system in Ref. [24] have been investigated.

In the present paper, we mainly study a type of structural equation and conserved quantity which are directly induced by Mei symmetry of Nielsen equations for a holonomic system.

2. Nielsen equations for a holonomic system and definition of Mei symmetry

Assuming that the configuration of the system is determined by n generalized coordinates q_s $(s = 1, \ldots, n)$ and it is subjected to the ideal holonomic constraints of bilateral constraints, the differential equations of motion for the system can be expressed as follows:

$$\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t} - 2\frac{\partial L}{\partial q_s} = Q_s, \quad (s = 1, \ldots, n). \tag{1}$$

Here, $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian function of the system and $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the non-potential generalized forces.

Introducing Nielsen operators

$$N_s = \frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s}, \quad (s = 1, \ldots, n), \tag{2}$$

equation (1) can be expressed as

$$N_s(L) = Q_s, \quad (s = 1, \ldots, n). \tag{3}$$

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Considering the system to be non-singular, and using Eq. (3), all generalized accelerations can be obtained as

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, \ldots, n). \tag{4}$$

Introducing the infinitesimal transformations of groups of time and the generalized coordinates

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \end{aligned} \tag{5}$$

where ε is an infinitesimal parameter; ξ_0 and ξ_s are infinitesimal generators.

After transformations (5), the functions L and Q_s respectively become L^* and Q_s^*. Taking the Taylor expansions of L^* and Q_s^* at a point of $(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, we have

$$\begin{aligned} L^* &= L\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \\ &= L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon X^{(1)}(L) + O(\varepsilon^2), \end{aligned} \tag{6}$$

$$\begin{aligned} Q_s^* &= Q_s\left(t^*, \boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right) \\ &= Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon X^{(1)}(Q_s) + O(\varepsilon^2), \\ &\quad (s = 1, \ldots, n), \end{aligned} \tag{7}$$

where

$$X^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}. \tag{8}$$

Definition 1 If the dynamic functions L^* and Q_s^*, which are transformed by Eq. (5), are used respectively to replace the dynamic functions L and Q_s which are those before transformation, equation (3) remains unchanged in form, namely

$$N_s(L^*) = Q_s^*, \quad (s = 1, \ldots, n). \tag{9}$$

Hence, this symmetry is called the Mei symmetry of Nielsen equations for a holonomic system.

3. Criterion of Mei symmetry

Substituting Eqs. (6) and (7) into Eq. (9) and neglecting terms that are of the order of $O(\varepsilon^2)$ or of higher orders, and using Eq. (3), we obtain

$$N_s[X^{(1)}(L)] = X^{(1)}(Q_s), \quad (s = 1, \ldots, n). \tag{10}$$

Hence, we have

Criterion For the Nielsen equation (3), if the infinitesimal generators ξ_0 and ξ_s under infinitesimal transformations (5) satisfy expression (10), then the relevant symmetry is called the Mei symmetry of the system.

Equation (10) is called the criterion equation of the Mei symmetry for a holonomic system.

4. Mei conserved quantity directly deduced from Mei symmetry

In our previous study, we usually induced the Mei conserved quantities from Proposition 1.[24]

Proposition 1 If the infinitesimal generators ξ_0 and ξ_s and the gauge function $G_{\mathrm{M}} = G_{\mathrm{M}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Mei symmetry satisfy the following structural equation:

$$\begin{aligned} &X^{(1)}(L)\dot{\xi}_0 + X^{(1)}[X^{(1)}(L)] \\ &\quad + X^{(1)}(Q_s)(\xi_s - \dot{q}_s\xi_0) + \dot{G}_{\mathrm{M}} = 0, \end{aligned} \tag{11}$$

then the Mei conserved quantities directly deduced from the Mei symmetries are expressed as

$$\begin{aligned} I_{\mathrm{M}} &= X^{(1)}(L)\xi_0 + \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{M}} \\ &= \text{const}. \end{aligned} \tag{12}$$

In this paper, by studying Proposition 2, both the conditions and the expression of the new types of structural equation and conserved quantity of Mei symmetry for Nielsen equations of a holonomic system are obtained.

Proposition 2 If the infinitesimal generators ξ_0 and ξ_s and the gauge function $G_{\mathrm{X}} = G_{\mathrm{X}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Mei symmetry for Nielsen equations of a holonomic system satisfy the following new types of structural equation:

$$\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} - X^{(1)}(Q_s)\dot{q}_s + \dot{G}_{\mathrm{X}} = 0, \tag{13}$$

then the new types of conserved quantities directly deduced from the Mei symmetries are expressed as

$$\begin{aligned} I_{\mathrm{X}} &= X^{(1)}(L)\xi_0 - \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\dot{q}_s + G_{\mathrm{X}} \\ &= \text{const}. \end{aligned} \tag{14}$$

Proof To calculate the derivatives of Eq. (14)

with respect to t,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{X}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[X^{(1)}(L)\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right]\dot{q}_s \\ &\quad - \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\ddot{q}_s + \dot{G}_{\mathrm{X}} \\ &= \frac{\partial X^{(1)}(L)}{\partial t} + \frac{\partial X^{(1)}(L)}{\partial q_s}\dot{q}_s \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right]\dot{q}_s + \dot{G}_{\mathrm{X}} \\ &= \frac{\partial X^{(1)}(L)}{\partial t} - E_s\left[X^{(1)}(L)\right]\dot{q}_s + \dot{G}_{\mathrm{X}}, \quad (15)\end{aligned}$$

where

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}$$

is the Euler operators. Pay attention to the equivalence between Euler operators and Nielsen operators.

