Stability and attractive basins of multiple equilibria in delayed two-neuron networks*

Huang Yu-Jiao(黄玉娇), Zhang Hua-Guang(张化光)†, and Wang Zhan-Shan(王占山)

College of Information Science and Engineering, Northeastern University, Shenyang 110819, *China*

(Received 6 January 2012; revised manuscript received 9 February 2012)

Multiple stability for two-dimensional delayed recurrent neural networks with piecewise linear activation functions of $2r$ ($r \geq 1$) corner points is studied. Sufficient conditions are established for checking the existence of $(2r+1)^2$ equilibria in delayed recurrent neural networks. Under these conditions, $(r+1)^2$ equilibria are locally exponentially stable, and $(2r+1)^2-(r+1)^2-r^2$ equilibria are unstable. Attractive basins of stable equilibria are estimated, which are larger than invariant sets derived by decomposing state space. One example is provided to illustrate the effectiveness of our results.

Keywords: delayed recurrent neural network, multiple equilibria, stability, attractive basin

PACS: 07.05.Mh, 02.30.Ks **DOI:** 10.1088/1674-1056/21/7/070701

1. Introduction

Recently, recurrent neural networks (RNNs) have been paid much attention due to their extensive applications, such as pattern recognition, signal processing, combinatorial optimization, and associative memories.[1−5] The time delay of neural networks is unavoidable because of the finite speed of a signal's switch and transmission. Therefore, the stability of delayed recurrent neural networks (DRNNs) has become a topic of great theoretical and practical importance. Marcus and Westervelt first introduced a single delay into Hopfield neural networks.[6] Many studies on stability for various DRNNs have been performed in recent years.[7−10]

Most relevant studies are mainly focused on the existence and stability of one unique equilibrium point for RNNs. However, for some applications including associative memories, pattern recognition, image processing, and many other fields, it is desirable that neural networks have multiple equilibria, and each of them represents an individual pattern.[11−13] Recently, stability of multiple equilibria for RNNs has been extensively studied.[14−22] Chen and Amari[14] reported that one-neuron neural network model has two locally stable equilibria. Recent studies generalized their work to n-neuron neural networks. Sufficient conditions were established in Refs. [15]–[17], which ensure the existence of 2^n stable equilibria in n-dimensional DRNNs with one step piecewise linear activation function. In order to increase storage capacity, Zeng *et al*. studied RNNs with time-varying delays and k ($k \geq 1$) steps piecewise linear activation functions in Ref. [20], and some sufficient conditions were established to ensure that n-neuron DRNNs with k-stair activation functions could have $(2k)^n$ locally exponentially stable equilibria.

In recent years, various estimates on attractive basins of stable equilibria were derived for some kinds of neural networks.[23−25] It is well known that time delay may affect attractive basins of equilibria for neural networks. Changing time delay can alter the attractive basin boundaries of stable equilibria, which can be of prime importance in associative memory networks where the positions of basin boundaries determine what information is retrieved for a given initial condition.[26] Hence, estimating the attractive basins of equilibria for delayed neural networks is of great significance. Since existing literatures estimating attractive basins of multiple stable equilibria for DRNNs are based on convergence analysis of partial regions of state space, these small attractive basins are only lo-

*Project supported by the National Natural Science Foundation of China (Grant Nos. 50977008, 61034005, and 61074073), the National Basic Research Program of China (Grant No. 2009CB320601), the Program for New Century Excellent Talents in Universities of China (Grant No. NCET-10-0306), and the Fundamental Research Funds for the Central Universities of China (Grant Nos. N110604005 and N110504001).

†Corresponding author. E-mail: zhanghuaguang@mail.neu.edu.cn

© 2012 Chinese Physical Society and IOP Publishing Ltd

cated in some invariant sets obtained by decomposing the state space (see Refs. [17] and [20]). However, how to enlarge the attractive basins of stable equilibria for DRNNs is an attractive topic and their potential extensive applications has motivated the present study.

In this paper, we will explore the stability of multiple equilibria and enlarge the attractive basins of stable equilibria in two-neuron networks with a time delay and piecewise linear activation functions of $2r$ ($r \geq 1$) corner points. Different from previous literatures, our work makes two main contributions as follows.

1) Based on the analysis of difference–differential equation and Lyapunov's first approximate theory, it is the first time that the stability of $(2r+1)^2-(r+1)^2-r^2$ equilibria in two-dimensional DRNNs is presented. These equilibria are unstable.

2) By analyzing the convergence of neural network states in some subregions of state space, we establish larger attractive basins of stable equilibria than invariant sets derived based on the decomposition of state space.

The remainder of this paper consists of the following sections. In Section 2, we describe the problem formulation. The main results are presented in Section 3. One example is used to test the effectiveness of obtained results in Section 4. In Section 5, the conclusion is drawn.

2. Problem formulation

A delayed two-neuron network can be formulated as follows:

$$\begin{aligned}\dot{x}_1(t) &= -c_1x_1(t) + a_{11}f_1(x_1(t)) \\ &\quad + a_{12}f_2(x_2(t)) + b_{11}f_1(x_1(t-\tau_{11})) \\ &\quad + b_{12}f_2(x_2(t-\tau_{12})) + I_1, \\ \dot{x}_2(t) &= -c_2x_2(t) + a_{21}f_1(x_1(t)) \\ &\quad + a_{22}f_2(x_2(t)) + b_{21}f_1(x_1(t-\tau_{21})) \\ &\quad + b_{22}f_2(x_2(t-\tau_{22})) + I_2, \end{aligned} \tag{1}$$

where $c_1 > 0$, $c_2 > 0$; a_{ij} and b_{ij} $(i,j = 1,2)$ are connection weights from neuron j to neuron i; f_1 and f_2 are activation functions; $0 \leq \tau_{ij} \leq \tau := \max\limits_{1\leq i,j\leq 2}\{\sup\{\tau_{ij}\}\}$ are time delays; I_1 and I_2 represent external inputs.

