Demonstrating an additional law of relativistic velocities based on squeezed light*

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The special relativity is the foundation for many branches of modern physics, of which the theoretical results are far beyond our daily experience and hard to realized in kinematic experiments. However, its outcomes could be demonstrated by making use of the convenient substitute, i.e., the squeezed light in the present paper. The squeezed light is very important in the field of quantum optics, and the corresponding transformation can be regarded as the coherent state of SU(1,1). In this paper, the connection between the squeezed operator and the Lorentz boost is built under certain conditions. Furthermore, the additional law of relativistic velocities and the angle of the Wigner rotation are deduced as well.

Keywords: squeezed states, special relativity, quantum optics, Lie groups

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1. Introduction

The special relativity discovered by Einstein in 1905 is the foundation for many branches of modern physics, of which relevant subjects[1] are studied even in recent years. Its outcomes can be observed prominently when the speed of an object approaches to that of light. So the theoretical results are far beyond our daily experience and hard to realized in kinematic experiments. Nevertheless, the squeezed light may supply a possible experimental test in modern optics laboratories. By using the canonical transformation of the Wigner distribution function in the phase space, the theoretical schemes of possible experimental tests for the Wigner rotation and the Thomas precession have been put forward.[2,3] However, the additional law of relativistic velocities, which is distinct from the usual additional law of vectors, has not been obtained. In the present paper, we not only acquire the corresponding additional law but also use a more direct method, which will enclose the essence of the analogue.

As a nonclassical light field, the squeezed light has a fundamental role in the development of quantum optics, which preserves the minimum-uncertainty product in the phase space and may exhibit many interesting properties, such as sub-Poissonian photon counting statistics and photon antibunching.[4,5] Its nonlinear generalization and the corresponding features were studied by Kwek and Kiang.[6] In addition, its noncyclic and nonunitary geometric phase was formulated by Yang *et al.*[7] And the extension to the two-mode squeezed light[8,9] was also studied.

This article is organized as follows. In Section 2, we review some indispensable concepts of squeezed optics and the special relativity necessary for the present paper. In Section 3, the connection between the squeezed operator and the Lorentz boost is built. In Section 4, the additional law of relativistic velocities and the angle of the Wigner rotation are deduced. Moreover, a possible experimental test on the additional law of relativistic velocities is put forward. At the end of this paper, a conclusion is drawn.

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2. Reviews of the squeezed state and the special relativity

Squeezed state operator

$$S(\beta) = \exp\left(\frac{1}{2}\beta a^{\dagger 2} - \frac{1}{2}\beta^* a^2\right) \tag{1}$$

can be regarded as a coset space of SU(1, 1), which is SU(1, 1)/U(1), where β is a complex number, $*$ denotes the conjugate operation, $a^\dagger$ and a are creation and annihilation operators, respectively, and they satisfy the canonical commutation relation $[a, a^\dagger] = 1$. In order to demonstrate this idea clearly, let us review the necessary knowledge about SU(1, 1).[10] It has generators satisfying the following commutation relations:

$$[K_1, K_2] = -\mathrm{i}K_0, \quad [K_0, K_1] = \mathrm{i}K_2,$$
$$[K_0, K_2] = -\mathrm{i}K_1. \tag{2}$$

Via choosing another appropriate basis, the generators become $K_\pm = \pm i(K_1 \pm iK_2)$ and K_0. Therefore, the commutation relations are transformed to be

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0. \tag{3}$$

Furthermore, the definition of Perelomov's SU(1, 1) coherent state is given by

$$S(\beta) = \exp(\beta K_+ - \beta^* K_-). \tag{4}$$

The boson realization of SU(1, 1) is applied, then the generators become

$$K_+ = \frac{1}{2}a^{\dagger 2}, \quad K_- = \frac{1}{2}a^2, \quad K_0 = \frac{1}{4}(aa^\dagger + a^\dagger a), \tag{5}$$

which satisfy the commutation relations in Eq. (3). Substituting Eq. (5) into Eq. (4), we recover the squeezed operator (1). As SU(1, 1) is locally isomorphic to SO(2, 1), the coherent state of SU(1, 1) in another word of the squeezed operator may have a connection with the Lorentz boost, which will be proved in Section 3. In the following paragraph, we will review the indispensable knowledge about the special relativity.

