Effects of anisotropy and magnetic fields on the specific heat of a quasi-two-dimensional Boltzmann gas in an elliptical parabolic quantum dot*

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We investigate the effects due to anisotropy and magnetic field interaction for a quasi-two-dimensional Boltzmann gas in an elliptical parabolic quantum dot. The specific heat is studied with varying temperature, anisotropy, and magnetic field strength. The cases without and with the inclusion of the spin Zeeman interaction are considered.

Keywords: specific heat, Zeeman interaction, anisotropy harmonic oscillator

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1. Introduction

In the past decades, due to the progress in semiconductor technology, low-dimensional electron systems created in semiconductor hetero-structures under a perpendicular magnetic field have revealed many interesting physical phenomena and attracted much research interest.[1−14] In particular, when the system is confined in a parabolic potential, then the model is the well known exact solvable Fock–Darwin (FD) model[15,16] and has wide applications in the studies of quantum Hall effects and quantum dots. The generalization of the FD model to the elliptical parabolic potential case can also be solved exactly,[17−22] various physical properties, such as the quantization of the conductance[19] and the electronic properties, can be studied.[21] This model also can be used to study rotating droplets of electrons trapped in quasi-two-dimensional quantum dots and rotating Bose–Einstein condensates.[23,24] In this paper, we calculate the specific heat for a quasi-two-dimensional Boltzmann gas in an elliptical parabolic quantum dot, then study the effects due to the anisotropy and the magnetic field interaction. The case with the inclusion of the spin effect is studied in Section 3. The conclusion is presented in Section 4.

2. Without spin

The system under investigation is a dot formed in a two-dimensional electron gas with an anisotropic two-dimension harmonic confining potential

$$V_{\rm D} = \frac{m^*}{2}(\omega_y^2 y^2 + \omega_x^2 x^2), \tag{1}$$

where m^* is the effective electron mass, $m^* = 0.067$ for GaAs dots . The two characteristic frequencies of the confinement $\omega_{x,y}$ may be related to the effective size of the dot L as[25]

$$\omega_{x,y} = \frac{v_{\rm F}}{L_{x,y}}, \tag{2}$$

where L_x and L_y are the semimajor axes of the elliptical dot, and $v_{\rm F}$ is the Fermi velocity of the electrons. We use the dimensionless parameter $\epsilon = (\omega_y - \omega_x)/\omega_0$ to define the anisotropy of the confining potential, with ${\omega_0}^2 = \omega_x\omega_y$ as the area of the dot been kept as a constant. For $\epsilon = 0$, the dot has a circular shape. With the increasing ϵ, it deforms into an elliptical shape and ends up as a wirelike dot.

The dot is placed in a perpendicular magnetic field $\boldsymbol{B}$ in the z direction. Choosing a symmetric gauge $\boldsymbol{A} = (1/2)\boldsymbol{B}\times\boldsymbol{r} = (-By/2, Bx/2, 0)$, we have the Hamiltonian

$$H = \frac{[p_x - (e/c)A_x]^2}{2m^*} + [p_y - (e/c)A_y]^2 2m^* + V_{\rm D}, \tag{3}$$

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where c is the speed of light, and e is the electron charge. The spectrum of the Hamiltonian can be obtained as

$$\varepsilon_{n_1,n_2} = \left(n_1 + \frac{1}{2}\right)\hbar\omega_+ + \left(n_2 + \frac{1}{2}\right)\hbar\omega_-, \quad (4)$$

where

$$\omega_\pm = \frac{1}{\sqrt{2}}\Big[\omega_x^2 + \omega_y^2 + \omega_{\rm c}^2 \pm \sqrt{(\omega_x^2 + \omega_y^2 + \omega_{\rm c}^2)^2 - 4\omega_x^2\omega_y^2}\,\Big]^{1/2}, \quad (5)$$

with $\omega_{\rm c} = |e|B/m^*$ being the cyclotron frequency. It is evident from the spectrum that the system is equivalent to two independent harmonic oscillators. Thus, the partition function can be obtained as

$$\mathscr{Z}(\beta) = \sum_{n_1,n_2} {\rm e}^{-\beta\varepsilon_{n_1,n_2}} = Z(\omega_+)Z(\omega_-) = \frac{1}{4\sinh(\beta\hbar\omega_+/2)\sinh(\beta\hbar\omega_-/2)}, \quad (6)$$

where $\beta = 1/k_{\rm B}T$, and $k_{\rm B}$ is the Boltzmann constant. Note that $Z(\omega) = 1/2\sinh(\beta\hbar\omega/2)$ is the partition function for a simple harmonic oscillator with frequency ω.