It is well known that activation functions play an important role in the dynamical analysis of recurrent neural networks. The existence and stability of equilibria for RNNs depend mainly on the characteristics of activation functions. In this paper, we shall consider the following general piecewise linear activation functions of $2r$ ($r \geq 1$) corner points:

$$f_j(\xi) = \begin{cases} m_j^1, & \xi \in (-\infty, p_j^1), \\ \dfrac{m_j^2 - m_j^1}{q_j^1 - p_j^1}(\xi - p_j^1) + m_j^1, & \xi \in [p_j^1, q_j^1], \\ m_j^2, & \xi \in (q_j^1, p_j^2), \\ \dfrac{m_j^3 - m_j^2}{q_j^2 - p_j^2}(\xi - p_j^2) + m_j^2, & \xi \in [p_j^2, q_j^2], \\ m_j^3, & \xi \in (q_j^2, p_j^3), \\ \cdots & \cdots \\ \dfrac{m_j^{r+1} - m_j^r}{q_j^r - p_j^r}(\xi - p_j^r) + m_j^r, & \xi \in [p_j^r, q_j^r], \\ m_j^{r+1}, & \xi \in (q_j^r, +\infty), \end{cases} \tag{2}$$

where $\{m_j^k\}$ $(k = 1, 2, \ldots, r+1)$ is an increasing constant series with $-\infty < m_j^1 < m_j^2 < \cdots < m_j^{r+1} < +\infty$, p_j^k and q_j^k $(k = 1, 2, \ldots, r; j = 1, 2)$ are constants satisfying $-\infty < p_j^1 < q_j^1 < p_j^2 < q_j^2 < \cdots < p_j^r < q_j^r < +\infty$. The typical configuration of activation function (2) is shown in Fig. 1.

By using piecewise linear activation functions with a few corner points, we can design neural networks with an arbitrary number of stable equilibria, which really enlarge the storage capacity of associative memories.

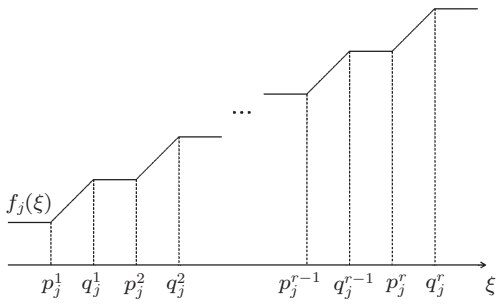


Fig. 1. Typical configuration of activation function (2).

Let $\mathcal{C}([-\tau, 0], \mathbb{R}^2)$ be the Banach space of functions mapping $[-\tau, 0]$ into $\mathbb{R}^2$. The initial condition of system (1) is assumed to be $\boldsymbol{\phi}(\theta) = (\phi_1(\theta), \phi_2(\theta))^{\mathrm{T}}$, where $\boldsymbol{\phi}(\theta) \in \mathcal{C}([-\tau, 0], \mathbb{R}^2)$. Assuming that

$$\begin{aligned} G_i(\xi) &= -c_i\xi + (a_{ii} + b_{ii})f_i(\xi) \\ &\quad + \max_{j\neq i}\{(a_{ij} + b_{ij})m_j^1, (a_{ij} + b_{ij})m_j^{r+1}\} + I_i, \\ H_i(\xi) &= -c_i\xi + (a_{ii} + b_{ii})f_i(\xi) \\ &\quad + \min_{j\neq i}\{(a_{ij} + b_{ij})m_j^1, (a_{ij} + b_{ij})m_j^{r+1}\} + I_i, \end{aligned}$$

where $i, j = 1, 2$, we have $H_i(\xi) < G_i(\xi)$. Next, we list two conditions needed throughout this paper.

Condition 1

$$-c_i + \min_{k\in\{1,2,\ldots,r\}} \left\{ (a_{ii} + b_{ii}) \frac{m_i^{k+1} - m_i^k}{q_i^k - p_i^k} \right\} > 0,$$
$$i = 1, 2.$$

Condition 2

$$G_i(p_i^k) < 0,\ H_i(q_i^k) > 0,$$
$$i = 1, 2, \quad k = 1, 2, \ldots, r.$$

If $\xi < p_i^1$ or $q_i^k < \xi < p_i^{k+1}$ $(k = 1, 2, \ldots, r-1)$ or $\xi > q_i^r$ $(i = 1, 2)$, then $f_i(\xi)$ is differential and $\dot{f}_i(\xi) = 0$. We obtain that $\dot{G}_i(\xi) = -c_i < 0$ and $\dot{H}_i(\xi) = -c_i < 0$, and $G_i(\xi)$ and $H_i(\xi)$ decrease. If $p_i^k < \xi < q_i^k$ $(k = 1, 2, \ldots, r;\ i = 1, 2)$, then $f_i(\xi)$ is differential and $\dot{f}_i(\xi) = (m_i^{k+1} - m_i^k)/(q_i^k - p_i^k)$. We derive from Condition 1 that

$$\begin{aligned}\dot{G}_i(\xi) &= -c_i + (a_{ii} + b_{ii}) \frac{m_i^{k+1} - m_i^k}{q_i^k - p_i^k} \\ &> -c_i + \min_{k\in\{1,2,\ldots,r\}} \left\{ (a_{ii} + b_{ii}) \frac{m_i^{k+1} - m_i^k}{q_i^k - p_i^k} \right\} \\ &> 0.\end{aligned}$$

Similarly, we have $\dot{H}_i(\xi) > 0$. Therefore, $G_i(\xi)$ and $H_i(\xi)$ increase. Based on Conditions 1 and 2 as well as the above discussion, $G_i(\xi)$ and $H_i(\xi)$ can be drawn in Fig. 2.

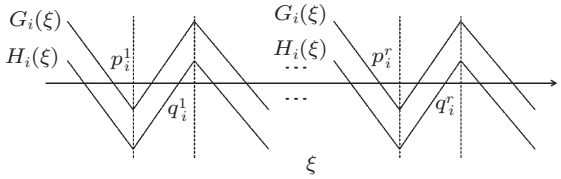


Fig. 2. Graphs for $G_i(\xi)$ and $H_i(\xi)$.

Assuming that

$$\begin{aligned}(-\infty, p_i^1) &= (-\infty, p_i^1)^1 \times [p_i^1, q_i^1]^0 \times (q_i^1, p_i^2)^0 \times \cdots \\ &\quad \times [p_i^r, q_i^r]^0 \times (q_i^r, +\infty)^0, \\ [p_i^1, q_i^1] &= (-\infty, p_i^1)^0 \times [p_i^1, q_i^1]^1 \times (q_i^1, p_i^2)^0 \times \cdots \\ &\quad \times [p_i^r, q_i^r]^0 \times (q_i^r, +\infty)^0, \\ (q_i^1, p_i^2) &= (-\infty, p_i^1)^0 \times [p_i^1, q_i^1]^0 \times (q_i^1, p_i^2)^1 \times \cdots \\ &\quad \times [p_i^r, q_i^r]^0 \times (q_i^r, +\infty)^0, \\ &\quad \cdots \\ [p_i^r, q_i^r] &= (-\infty, p_i^1)^0 \times [p_i^1, q_i^1]^0 \times (q_i^1, p_i^2)^0 \times \cdots \\ &\quad \times [p_i^r, q_i^r]^1 \times (q_i^r, +\infty)^0, \\ (q_i^r, +\infty) &= (-\infty, p_i^1)^0 \times [p_i^1, q_i^1]^0 \times (q_i^1, p_i^2)^0 \times \cdots \\ &\quad \times [p_i^r, q_i^r]^0 \times (q_i^r, +\infty)^1,\end{aligned}$$

we divide $\mathbb{R}^2$ into $(2r+1)^2$ regions

$$\begin{aligned}\boldsymbol{\Phi} \triangleq \Big\{ &\prod_{i=1}^{2} (-\infty, p_i^1)^{\delta_i^{(1)}} \times [p_i^1, q_i^1]^{\delta_i^{(2)}} \times (q_i^1, p_i^2)^{\delta_i^{(3)}} \times \cdots \\ &\times [p_i^r, q_i^r]^{\delta_i^{(2r)}} \times (q_i^r, +\infty)^{\delta_i^{(2r+1)}} \Big\},\end{aligned}$$

where $(\delta_i^{(1)}, \delta_i^{(2)}, \ldots, \delta_i^{(2r+1)}) = (1, 0, \ldots, 0)$ or $(0, 1, \ldots, 0)$ or $\cdots$ or $(0, 0, \ldots, 1)$.