Let us consider about $(2+1)$-dimensional space–time. There are two inertial frames of reference Σ and Σ' that are coincident at time $t = 0$, and the velocity of Σ' relative to Σ is $\boldsymbol{v} = (v^1, v^2)$. An event is denoted as $(x^0, x^1, x^2)^{\mathrm{T}}$ in Σ and $(y^0, y^1, y^2)^{\mathrm{T}}$ in Σ'. The two teams of coordinates are connected by a Lorentz boost $\mathscr{L}$, i.e., $(y^0, y^1, y^2)^{\mathrm{T}} = \mathscr{L}(x^0, x^1, x^2)^{\mathrm{T}}$, where $x^0 = ct$, and T denotes the operation of transposition. The $\mathscr{L}$ takes the form[11]

$$\mathscr{L} = \begin{pmatrix} \gamma & -\eta\gamma\dfrac{v^1}{v} & -\eta\gamma\dfrac{v^2}{v} \\ -\eta\gamma\dfrac{v^1}{v} & 1+\eta^2\dfrac{\gamma^2}{1+\gamma}\left(\dfrac{v^1}{v}\right)^2 & \eta^2\dfrac{\gamma^2}{1+\gamma}\dfrac{v^1}{v}\dfrac{v^2}{v} \\ -\eta\gamma\dfrac{v^2}{v} & \eta^2\dfrac{\gamma^2}{1+\gamma}\dfrac{v^1}{v}\dfrac{v^2}{v} & 1+\eta^2\dfrac{\gamma^2}{1+\gamma}\left(\dfrac{v^2}{v}\right)^2 \end{pmatrix}, \tag{6}$$

where $v = \sqrt{(v^1)^2 + (v^2)^2}$, $\eta = v/c$, and $\gamma = 1/\sqrt{1-\eta^2}$. To simplify Eq. (6), let us introduce auxiliary variables $\cosh\rho = \gamma$, $\sinh\rho = \eta\gamma$, $\cos\varphi = v^1/v$, and $\sin\varphi = v^2/v$, so equation (6) becomes

$$\mathscr{L} = \begin{pmatrix} \cosh\rho & -\sinh\rho\cos\varphi & -\sinh\rho\sin\varphi \\ -\sinh\rho\cos\varphi & 1+(\cosh\rho-1)\cos^2\varphi & (\cosh\rho-1)\cos\varphi\sin\varphi \\ -\sinh\rho\sin\varphi & (\cosh\rho-1)\cos\varphi\sin\varphi & 1+(\cosh\rho-1)\sin^2\varphi \end{pmatrix}, \tag{7}$$

where the formulae $\cosh^2\rho - \sinh^2\rho = 1$ and $\sinh^2\rho/(1+\cosh\rho) = \cosh\rho - 1$ are useful. If $\varphi = 0$ or $\pi/2$, we can check that the common Lorentz boost, which commonly appears in the standard text book about the special relativity, is obtained.

Further more, let us recall the additional law of relativistic velocities. The Σ, Σ', and Σ'' are arbitrary three inertial frames of reference that are coincident at the initial time. If the velocity of Σ' relative to Σ is $\boldsymbol{u}$, and the velocity of Σ'' relative to Σ' is $\boldsymbol{v}$, the velocity of Σ'' relative to Σ is $\boldsymbol{u} \oplus \boldsymbol{v}$, where $\oplus$ is the additional operation of relativistic velocities, which is distinct from the parallelogram law of velocities. Its concrete formula[11] reads

$$\boldsymbol{u} \oplus \boldsymbol{v} = \frac{1}{1+\dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{c^2}}\left[\boldsymbol{u} + \frac{1}{\gamma_u}\boldsymbol{v}\right.$$

$$+\frac{1}{c^2}\frac{\gamma_u}{1+\gamma_u}(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u}\Big]. \tag{8}$$

Observers at rest relative to Σ (relative to Σ') agree with observers at rest relative to Σ' (relative to Σ'') that their space–time coordinates are linked by a pure Lorentz transformation without rotation. However, if the two successive Lorentz boosts are in noncollinear directions, observers at rest relative to Σ agree with observers at rest relative to Σ'' that their coordinates are linked not only by a Lorentz transformation but also by a rotation, which is called the Wigner rotation.