In the following, we consider the nondegenerate gas, which obeys the Boltzmann statistics, i.e., the equilibrium distribution function $f_0(\varepsilon_{n_1,n_2})$ considered in this case is

$$f_0(\varepsilon_{n_1,n_2}) = 4D^{-1}n_0\frac{{\rm e}^{-\beta\varepsilon_{n_1,n_2}}}{\mathscr{Z}(\beta)}, \quad (7)$$

where n_0 is the electron concentration, D is a parameter determined by the effective size of the dot.[25,26] The specific heat is

$$c_V = \left(\frac{\partial E}{\partial T}\right)\Big|_V, \quad (8)$$

where E is the energy of a unit volume of gas, $E = D\sum_{m,n}\epsilon_{m,n}f_0(\epsilon_{m,n})$. Using the above equations, we obtain

$$\frac{c_V}{k_{\rm B}n_0} = \left(\frac{\beta\hbar\omega_+}{2}\right)^2\frac{1}{\sinh^2(\beta\hbar\omega_+/2)} + \left(\frac{\beta\hbar\omega_-}{2}\right)^2\frac{1}{\sinh^2(\beta\hbar\omega_-/2)}. \quad (9)$$

We should note that the results in Ref. [21] are incorrect.

As $T \to 0$, the specific heat vanishes as

$$\frac{c_V}{k_{\rm B}n_0} \approx (\beta\hbar\omega_+)^2\,{\rm e}^{-\beta\hbar\omega_+} + (\beta\hbar\omega_-)^2\,{\rm e}^{-\beta\hbar\omega_-}. \quad (10)$$

This is a characteristic feature of the quantum system with a discrete energy spectrum. If the thermal energy $k_{\rm B}T$ is much smaller than the two oscillator energies, i.e., $k_{\rm B}T \ll \hbar\omega_- \leq \hbar\omega_+$ (note that the parameter ϵ and the strength of the magnetic field only adjust the ratio of ω_- and ω_+), then the two oscillators with frequencies ω_- and ω_+ can not (or only with an extremely small probability) be excited to the higher energy levels, and the system is absolutely incapable of absorbing energy provided by the heat bath. Thus the specific heat is almost zero, while the classical systems can absorb arbitrarily small energy $k_{\rm B}T$. As the temperature increases, if ω_+ is close to ω_-, then the two oscillators of the system can be excited by the thermal energy almost simultaneously and behave as one simple harmonic oscillator, which is reflected in the specific heat curve with $\epsilon = 0$ in Fig. 1(a). For simplicity, we fix $\omega_0 = 1$, then $\omega_x = \omega_y = 1, \omega_- = 0.905, \omega_+ = 1.105$, the specific heat curve is very similar to that of a simple harmonic oscillator. If ω_+ is much larger than ω_-, then the oscillator with frequency ω_- can be excited to the higher energy levels by the thermal energy first, and the oscillator with frequency ω_+ has a much smaller probability to be excited. There is an interplay between the two frequencies when the temperature varies. As shown in the curve with $\epsilon = 10$ in Fig. 1(a), we can find that $\omega_- = 0.100$ and $\omega_+ = 10.101$ if we set $\omega_0 = 1$. Similar results can be found for the other curves with different parameters in Fig. 1. Apparently, the competition (adjusted by parameter ϵ and the value of the magnetic field) between the two oscillators to absorb the thermal energy makes the specific heat have a peak, where the system absorbs the thermal energy with a local maximum efficiency.

In the limit of vanishing magnetic field ($\omega_{\rm c} = 0$), this problem reduces to the problem of two anisotropic harmonic oscillators. While in the limit of vanishing oscillator potential ($\omega_0 = 0$), we recover the specific heat of the Landau problem.

It is clear from Fig. 1 that the specific heat usually increases as the temperature increases. At low temperature, c_V increases very rapidly, then slowly, at certain temperature even with almost zero slope. Finally, it reaches the constant value of $2n_0k_{\rm B}$ when $T \to \infty$, and the classical value is recovered. When the value of ϵ increases, the temperature taken to reach the constant value becomes larger. And when the magnetic field increases, the temperature taken to reach the constant value also becomes larger. Thus the role played

by the magnetic field is similar to the anisotropy.