Define

$$\begin{aligned}\boldsymbol{\Phi}_1 = \Big\{ &\prod_{i=1}^{2} (-\infty, p_i^1)^{\delta_i^{(1)}} \times [p_i^1, q_i^1]^0 \times (q_i^1, p_i^2)^{\delta_i^{(2)}} \times \cdots \\ &\times [p_i^r, q_i^r]^0 \times (q_i^r, +\infty)^{\delta_i^{(r+1)}} \Big\},\end{aligned}$$

where $(\delta_i^{(1)}, \delta_i^{(2)}, \ldots, \delta_i^{(r+1)}) = (1, 0, \ldots, 0)$ or $(0, 1, \ldots, 0)$ or $\cdots$ or $(0, 0, \ldots, 1)$,

$$\begin{aligned}\boldsymbol{\Phi}_2 = \Big\{ &\prod_{i=1}^{2} (-\infty, p_i^1)^0 \times [p_i^1, q_i^1]^{\delta_i^{(1)}} \times (q_i^1, p_i^2)^0 \times \cdots \\ &\times [p_i^r, q_i^r]^{\delta_i^{(r)}} \times (q_i^r, +\infty)^0 \Big\},\end{aligned}$$

where $(\delta_i^{(1)}, \delta_i^{(2)}, \ldots, \delta_i^{(r)}) = (1, 0, \ldots, 0)$ or $(0, 1, \ldots, 0)$ or $\cdots$ or $(0, 0, \ldots, 1)$,

$$\boldsymbol{\Phi}_3 = \{\boldsymbol{\Phi} - \boldsymbol{\Phi}_1 - \boldsymbol{\Phi}_2\}.$$

It can be seen that $\boldsymbol{\Phi}_1$ is composed of $(r+1)^2$ regions, $\boldsymbol{\Phi}_2$ is composed of r^2 regions, and $\boldsymbol{\Phi}_3$ is composed of $(2r+1)^2 - (r+1)^2 - r^2$ regions.

Lemma 1 Let f be a continuous real function in the interval $[a, b]$. If $f(a) < f(b)$ and c is a real number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.

3. Main results

In this section, we will investigate the existence and stability of equilibria for system (1) with activation function (2). Moreover, we will establish attractive basins of stable equilibria.

Theorem 1 Under Conditions 1 and 2, the following propositions are true:

i) There exist $(2r+1)^2$ equilibria for system (1) with activation function (2), which are located in each subset of $\boldsymbol{\Phi}$.

ii) For any subset $\boldsymbol{\Omega}$ of $\boldsymbol{\Phi}_1$, $\boldsymbol{\Omega}$ is positive invariant.

iii) For any subset $\boldsymbol{\Xi}$ of $\boldsymbol{\Phi}_1$, the equilibrium point in $\boldsymbol{\Xi}$ is exponentially stable.

iv) For any subset $\boldsymbol{\Theta}$ of $\boldsymbol{\Phi}_3$, the equilibrium point in $\boldsymbol{\Theta}$ is unstable.

Proof a) The stationary equation of system (1) is expressed as follows:

$$\begin{aligned} &-c_i x_i + (a_{ii} + b_{ii}) f_i(x_i) \\ &+ (a_{ij} + b_{ij}) f_j(x_j) + I_i = 0, \\ &i \neq j, \quad i, j = 1, 2. \end{aligned} \tag{3}$$

For any given $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T \in \boldsymbol{\Phi}$, we define

$$\begin{aligned} F_i(\xi) = &-c_i \xi + (a_{ii} + b_{ii}) f_i(\xi) \\ &+ (a_{ij} + b_{ij}) f_j(x_j) + I_i, \quad i \neq j, \quad i, j = 1, 2. \end{aligned}$$

From the definitions of $F_i(\cdot)$, $G_i(\cdot)$, and $H_i(\cdot)$, it is easy to see that

$$\begin{aligned} &F_i(-\infty) > H_i(-\infty) > 0, \\ &F_i(p_i^k) < G_i(p_i^k) < 0, \\ &F_i(q_i^k) > H_i(q_i^k) > 0, \\ &F_i(+\infty) < G_i(+\infty) < 0, \end{aligned}$$

where $i = 1, 2$, $k = 1, 2, \ldots, r$. According to Lemma 1, the function $F_i(\xi) = 0$ has $2r+1$ solutions in $\mathbb{R}$. They are in $(-\infty, p_i^1)$, (p_i^k, q_i^k), and $(q_i^r, +\infty)$, respectively. Therefore, the stationary equation (3) has $(2r+1)^2$ equilibria in $\mathbb{R}^2$. They are located in every subset of $\boldsymbol{\Phi}$.

Proof b) Suppose that $\boldsymbol{\Omega}$ is a subset of $\boldsymbol{\Phi}_1$. For any initial condition $\boldsymbol{\phi}(\theta) = (\phi_1(\theta), \phi_2(\theta))^{\mathrm{T}} \in \boldsymbol{\Omega}$, we claim that the solution $\boldsymbol{x}(t)$ will stay in $\boldsymbol{\Omega}$ for all $t \geq 0$. If this is not true, then there exists a t_0 larger than 0 such that $\boldsymbol{x}(t_0)$ escapes from $\boldsymbol{\Omega}$. That is, there exist $i_0 \in \{1, 2\}$ and $k_0 \in \{1, 2, \ldots, r\}$ such that $x_{i_0}(t_0) = p_{i_0}^{k_0}$, $\dot{x}_{i_0}(t_0) > 0$ or $x_{i_0}(t_0) = q_{i_0}^{k_0}$, $\dot{x}_{i_0}(t_0) < 0$. For the first case, we derive from system (1) that