3. Connecting squeezed transformation with Lorentz boost

In this section, our main goal is to prove that the squeezed transformation corresponds to the Lorentz boost. To begin with, we express a vector $(x^0\,,\,x^1\,,\,x^2)$ in $(2+1)$-dimensional space–time as

$$x = x^0K_0 - x^1K_1 - x^2K_2, \tag{9}$$

whose basis is (K_0, K_1, K_2), which are the generators of SU(1, 1). Moreover, let us perform the operation

$$\begin{aligned} &y^0K_0 - y^1K_1 - y^2K_2 \\ &= S(\beta)(x^0K_0 - x^1K_1 - x^2K_2)S^\dagger(\beta). \end{aligned} \tag{10}$$

In order to obtain the specific connection between (y^0, y^1, y^2) and (x^0, x^1, x^2), we must calculate the right-hand side of Eq. (10). During the calculation, the BCH formula[12] may be useful, which takes the form

$$\mathrm{e}^A B\,\mathrm{e}^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots. \tag{11}$$

By using Eq. (11) and the commutation relation (2) of SU(1, 1), we can obtain

$$\begin{aligned} &S(\beta)K_0S^\dagger(\beta) \\ &= \cosh\rho K_0 + \cos\varphi\sinh\rho K_1 + \sin\varphi\sinh\rho K_2, \\ &S(\beta)K_1S^\dagger(\beta) \\ &= \cos\varphi\sinh\rho K_0 + [1+(\cosh\rho-1)\cos^2\varphi]K_1 \\ &\quad + (\cosh\rho-1)\cos\varphi\sin\varphi K_2, \\ &S(\beta)K_2S^\dagger(\beta) \\ &= \sin\varphi\sinh\rho K_0 + (\cosh\rho-1)\cos\varphi\sin\varphi K_1 \\ &\quad + [1+(\cosh\rho-1)\sin^2\varphi]K_2, \end{aligned} \tag{12}$$

where

$$\beta = \frac{\rho}{2}\,\mathrm{e}^{\,\mathrm{i}(\pi/2-\varphi)} = \frac{\rho}{2}(\sin\varphi + \mathrm{i}\cos\varphi). \tag{13}$$

Substituting Eq. (12) into Eq. (10), we obtain

$$\begin{aligned} &y^0K_0 - y^1K_1 - y^2K_2 \\ &= (\cosh\rho x^0 - \sinh\rho\cos\varphi x^1 - \sinh\rho\sin\varphi x^2)K^0 \\ &\quad - \{-\sinh\rho\cos\varphi x^0 + [1+(\cosh\rho-1)\cos^2\varphi]x^1 \\ &\quad + [(\cosh\rho-1)\cos\varphi\sin\varphi]x^2\}K^1 \\ &\quad - \{-\sinh\rho\sin\varphi x^0 + [(\cosh\rho-1)\cos\varphi\sin\varphi]x^1 \\ &\quad + [1+(\cosh\rho-1)\sin^2\varphi]x^2\}K^2. \end{aligned}$$

Because the linear independence of the generators, the above equation can be written in a matrix form, which is

$$\begin{pmatrix} y^0 \\ y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} \cosh\rho & -\sinh\rho\cos\varphi & -\sinh\rho\sin\varphi \\ -\sinh\rho\cos\varphi & 1+(\cosh\rho-1)\cos^2\varphi & (\cosh\rho-1)\cos\varphi\sin\varphi \\ -\sinh\rho\sin\varphi & (\cosh\rho-1)\cos\varphi\sin\varphi & 1+(\cosh\rho-1)\sin^2\varphi \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix}. \tag{14}$$

By comparing with Eq. (7), we find that the above 3×3 coefficient matrix is identical to Lorentz boost (7). So it is proved that $S(\beta)xS^\dagger(\beta)$ can be regarded as the Lorentz boost under the condition $\beta = \frac{\rho}{2}(\sin\varphi + \mathrm{i}\cos\varphi)$. If we do the above calculations on the boson representation, the Lorentz boost can be realized by the above operation of quantum optics.