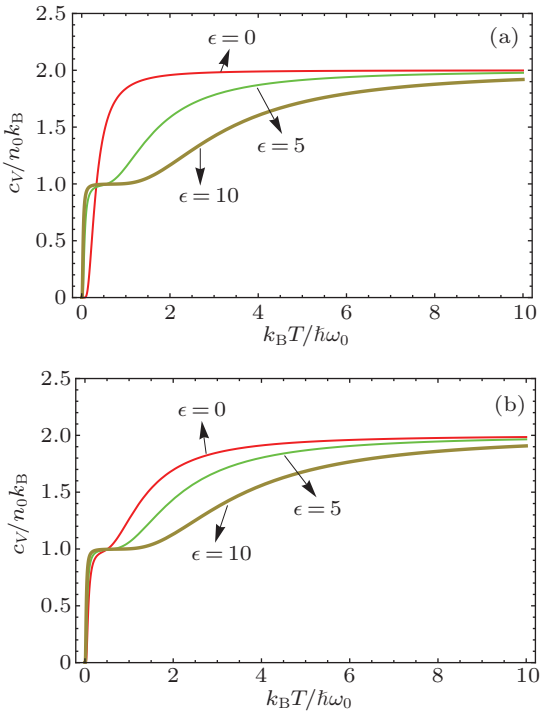


Fig. 1. (colour online) Variations of specific heat with temperature for different values of asymmetry parameter ϵ and different magnetic fields. Here $\omega_c = 0.2\omega_0$ in panel (a), and $\omega_c = 4.0\omega_0$ in panel (b).

In Fig. 2, we plot the specific heat c_V as a function of applied magnetic field (expressed as ω_c/ω_0) at different values of temperature. From the graphs, we can see that when the magnetic field is very strong, i.e., in the limit of $\omega_c \to \infty$, $c_V \to n_0k_B$, which is half of the value when $T \to \infty$. At low temperature, c_V approaches the limit from below, while it approaches the limit from above at high temperature. There exists a region of temperature as shown in Fig. 2(c) ($\epsilon = 0$ curve): c_V/n_0k_B is greater than 1 at low magnetic field and starts to decrease as the magnetic field increases until it reaches a certain value which is less than 1, then it starts to increase and approaches 1 from below. When the value of temperature continues to increase as shown in Fig. 2(d), the value of c_V/n_0k_B approaches 1 from above. When the value of ϵ increases, it always makes the specific heat curve closer to the constant value 1. At low temperature, it moves up as shown in Figs. 2(a) and 2(b), while it moves down at high temperature as shown in Figs. 2(c) and 2(d). When the temperature increases, it always moves the specific heat curve up. After it is higher than a certain value, the effect of anisotropy is dominated by the temperature. As shown in Figs. 2(d) and 3, the distance between the specific heat curves for different values of ϵ become smaller and smaller when the temperature increases. In Fig. 4, we plot the specific curves at different values of temperature with a fixed value of $\epsilon = 2$. Notice that as the temperature increases, it moves the curve up, and it takes a stronger magnetic field to pull the curve down to the constant value 1, as reflected by the scale difference between Figs. 3 and 4.