$$\begin{aligned} \dot{x}_{i_0}(t_0) = &-c_{i_0} x_{i_0}(t_0) + (a_{i_0 i_0} + b_{i_0 i_0}) f_{i_0}(x_{i_0}(t_0)) \\ &+ (a_{i_0 j} + b_{i_0 j}) f_j(x_j(t_0)) + I_{i_0} \\ = &-c_{i_0} p_{i_0}^{k_0}(t_0) + (a_{i_0 i_0} + b_{i_0 i_0}) f_{i_0}(p_{i_0}^{k_0}) \\ &+ (a_{i_0 j} + b_{i_0 j}) f_j(x_j(t_0)) + I_{i_0} \\ < &-c_{i_0} p_{i_0}^{k_0}(t_0) + (a_{i_0 i_0} + b_{i_0 i_0}) f_{i_0}(p_{i_0}^{k_0}) \\ &+ \max\{(a_{i_0 j} + b_{i_0 j}) m_j^1, (a_{i_0 j} + b_{i_0 j}) m_j^r\} + I_{i_0} \\ = &\, G_{i_0}(p_{i_0}^{k_0}) \\ < &\, 0, \end{aligned} \tag{4}$$

where $j \neq i_0$. This yields a contradiction to $\dot{x}_{i_0}(t_0) > 0$. Similarly, the second case is either not true. Therefore, $\boldsymbol{x}(t)$ will stay in $\boldsymbol{\Omega}$ for all $t \geq 0$. That is, $\boldsymbol{\Omega}$ is positively invariant.

Proof c) Suppose $\boldsymbol{\Xi}$ is any subset of $\boldsymbol{\Phi}_1$. $\boldsymbol{x}^* = (x_1^*, x_2^*)^{\mathrm{T}}$ is the equilibrium point in $\boldsymbol{\Xi}$. Let $z_i(t) = x_i(t) - x_i^*$ $(i = 1, 2)$. From Eq. (2), when $\boldsymbol{x}(t) \in \boldsymbol{\Xi}$, $\sum_{j=1}^{n} f_j(x_j(t))$ is a constant, i.e., $\sum_{j=1}^{n} f_j(x_j(t)) = \sum_{j=1}^{n} f_j(x_j^*)$. Therefore, when $\boldsymbol{x}(t)$, $\boldsymbol{x}(t-\tau)$, and $\boldsymbol{x}^* \in \boldsymbol{\Xi}$, system (1) can be rewritten as

$$\dot{z}_1(t) = -c_1 z_1(t), \quad \dot{z}_2(t) = -c_2 z_2(t), \tag{5}$$

which indicates that $\boldsymbol{x}^* \in \boldsymbol{\Xi}$ is exponentially stable.

Proof d) Suppose $\boldsymbol{\Theta}$ is a subset of $\boldsymbol{\Phi}_3$. $\boldsymbol{x}^* = (x_1^*, x_2^*)^{\mathrm{T}}$ is the equilibrium point in $\boldsymbol{\Theta}$. The linear equation of system (1) at the equilibrium point $\boldsymbol{x} = \boldsymbol{x}^*$ reads

$$\begin{aligned} \dot{u_1}(t) = &-c_1 u_1(t) + a_{11} \dot{f}_1(x_1^*) u_1(t) + a_{12} \dot{f}_2(x_2^*) u_2(t) \\ &+ b_{11} \dot{f}_1(x_1^*) u_1(t - \tau_{11}) \\ &+ b_{12} \dot{f}_2(x_2^*) u_2(t - \tau_{12}), \\ \dot{u_2}(t) = &-c_2 u_2(t) + a_{21} \dot{f}_1(x_1^*) u_1(t) + a_{22} \dot{f}_2(x_2^*) u_2(t) \\ &+ b_{21} \dot{f}_1(x_1^*) u_1(t - \tau_{21}) \\ &+ b_{22} \dot{f}_2(x_2^*) u_2(t - \tau_{22}). \end{aligned} \tag{6}$$

The characteristic equation of Eq. (6) is written as follows:

$$\begin{vmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_4 \end{vmatrix} = 0, \tag{7}$$

where

$$\begin{aligned} \Lambda_1 &= \lambda - [-c_1 + a_{11} \dot{f}_1(x_1^*) + b_{11} \dot{f}_1(x_1^*) \exp(-\lambda \tau_{11})], \\ \Lambda_2 &= -[a_{12} \dot{f}_2(x_2^*) + b_{12} \dot{f}_2(x_2^*) \exp(-\lambda \tau_{12})], \\ \Lambda_3 &= -[a_{21} \dot{f}_1(x_1^*) + b_{21} \dot{f}_1(x_1^*) \exp(-\lambda \tau_{21})], \\ \Lambda_4 &= \lambda - [-c_2 + a_{22} \dot{f}_2(x_2^*) + b_{22} \dot{f}_2(x_2^*) \exp(-\lambda \tau_{22})]. \end{aligned}$$

The eigenvalues are the solutions of the following equation:

$$\begin{aligned} &[\lambda + c_1 - a_{11} \dot{f}_1(x_1^*) - b_{11} \dot{f}_1(x_1^*) \, \mathrm{e}^{-\lambda \tau_{11}}] \\ &\times [\lambda + c_2 - a_{22} \dot{f}_2(x_2^*) - b_{22} \dot{f}_2(x_2^*) \, \mathrm{e}^{-\lambda \tau_{22}}] \\ &- [a_{12} \dot{f}_2(x_2^*) + b_{12} \dot{f}_2(x_2^*) \, \mathrm{e}^{-\lambda \tau_{12}}] \\ &\times [a_{21} \dot{f}_1(x_1^*) + b_{21} \dot{f}_1(x_1^*) \, \mathrm{e}^{-\lambda \tau_{21}}] = 0. \end{aligned}$$

Assuming that

$$\begin{aligned} \Delta(\lambda) = &[\lambda + c_1 - a_{11} \dot{f}_1(x_1^*) - b_{11} \dot{f}_1(x_1^*) \, \mathrm{e}^{-\lambda \tau_{11}}] \\ &\times [\lambda + c_2 - a_{22} \dot{f}_2(x_2^*) - b_{22} \dot{f}_2(x_2^*) \, \mathrm{e}^{-\lambda \tau_{22}}] \\ &- [a_{12} \dot{f}_2(x_2^*) + b_{12} \dot{f}_2(x_2^*) \, \mathrm{e}^{-\lambda \tau_{12}}] \\ &\times [a_{21} \dot{f}_1(x_1^*) + b_{21} \dot{f}_1(x_1^*) \, \mathrm{e}^{-\lambda \tau_{21}}], \end{aligned}$$

we have

$$\begin{aligned}\Delta(0) = {} & [c_1 - a_{11}\dot{f}_1(x_1^*) - b_{11}\dot{f}_1(x_1^*)] \\ & \times [c_2 - a_{22}\dot{f}_2(x_2^*) - b_{22}\dot{f}_2(x_2^*)] \\ & - [a_{12}\dot{f}_2(x_2^*) + b_{12}\dot{f}_2(x_2^*)] \\ & \times [a_{21}\dot{f}_1(x_1^*) + b_{21}\dot{f}_1(x_1^*)].\end{aligned}$$