4. Squeezed transformation and additional law of relativistic velocities

The next task is to demonstrate the additional law of relativistic velocities by using the squeezed operator. We start with the formula

$$S(\beta_1)S(\beta_2) = S(\beta_3)R(\delta), \tag{15}$$

where $R(\delta) = \exp(\mathrm{i}K_0\delta) = \exp[\mathrm{i}\frac{1}{4}(2a^\dagger a + 1)\delta]$. From the two previous sections, we obtain the idea that equation (15) means that two Lorentz boosts are equal to not only a boost but also a rotation. In order to represent β_3 and δ by β_1 and β_2, let us perform the following operations on Eq. (15):

$$\begin{aligned} & S(\beta_1)S(\beta_2)aS(\beta_2)^\dagger S(\beta_1)^\dagger \\ = \; & S(\beta_3)R(\delta)aR(\delta)^\dagger S(\beta_3)^\dagger. \end{aligned} \tag{16}$$

With the help of the equations

$$\begin{aligned} S(\beta)aS(\beta)^\dagger &= a\cosh\frac{\rho}{2} - a^\dagger \mathrm{e}^{\mathrm{i}(\pi/2-\varphi)}\sinh\frac{\rho}{2}, \\ S(\beta)a^\dagger S(\beta)^\dagger &= a^\dagger\cosh r - a\,\mathrm{e}^{-\mathrm{i}\theta}\sinh r, \\ R(\delta)aR(\delta)^\dagger &= a\,\mathrm{e}^{-\mathrm{i}\delta/2}, \end{aligned}$$

the left-hand and the right-hand sides of Eq. (16) can be written as linear combinations of a and $a^\dagger$ respectively. As a and $a^\dagger$ are linear independent, the coefficient of a in the left-hand side of Eq. (16) coincides with the counterpart in the right-hand side, and the same is true for $a^\dagger$. Hence, we obtain

$$\begin{aligned} a\colon\; & \cosh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2} + \mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)}\sinh\frac{\rho_1}{2}\sinh\frac{\rho_2}{2} \\ = \; & \mathrm{e}^{-\mathrm{i}\delta/2}\cosh\frac{\rho_3}{2}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} a^\dagger\colon\; & \mathrm{e}^{-\mathrm{i}\varphi_1}\sinh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2} + \mathrm{e}^{-\mathrm{i}\varphi_2}\sinh\frac{\rho_2}{2}\cosh\frac{\rho_1}{2} \\ = \; & \mathrm{e}^{-\mathrm{i}\delta/2}\,\mathrm{e}^{-\mathrm{i}\varphi_3}\sinh\frac{\rho_3}{2}. \end{aligned} \tag{18}$$

Moreover, the above two complex Eqs. (17) and (18) can be converted into four real equations, which are

$$\begin{aligned} \cosh\frac{\rho_3}{2}\cos\frac{\delta}{2} &= \cosh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2} \\ &\quad + \cos(\varphi_2-\varphi_1)\sinh\frac{\rho_1}{2}\sinh\frac{\rho_2}{2}, \end{aligned} \tag{19}$$

$$\sin\frac{\delta}{2}\cosh\frac{\rho_3}{2} = \sinh(\varphi_2-\varphi_1)\sinh\frac{\rho_1}{2}\sinh\frac{\rho_2}{2}, \tag{20}$$

$$\begin{aligned} \cos\left(\theta_3+\frac{\delta}{2}\right)\sinh\frac{\rho_3}{2} &= \cos\varphi_1\sinh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2} \\ &\quad + \cos\varphi_2\sinh\frac{\rho_2}{2}\cosh\frac{\rho_1}{2}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \sin\left(\theta_3+\frac{\delta}{2}\right)\sinh\frac{\rho_3}{2} &= \sin\varphi_1\sinh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2} \\ &\quad + \sin\varphi_2\sinh\frac{\rho_2}{2}\cosh\frac{\rho_1}{2}. \end{aligned} \tag{22}$$

By using Eqs. (19) and (20), we can reach the following results:

$$\tan\frac{\delta}{2} = \frac{\sinh(\varphi_2-\varphi_1)\sinh\frac{\rho_1}{2}\sinh\frac{\rho_2}{2}}{\cosh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2} + \cos(\varphi_2-\varphi_1)\sinh\frac{\rho_1}{2}\sinh\frac{\rho_2}{2}}, \tag{23}$$

and

$$\begin{aligned} \cosh^2\frac{\rho_3}{2} &= \sinh^2\frac{\rho_1}{2}\sinh^2\frac{\rho_2}{2} + \cosh^2\frac{\rho_1}{2}\cosh^2\frac{\rho_2}{2} \\ &\quad + 2\cos(\varphi_2-\varphi_1)\sinh\frac{\rho_1}{2}\sinh\frac{\rho_2}{2} \\ &\quad \times \cosh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2}. \end{aligned} \tag{24}$$