Substituting Eq. (13) into Eq. (15) and note Eq. (10), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{X}} &= \frac{\partial X^{(1)}(L)}{\partial t} - E_s\left[X^{(1)}(L)\right]\dot{q}_s + \dot{G}_{\mathrm{X}} \\ &= \frac{\partial X^{(1)}(L)}{\partial t} - N_s\left[X^{(1)}(L)\right]\dot{q}_s + \dot{G}_{\mathrm{X}} \\ &= \left\{X^{(1)}(Q_s) - N_s\left[X^{(1)}(L)\right]\right\}\dot{q}_s \\ &= 0. \quad (16)\end{aligned}$$

For Nielsen equations of conservative systems, we know that $Q_s = 0$, then we have the following inferences.

Inference 1 If the infinitesimal generators ξ_0 and ξ_s of the Mei symmetry of Nielsen equations for conservative systems satisfy the equation $N_s[X^{(1)}(L)] = 0$, and there exists a gauge function $G_{\mathrm{X}} = G_{\mathrm{X}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ which satisfies the following equation:

$$\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} + \dot{G}_{\mathrm{X}} = 0, \quad (17)$$

then the conserved quantities directly deduced from the Mei symmetries are expressed as

$$\begin{aligned}I_{\mathrm{X}} &= X^{(1)}(L)\xi_0 - \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\dot{q}_s + G_{\mathrm{X}} \\ &= \mathrm{const}. \quad (18)\end{aligned}$$

Inference 2 If the infinitesimal generators ξ_0 and ξ_s of the Mei symmetry of Nielsen equations for a holonomic system satisfy the equation $X^{(1)}(L) = 0$, then the trivial conserved quantity deduced from Proposition 1, Proposition 2, and Inference 1 are expressed as

$$I_{\mathrm{X}} = 0. \quad (19)$$

5. An illustrative example

Consider a Lagrangian function of a holonomic system with three degrees of freedom, which is expressed as

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - q_3, \quad (20)$$

then the non-potential generalized forces will be

$$Q_1 = -\dot{q}_2^2, \quad Q_2 = \dot{q}_1^2, \quad Q_3 = 0. \quad (21)$$

Now study the Mei conserved quantity and new types of conserved quantity deduced directly from the Mei symmetry.

The differential equations of motion of the system are

$$\ddot{q}_1 = -\dot{q}_2^2, \quad \ddot{q}_2 = \dot{q}_1^2, \quad \ddot{q}_3 + 1 = 0. \quad (22)$$

Take the infinitesimal generators as follows:

$$\xi_0 = \xi_1 = \xi_2 = 0, \quad \xi_3 = -\dot{q}_3 - t. \quad (23)$$

Using Eq. (8), we have

$$\begin{aligned}&X^{(1)}(L) = \dot{q}_3 + t, \\ &N_s\left[X^{(1)}(L)\right] = X^{(1)}(Q_s) = 0, \\ &(s = 1, 2, 3). \quad (24)\end{aligned}$$

Using Eq. (23), we can easily verify criterion equation (10) to be true. We know that the infinitesimal generators in Eq. (22) are of Mei symmetries.

From structural equation (11), we have

$$G_{\mathrm{M}} = 0. \quad (25)$$

From expression (12) of the Mei conserved quantity, one can obtain

$$I_{\mathrm{M}} = -\dot{q}_3 - t = \mathrm{const}. \quad (26)$$

From structural equation (13), we have

$$G_{\mathrm{X}} = -t. \quad (27)$$

By virtue of expression (14) of new types of conserved quantities, one obtains

$$I_{\mathrm{X}} = -\dot{q}_3 - t = \mathrm{const}. \quad (28)$$

6. Conclusions

It can be seen that the Mei conserved quantity (25) from the example is the same as the new type of conserved quantity (27), but it is simpler to calculate the structural equation (13) of Proposition 2 than Eq. (11) of Proposition 1. It shows the correctness and the superiority of Proposition 2. At the same time, for the same group of infinitesimal generators ξ_0 and ξ_s, we must point out two points as follows:

(i) generally speaking, the conserved quantities deduced from are different;

(ii) sometimes, using Proposition 1, we obtain only trivial conserved quantity, but we can obtain nontrivial conserved quantity from Proposition 2.

The above-mentioned two points show the difference between Proposition 1 and Proposition 2.

The result will develop the theories of Mei symmetries and Mei conserved quantities of Nielsen equations for a holonomic system, and will spread to the nonholonomic constrained mechanic system, etc.

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