Since $(x_1^*, x_2^*)^{\mathrm{T}} \in \boldsymbol{\Theta}$ and according to activation function (2), we have $\dot{f}_1(x_1^*) = 0$, $\dot{f}_2(x_2^*) \neq 0$ or $\dot{f}_1(x_1^*) \neq 0$, $\dot{f}_2(x_2^*) = 0$. Without loss of generality, we suppose that $\dot{f}_1(x_1^*) \neq 0$ and $\dot{f}_2(x_2^*) = 0$. Then

$$\Delta(0) = [c_1 - a_{11}\dot{f}_1(x_1^*) - b_{11}\dot{f}_1(x_1^*)]c_2.$$

From Condition 1, we know that $\Delta(0) < 0$. When $\lambda = +\infty$, we have $\Delta(+\infty) = +\infty$. According to Lemma 1, there exists $\lambda_0 > 0$ such that $\Delta(\lambda_0) = 0$. In other words, there exists one positive real root of characteristic equation (7). Thus, $\boldsymbol{x}^* \in \boldsymbol{\Theta}$ is unstable.

Remark 1 According to Theorem 1, we know system (1) with activation function (2) has $(2r+1)^2$ equilibria, $(r+1)^2$ of which are exponentially stable, and $(2r+1)^2 - (r+1)^2 - r^2$ of which are unstable.

Remark 2 Based on Lyapunov's first approximate theory, we have investigated the instability of $(2r+1)^2 - (r+1)^2 - r^2$ equilibria for system (1). Since there have not been results about stability of $(2r+1)^2 - (r+1)^2$ equilibria in DRNNs, our results are latest.

Remark 3 Every subset of $\boldsymbol{\Phi}_1$ is an attractive basin of the corresponding stable equilibrium point of system (1).

Remark 4 In this paper, the activation function has $2r$ corner points. Since r is arbitrary, the number of equilibria for DRNNs can be infinitely large. Therefore, in associative memories, the memory capacity can be as large as you need.

Condition 3

$$-c_i + \min_{k\in\{1,2,\cdots,r\}} \left\{ a_{ii} \frac{m_i^{k+1} - m_i^k}{q_i^k - p_i^k} \right\} > 0,$$
$$b_{ii} > 0,\ i = 1, 2.$$

Corollary 1 Under Conditions 2 and 3, the following propositions are true.

i) There are $(2r+1)^2$ equilibria for system (1) with activation function (2), which are located in each subset of $\boldsymbol{\Phi}$.

ii) For any subset $\boldsymbol{\Omega}$ of $\boldsymbol{\Phi}_1$, $\boldsymbol{\Omega}$ is a positive invariant.

iii) For any subset $\boldsymbol{\Xi}$ of $\boldsymbol{\Phi}_1$, the equilibrium point in $\boldsymbol{\Xi}$ is exponentially stable.

iv) For any subset $\boldsymbol{\Theta}$ of $\boldsymbol{\Phi}_3$, the equilibrium point in $\boldsymbol{\Theta}$ is unstable.

The proof is similar to Theorem 1.

Remark 5 Compared with Conditions 1 and 3, Condition 1 in Theorem 1 is less conservative.

According to Corollary 1, for any subset $\boldsymbol{\Xi} \subset \boldsymbol{\Phi}_1$, there exists one stable equilibrium point in $\boldsymbol{\Xi}$. The equilibrium point in $\boldsymbol{\Xi}$ is denoted as $\boldsymbol{x}^{\boldsymbol{\Xi}}$. The intersection between adjacent regions of $\boldsymbol{\Xi}$ and $\boldsymbol{\Omega}_3$ are denoted as $\boldsymbol{\Theta}_1$, $\boldsymbol{\Theta}_2$, $\boldsymbol{\Theta}_3$, and $\boldsymbol{\Theta}_4$, and the equilibria are denoted as $\boldsymbol{x}^{\boldsymbol{\Theta}_1}$, $\boldsymbol{x}^{\boldsymbol{\Theta}_2}$, $\boldsymbol{x}^{\boldsymbol{\Theta}_3}$, and $\boldsymbol{x}^{\boldsymbol{\Theta}_4}$, respectively. They are unstable equilibria. Relationships between $\boldsymbol{\Xi}$, $\boldsymbol{\Theta}_1$, $\boldsymbol{\Theta}_2$, $\boldsymbol{\Theta}_3$, and $\boldsymbol{\Theta}_4$ are shown in Fig. 3.

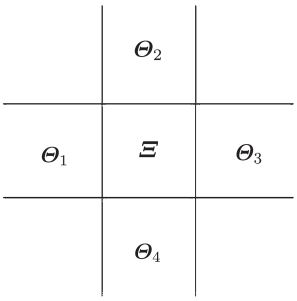


Fig. 3. Schematic plot of two-dimensional space.

Next we will present that the attractive basin of stable equilibrium point $\boldsymbol{x}^{\boldsymbol{\Xi}}$ can be enlarged into a larger region than invariant set $\boldsymbol{\Xi}$. We can obtain the following result.

Theorem 2 Under Conditions 2 and 3, the attractive basin of stable equilibrium point $\boldsymbol{x}^{\boldsymbol{\Xi}}$ is

$$\begin{aligned}&\boldsymbol{\Xi} \cup (\boldsymbol{\Theta}_1 \cap \{\boldsymbol{x} : x_1 > x_1^{\boldsymbol{\Theta}_1}\}) \cup (\boldsymbol{\Theta}_2 \cap \{\boldsymbol{x} : x_2 < x_2^{\boldsymbol{\Theta}_2}\}) \\ &\cup (\boldsymbol{\Theta}_3 \cap \{\boldsymbol{x} : x_1 < x_1^{\boldsymbol{\Theta}_3}\}) \cup (\boldsymbol{\Theta}_4 \cap \{\boldsymbol{x} : x_2 > x_2^{\boldsymbol{\Theta}_4}\}).\end{aligned}$$