We should note that equation (23) is just the formula for the Wigner angle.[13] Substituting formulae $\cosh 2x = \cosh^2 x + \sinh^2 x$, $\cosh^2 x - \sinh^2 x = 1$, and $\sinh 2x = 2\sinh x\cosh x$ into Eq. (24), we can obtain a more elegant and pragmatic expression about $\cosh\rho_3$, which is

$$\begin{aligned} \cosh\rho_3 &= \cosh\rho_1\cosh\rho_2 \\ &\quad + \cos(\varphi_2-\varphi_1)\sinh\rho_1\sinh\rho_2. \end{aligned} \tag{25}$$

In addition, let us introduce the following auxiliary variables:

$$\gamma_u = \cosh\rho_1, \quad \gamma_v = \cosh\rho_2, \quad \gamma_w = \cosh\rho_3,$$

$$\begin{aligned} \boldsymbol{u} &= c\tanh\rho_1\hat{\boldsymbol{u}}, \quad \boldsymbol{v} = c\tanh\rho_2\hat{\boldsymbol{v}}, \\ \boldsymbol{w} &= c\tanh\rho_3\hat{\boldsymbol{w}}, \end{aligned} \tag{26}$$

where $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{w}}$ are unit vectors along the directions of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, respectively, and c is the velocity of light propagating in the vacuum. Substituting Eq. (26) into Eq. (25), we can obtain a more meaningful formula

$$\gamma_w = \gamma_u\gamma_v\left(1 + \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{c^2}\right). \tag{27}$$

Now, we concentrate on Eqs. (21) and (22). By substituting $\cos\delta/2$ from Eq. (19) and $\sin\delta/2$ from Eq. (20) into them, they compose a set of equations about unknown elements $\cos\varphi_3$ and $\sin\varphi_3$, i.e.,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \cos\varphi_3 \\ \sin\varphi_3 \end{pmatrix} = \begin{pmatrix} E \\ F \end{pmatrix}, \tag{28}$$

where

$$A = -D = \tanh\frac{\rho_3}{2}\sin(\varphi_2-\varphi_1)\sinh\frac{\rho_1}{2}\sinh\frac{\rho_2}{2},$$
$$B = C = \tanh\frac{\rho_3}{2}\Bigg[\cosh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2}+\cos(\varphi_2-\varphi_1) \times \sinh\frac{\rho_1}{2}\sinh\frac{\rho_2}{2}\Bigg],$$
$$E = \cos\varphi_1\sinh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2}+\cos\varphi_2\sinh\frac{\rho_2}{2}\cosh\frac{\rho_1}{2},$$
$$F = \sin\varphi_1\sinh\frac{\rho_1}{2}\cosh\frac{\rho_2}{2}+\sin\varphi_2\sinh\frac{\rho_2}{2}\cosh\frac{\rho_1}{2}.$$

According to Cramer's rules, equation (28) can be solved. Hence we obtain

$$\begin{aligned}\cos\varphi_3 &= \begin{vmatrix} E & B \\ F & D \end{vmatrix}\Big/\begin{vmatrix} A & B \\ C & D \end{vmatrix} \\ &= \frac{1}{\sinh\rho_3}\{[\sinh\rho_1\cosh\rho_2+\sinh\rho_2(\cosh\rho_1-1) \\ &\quad\times(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)]\cos\varphi_1 \\ &\quad+\sinh\rho_2\cos\varphi_2\}, \qquad (29)\end{aligned}$$

$$\begin{aligned}\sin\varphi_3 &= \begin{vmatrix} A & B \\ C & D \end{vmatrix}\Big/\begin{vmatrix} A & B \\ C & D \end{vmatrix} \\ &= \frac{1}{\sinh\rho_3}\{[\sinh\rho_1\cosh\rho_2+\sinh\rho_2(\cosh\rho_1-1) \\ &\quad\times(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)]\sin\varphi_1 \\ &\quad+\sinh\rho_2\sin\varphi_2\}. \qquad (30)\end{aligned}$$

In order to disclose the meaning of the additional law of relativistic velocities, we introduce three unit vectors, which are

$$\hat{\boldsymbol{u}} = \cos\varphi_1\hat{\boldsymbol{i}}+\sin\varphi_1\hat{\boldsymbol{j}}, \quad \hat{\boldsymbol{v}} = \cos\varphi_2\hat{\boldsymbol{i}}+\sin\varphi_2\hat{\boldsymbol{j}},$$
$$\hat{\boldsymbol{w}} = \cos\varphi_3\hat{\boldsymbol{i}}+\sin\varphi_3\hat{\boldsymbol{j}}.$$