Proof According to Corollary 1, we know $\boldsymbol{\Xi}$ is a partial attractive basin of equilibrium point $\boldsymbol{x}^{\boldsymbol{\Xi}}$. Next, we will show that $\boldsymbol{\Theta}_1 \cap \{\boldsymbol{x} : x_1 > x_1^{\boldsymbol{\Theta}_1}\}$, $\boldsymbol{\Theta}_2 \cap \{\boldsymbol{x} : x_2 < x_2^{\boldsymbol{\Theta}_2}\}$, $\boldsymbol{\Theta}_3 \cap \{\boldsymbol{x} : x_1 < x_1^{\boldsymbol{\Theta}_3}\}$, and $\boldsymbol{\Theta}_4 \cap \{\boldsymbol{x} : x_2 > x_2^{\boldsymbol{\Theta}_4}\}$ are also attractive basins of equilibrium point $\boldsymbol{x}^{\boldsymbol{\Xi}}$. Without lose of generality, we only discuss the convergence of states in $\boldsymbol{\Theta}_1 \cap \{\boldsymbol{x} : x_1 > x_1^{\boldsymbol{\Theta}_1}\}$. Let $\boldsymbol{x}(t)$ be any state with initial condition $\boldsymbol{x}(\theta) \in \boldsymbol{\Theta}_1$ ($\theta \in [-\tau, 0]$) with $x_1(\theta) > x_1^{\boldsymbol{\Theta}_1}$. Let

$$y_i(t) = x_i(t) - x_i^{\boldsymbol{\Theta}_1}, \quad i = 1, 2,$$

then

$$\begin{aligned}\dot{y}_1(t) =& -c_1y_1(t) + a_{11}[f_1(x_1(t)) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ &+ a_{12}[f_2(x_2(t)) - f_2(x_2^{\boldsymbol{\Theta}_1})]\\ &+ b_{11}[f_1(x_1(t-\tau_{11})) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ &+ b_{12}[f_2(x_2(t-\tau_{12})) - f_2(x_2^{\boldsymbol{\Theta}_1})], \qquad (8)\\ \dot{y}_2(t) =& -c_2y_2(t) + a_{21}[f_1(x_1(t)) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ &+ a_{22}[f_2(x_2(t)) - f_2(x_2^{\boldsymbol{\Theta}_1})]\\ &+ b_{21}[f_1(x_1(t-\tau_{21})) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ &+ b_{22}[f_2(x_2(t-\tau_{22})) - f_2(x_2^{\boldsymbol{\Theta}_1})]. \qquad (9)\end{aligned}$$

Using the similar proof of Proposition ii) in Theorem 1, we know that $x_2(t)$ will not escape from the upper and lower boundaries of $\boldsymbol{\Theta}_1$. According to activation function (2), we have $f_2(x_2(t)) = f_2(x_2(t-\tau_{12})) = f_2(x_2(t-\tau_{22})) = f_2(x_2^{\boldsymbol{\Theta}_1})$. Therefore, equations (8) and (9) can be rewritten as follows:

$$\begin{aligned}\dot{y}_1(t) =& -c_1y_1(t) + a_{11}[f_1(x_1(t)) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ &+ b_{11}[f_1(x_1(t-\tau_{11})) - f_1(x_1^{\boldsymbol{\Theta}_1})], \qquad (10)\\ \dot{y}_2(t) =& -c_2y_2(t) + a_{21}[f_1(x_1(t)) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ &+ b_{21}[f_1(x_1(t-\tau_{21})) - f_1(x_1^{\boldsymbol{\Theta}_1})]. \qquad (11)\end{aligned}$$

We assume that $x_1(t)$ will not pass through $x_1 = x_1^{\boldsymbol{\Theta}_1}$. If it is not true, then $t_0 > 0$ so that $x_1(t) > x_1^{\boldsymbol{\Theta}_1}$ for $-\tau \le t < t_0$, $x_1(t_0) = x_1^{\boldsymbol{\Theta}_1}$, and $\dot{x}_1(t_0) < 0$. In other words, $y_1(t) > 0$ for $-\tau \le t < t_0$, $y_1(t_0) = 0$, and $\dot{y}_1(t_0) < 0$. However, we derive from Eq. (10) that

$$\begin{aligned}\dot{y}_1(t_0) =& -c_1y_1(t_0) + a_{11}[f_1(x_1(t_0)) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ &+ b_{11}[f_1(x_1(t_0-\tau_{11})) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ =& b_{11}[f_1(x_1(t_0-\tau_{11})) - f_1(x_1^{\boldsymbol{\Theta}_1})]\\ >& 0.\end{aligned}$$

This yields a contradiction to $\dot{y}_1(t_0) < 0$. Therefore, $x_1(t)$ will not pass through $x_1 = x_1^{\boldsymbol{\Theta}_1}$, and $y_1(t) > 0$ for $t \ge -\tau$. We derive from Condition 3 and Eq. (10) that

$$\begin{aligned}\dot{y}_1(t) =& -c_1y_1(t) + a_{11}\frac{m_1^{k+1} - m_1^k}{q_1^k - p_1^k}y_1(t)\\ &+ b_{11}\frac{m_1^{k+1} - m_1^k}{q_1^k - p_1^k}y_1(t-\tau_{11})\\ >& 0,\end{aligned}$$

where $k \in \{1, 2, \ldots, r\}$, implying that $x_1(t)$ increases. There exists t_1 such that $x_1(t_1)$ will pass through the left boundary of $\boldsymbol{\Xi}$. Hence, $(x_1(\theta),\ x_2(\theta))$ will be attracted to $\boldsymbol{x}^{\boldsymbol{\Xi}}$. Other cases can be analyzed similarly. Thus, the attractive basin of $\boldsymbol{x}^{\boldsymbol{\Xi}}$ is

$$\begin{aligned}&\boldsymbol{\Xi} \cup (\boldsymbol{\Theta}_1 \cap \{\boldsymbol{x} : x_1 > x_1^{\boldsymbol{\Theta}_1}\}) \cup (\boldsymbol{\Theta}_2 \cap \{\boldsymbol{x} : x_2 < x_2^{\boldsymbol{\Theta}_2}\})\\ &\cup(\boldsymbol{\Theta}_3 \cap \{\boldsymbol{x} : x_1 < x_1^{\boldsymbol{\Theta}_3}\}) \cup (\boldsymbol{\Theta}_4 \cap \{\boldsymbol{x} : x_2 > x_2^{\boldsymbol{\Theta}_4}\}).\end{aligned}$$

The proof ends here.

Remark 6 In Ref. [20], Zeng *et al.* derived the existence of multiple stable equilibria for DRNNs. Attractive basins of stable equilibria located in some invariant sets derived by decomposing state space. In this paper, we have established larger attractive basins than the invariant sets. Therefore, our results are less conservative.

Remark 7 Although there have been some papers published on the instability analysis problems and attractive basins of equilibria for neural networks (e.g., Ref. [21]), the results were based on neural networks without delay. To the best of our knowledge, there are no results concerning instability and attractive basins of stable equilibria for delayed neural networks. Hence, the results in this paper are still new.

4. Numerical example

In this section, we shall provide one example to illustrate the effectiveness of our results.

Let us consider the following delayed neural network model

$$\begin{aligned}\dot{x}_1(t) =& -x_1(t) + 0.7f_1(x_1(t))\\ &+ 0.5f_2(x_2(t)) + 0.5f_1(x_1(t-0.1))\\ &- 0.51f_2(x_2(t-0.1)) + 0.1,\\ \dot{x}_2(t) =& -x_2(t) - 0.62f_1(x_1(t))\\ &+ 0.6f_2(x_2(t)) + 0.6f_1(x_1(t-0.1))\\ &+ 0.7f_2(x_2(t-0.1)) - 0.1, \qquad (12)\end{aligned}$$

where

$$f_1(\xi) = f_2(\xi) = f(\xi) = \begin{cases} -6, & \xi \in (-\infty, -5),\\ 2\xi + 4, & \xi \in [-5, -3],\\ -2, & \xi \in (-3, -1),\\ 2\xi, & \xi \in [-1, 1],\\ 2, & \xi \in (1, 3),\\ 2\xi - 4, & \xi \in [3, 5],\\ 6, & \xi \in (5, +\infty). \end{cases} \qquad (13)$$

We can check that Conditions 1, 2, and 3 hold. According to Theorem 1, system (12) has 49 equilibria, 16 of which are exponentially stable, and 24 of which are unstable. Simulation results are shown in Figs. 4, 5, and 6. Based on Theorem 2, larger attractive basins of exponentially stable equilibria are established as follows:

D1 : $\{x_1 < -5, x_2 > 5\} \cup \{-5 \leq x_1 < -3.4571, x_2 > 5\} \cup \{x_1 < -5, 3.2375 < x_2 \leq 5\}$;

D2 : $\{x_1 < -5, 1 < x_2 < 3\} \cup \{x_1 < -5, 3 \leq x_2 < 3.2375\} \cup \{-5 \leq x_1 < -3.4857, 1 < x_2 < 3\} \cup \{x_1 < -5, -0.0125 < x_2 \leq 1\}$;

D3 : $\{x_1 < -5, -3 < x_2 < -1\} \cup \{x_1 < -5, -1 \leq x_2 < -0.0125\} \cup \{-5 \leq x_1 < -3.5143, -3 < x_2 < -1\} \cup \{x_1 < -5, -3.2625 < x_2 \leq -3\}$;

D4 : $\{x_1 < -5, x_2 < -5\} \cup \{x_1 < -5, -5 \leq x_2 < -3.2625\} \cup \{-5 \leq x_1 < -3.5429, x_2 < -5\}$;

D5 : $\{-3 < x_1 < -1, x_2 > 5\} \cup \{-3.4571 < x_1 \leq -3, x_2 > 5\} \cup \{-1 \leq x_1 < -0.0286, x_2 > 5\} \cup \{-3 < x_1 < -1, 3.2875 < x_2 \leq 5\}$;

D6 : $\{-3 < x_1 < -1, 1 < x_2 < 3\} \cup \{-3 < x_1 < -1, 3 \leq x_2 < 3.2875\} \cup \{-3.4857 < x_1 \leq -3, 1 < x_2 < 3\} \cup \{-1 \leq x_1 < -0.0571, 1 < x_2 < 3\} \cup \{-3 < x_1 < -1, 0.0375 < x_2 \leq 1\}$;

D7 : $\{-3 < x_1 < -1, -3 < x_2 < -1\} \cup \{-3 < x_1 < -1, -1 \leq x_2 < 0.0375\} \cup \{-3.5143 < x_1 \leq -3, -3 < x_2 < -1\} \cup \{-1 \leq x_1 < -0.0857, -3 < x_2 < -1\} \cup \{-3 < x_1 < -1, -3.2125 < x_2 \leq -3\}$;

D8 : $\{-3 < x_1 < -1, x_2 < -5\} \cup \{-3 < x_1 < -1, -5 \leq x_2 < -3.2125\} \cup \{-3.5429 < x_1 \leq -3, x_2 < -5\} \cup \{-1 \leq x_1 < -0.1143, x_2 < -5\}$;

D9 : $\{1 < x_1 < 3, x_2 > 5\} \cup \{-0.0286 < x_1 \leq 1, x_2 > 5\} \cup \{3 \leq x_1 < 3.4, x_2 > 5\} \cup \{1 < x_1 < 3, 3.3375 < x_2 \leq 5\}$;

D10 : $\{1 < x_1 < 3, 1 < x_2 < 3\} \cup \{1 < x_1 < 3, 3 \leq x_2 < 3.3375\} \cup \{-0.0571 < x_1 \leq 1, 1 < x_2 < 3\} \cup \{3 \leq x_1 < 3.3714, 1 < x_2 < 3\} \cup \{1 < x_1 < 3, 0.0875 < x_2 \leq 1\}$;

D11 : $\{1 < x_1 < 3, -3 < x_2 < -1\} \cup \{1 < x_1 < 3, -3 \leq x_2 < 0.0875\} \cup \{-0.0857 < x_1 \leq 1, -3 < x_2 < -1\} \cup \{-3 \leq x_1 < 3.3429, -3 < x_2 < -1\} \cup \{1 < x_1 < 3, -3.1625 < x_2 \leq -3\}$;

D12 : $\{1 < x_1 < 3, x_2 < -5\} \cup \{1 < x_1 < 3, -5 \leq x_2 < -3.1625\} \cup \{-0.1143 < x_1 \leq 1, x_2 < -5\} \cup \{3 \leq x_1 < 3.3143, x_2 < -5\}$;

D13 : $\{x_1 > 5, x_2 > 5\} \cup \{3.4 < x_1 \leq 5, x_2 > 5\} \cup \{x_1 > 5, 3.3875 < x_2 \leq 5\}$;

D14 : $\{x_1 > 5, 1 < x_2 < 3\} \cup \{x_1 > 5, 3 \leq x_2 < 3.3875\} \cup \{3.3714 < x_1 \leq 5, 1 < x_2 < 3\} \cup \{x_1 > 5, 0.1375 < x_2 \leq 1\}$;

D15 : $\{x_1 > 5, -3 < x_2 < -1\} \cup \{x_1 > 5, -1 \leq x_2 < 0.1375\} \cup \{3.3429 < x_1 \leq 5, -3 < x_2 < -1\} \cup \{x_1 > 5, -3.1125 < x_2 \leq -3\}$;

D16 : $\{x_1 > 5, x_2 < -5\} \cup \{x_1 > 5, -5 \leq x_2 < -3.1125\} \cup \{3.3143 < x_1 \leq 5, x_2 < -5\}$. The illustration with 150 initial states is drawn in Fig. 7.