Hence equations (29) and (30) become

$$\begin{aligned}\hat{\boldsymbol{w}} &= \frac{1}{\sinh\rho_3}\{[\sinh\rho_1\cosh\rho_2 \\ &\quad+\sinh\rho_2(\cosh\rho_1-1)\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{v}}]\hat{\boldsymbol{u}}+\sinh\rho_2\hat{\boldsymbol{v}}\}. \qquad (31)\end{aligned}$$

Moreover, substituting Eqs. (26) and (27) into Eq. (31), we obtain the final addition law

$$\boldsymbol{w} = \frac{1}{1+\dfrac{\boldsymbol{u}\cdot\boldsymbol{v}}{c^2}}\left(\boldsymbol{u}+\frac{\gamma_u}{\gamma_u+1}\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{c^2}\boldsymbol{u}+\frac{1}{\gamma_u}\boldsymbol{v}\right), \quad (32)$$

which is identical with Eq. (8). When $c\to\infty$, the common additional law of velocities

$$\boldsymbol{w} = \boldsymbol{u}+\boldsymbol{v}$$

in the Galilean transformation is obtained. The relations between squeezed parameters ($\beta_1 = \frac{\rho_1}{2}\,\mathrm{e}^{\mathrm{i}(\pi/2-\varphi_1)}$, $\beta_2 = \frac{\rho_2}{2}\,\mathrm{e}^{\mathrm{i}(\pi/2-\varphi_2)}$, $\beta_3 = \frac{\rho_3}{2}\,\mathrm{e}^{\mathrm{i}(\pi/2-\varphi_3)}$ in Eq. (15)) and velocities ($\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in Eq. (32)) are illustrated in the following table.

Table 1. Relations between squeezed parameters and velocities.

Velocity	Modulo	Directional vector
$\boldsymbol{u}$	$c\tanh\rho_1$	$(\cos\varphi_1,\sin\varphi_1)$
$\boldsymbol{v}$	$c\tanh\rho_2$	$(\cos\varphi_2,\sin\varphi_2)$
$\boldsymbol{w}$	$c\tanh\rho_3$	$(\cos\varphi_3,\sin\varphi_3)$

Depending on the analysis above, the additional law of relativistic velocities has been successfully obtained from squeezed optics. Furthermore, the theoretical results can be possibly verified by the following experiment. In practice, it is much easier to realize the two-mode squeezed state

$$S^{\mathrm{tm}}(\beta) = \exp\left[\frac{1}{2}(\beta a_1^\dagger a_2^\dagger+\beta^* a_1 a_2)\right]$$

than the one-mode one, where subscripts 1 and 2 denote the two different photons, and superscript tm represents two-mode, thus we choose the two-mode process. Because the generators of the two-mode squeezed operator obey the same Lie algebra as that of the one-mode squeezed operator, our analysis above also applies. At first, a light beam is splitted into two beams. Then beam 1 goes through a nondegenerate optical parametric amplifier (NOPA), which can generate a two-mode squeezed vacuum state.[14] As a result, beam 1 becomes $S^{\mathrm{tm}}(\beta_2)|00\rangle$. Moreover, it continues to pass the equipment that makes the light become $S^{\mathrm{tm}}(\beta_1)S^{\mathrm{tm}}(\beta_2)|00\rangle$. Beam 2 is changed to be $S^{\mathrm{tm}}(\beta_3)|00\rangle$ by the NOPA, where β_3 is calculated from β_1 and β_2 according to Eqs. (25), (29), and (30). After all the transformations on beams 1 and 2, let them interfere with each other. Then the dark and bright fringes can be observed. The separation of adjacent bright fringes is δ'. If δ' coincides with δ of Eq. (23), then the additional law of relativistic velocities is demonstrated.

5. Conclusion

In summary, since the squeezed transformations constitute a coset spac of SU(1, 1), the Lorentz transformations make up a Lorentz group, and the SU(1, 1) group is locally isomorphic to the (2 + 1)-dimensional Lorentz group.[10] We demonstrate the phenomenon of the special relativity by using squeezed optics. We

prove that squeezed transformation (10) is equivalent to Lorentz boost (14) under condition (13). Furthermore, the additional law of relativistic velocities (32) and the angle of the Wigner rotation (23) are deduced as well.

Moreover, a possible experimental test on the additional law of relativistic velocities is also discussed.

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