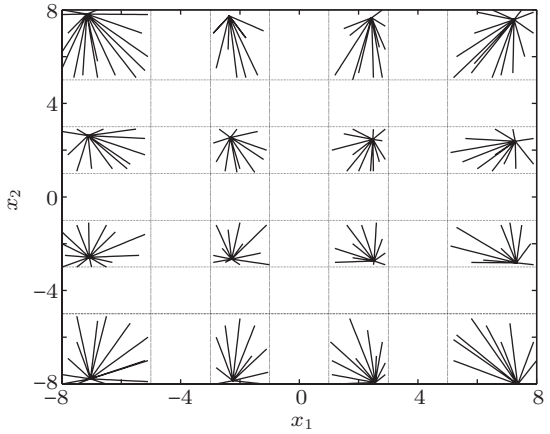


Fig. 4. Illustration for the dynamics of system (12).

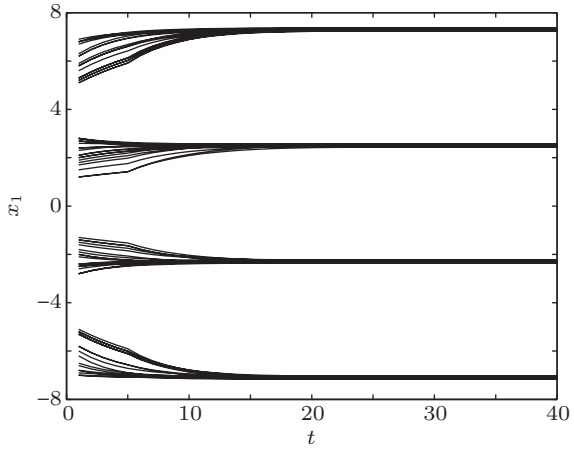


Fig. 5. Temporal evolution of x_1 of system (12).

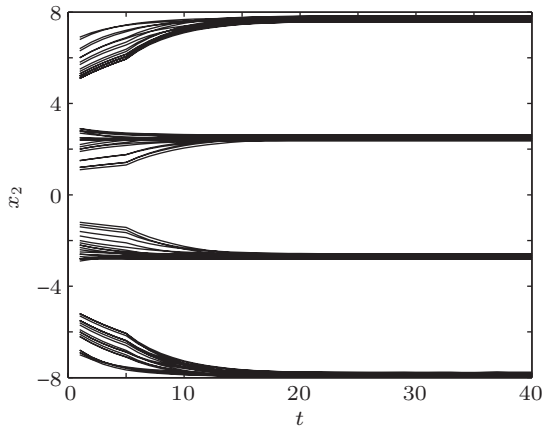


Fig. 6. Temporal evolution of x_2 of system (12).

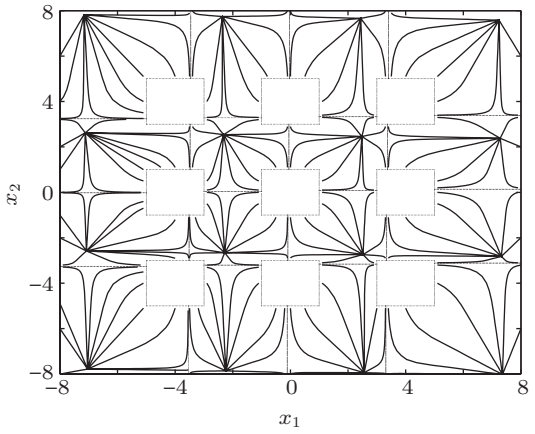


Fig. 7. Illustration for the dynamics of system (12).

5. Conclusion

Multiple stability of two-dimensional DRNNs with piecewise linear activation functions with $2r$ ($r \geq 1$) corner points has been studied in this paper. Sufficient conditions have been derived for checking the existence of $(2r+1)^2$ equilibria. Under these conditions, $(r+1)^2$ equilibria are locally exponentially stable, and $(2r+1)^2-(r+1)^2-r^2$ equilibria are unstable. Attractive basins of stable equilibria have been estimated, which are larger than the invariant sets derived by decomposing the state space. Several extensions, such as studying stability of equilibria in $\boldsymbol{\Phi}_2$ for DRNNs (1), establishing complete attractive basins of stable equilibria, and generalizing the results in this paper to n-dimensional DRNNs with piecewise linear activation functions, would be worthy of increasing efforts.

References

[1] Zeng Z G and Wang J 2006 *IEEE Trans. Neural Netw.* **17** 623

[2] Zhang H G, Wang Z S and Liu D R 2008 *IEEE Trans. Neural Netw.* **19** 855

[3] Zhang H G, Ma T D, Fu J and Tong S C 2009 *Chin. Phys.* B **18** 3742

[4] Qiu F, Cui B T and Ji Y 2009 *Chin. Phys.* B **18** 5203

[5] Wang J A 2011 *Chin. Phys.* B **20** 120701

[6] Marcus C M and Westervelt R M 1989 *Phys. Lett.* A **39** 347

[7] Marcus C M, Wagh F R and Westervelt R M 1991 *Physica* D **51** 234

[8] Gopalsamy K and Leung I K C 1996 *Physica* D **89** 395

[9] Ruan S G and Wei J J 2003 *Dyn. Contin. Discrete Impuls. Syst.* **10** 863

[10] Kwon O M, Kwon J W and Kim S H 2011 *Chin. Phys.* B **20** 050505

[11] Chua L O and Yang L 1988 *IEEE Trans. Circuits Syst.* **35** 1257

[12] Morita M 1993 *Neural Netw.* **6** 115

[13] Foss J, Longtin A, Mensour B and Milton J G 1996 *Phys. Rev. Lett.* **76** 708

[14] Chen T P and Amari S 2001 *Neural Netw.* **14** 1377

[15] Cheng C Y, Lin K H and Shih C W 2007 *Physica* D **225** 61

[16] Huang G and Cao J D 2008 *Phys. Lett.* A **372** 2842

[17] Cao J D, Feng G and Wang Y Y 2008 *Physica* D **237** 1734

[18] Huang Y J, Wang Z S, Zhang H G and Zhang T 2010 *IEEE Internation Conference on Networking, Sensing and Control* April 11–13, 2010 Chicago, USA, p. 376

[19] Huang Y J and Zhang X Y 2011 *Int. J. Info. Syst. Sci.* **7** 1

[20] Zeng Z G, Huang T W and Zheng W X 2010 *IEEE Trans. Neural Netw.* **21** 1371

[21] Wang L L, Lu W L and Chen T P 2010 *Neural Netw.* **23** 189

[22] Lu W L, Wang L L and Chen T P 2011 *IEEE Trans. Neural Netw.* **22** 381

[23] Yang X F, Liao X F, Bai S and Evans D J 2005 *Chaos Soliton. Fract.* **26** 445

[24] Yang X F, Liao X F, Tang Y Y and Evans D J 2006 *Int. J. Bifurcat. Chaos* **16** 2737

[25] Huang T W, Li C D and Zeng Z G 2009 *Comput. Math. Appl.* **58** 508

[26] Pakdaman K, Grotta-Ragazzo C, Malta C P, Arino O and Vibert J F 1998 *Neural Netw.* **11** 509