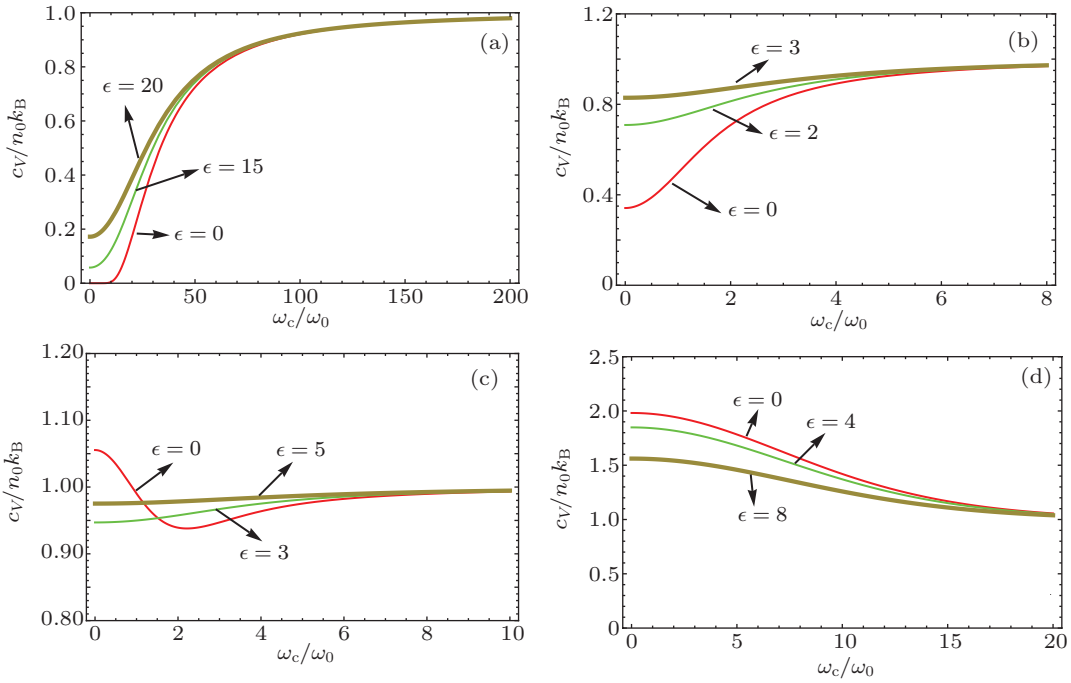


Fig. 2. (colour online) Variations of specific heat with magnetic field for different values of asymmetry parameter ϵ at the temperatures of (a) $k_BT = 0.01\omega_0$, (b) $k_BT = 0.2\omega_0$, (c) $k_BT = 0.35\omega_0$, and (d) $k_BT = 3\omega_0$.

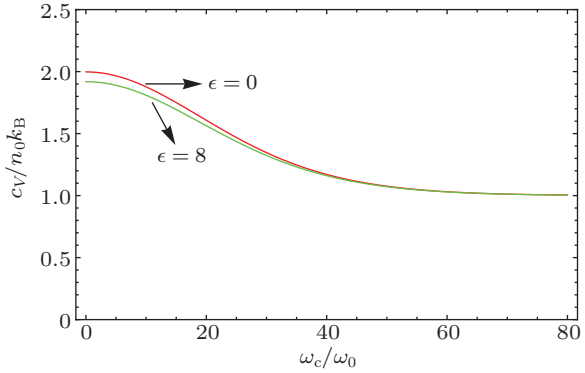


Fig. 3. (colour online) Variations of specific heat with magnetic field for $k_BT = 8\omega_0$ and two different values of asymmetry parameter ϵ.

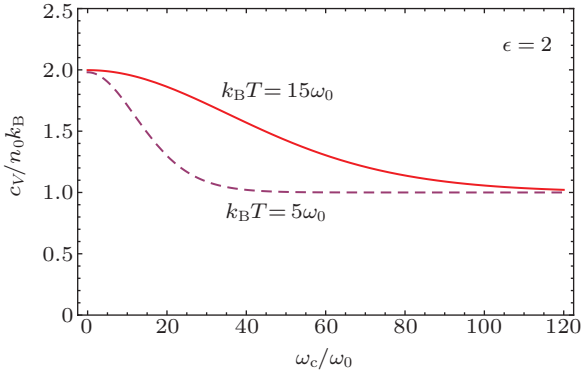


Fig. 4. (colour online) Variations of specific heat with magnetic field when $k_BT = 5\omega_0, 15\omega_0$.

3. With Spin

In this section, we study the effect of spin. For convenience, we choose the applied magnetic field direction as the z direction, the Hamiltonian with the inclusion of spin can be obtained by the addition of Zeeman term $H_z = -(g^*/2)\mu_B(\hat{\sigma}_z \cdot B)$

$$H = \frac{[p_x - (e/c)A_x]^2}{2m^*} + [p_y - (e/c)A_y]^2 2m^* + \frac{m^*}{2}(\omega_y^2 y^2 + \omega_x^2 x^2) + H_z, \quad (11)$$

where g^* is the effective Lande g factor, which is 2.0023 for free electrons and 0.44 for GaAs dots. Now the expression for the eigenvalues is

$$\epsilon_{n_1,n_2,\sigma} = \epsilon_{n_1,n_2} - \frac{g^*}{2}\mu_B(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{B}), \quad (12)$$

and the total partition function is given by

$$\mathscr{Z} = \mathscr{Z}(\beta)\mathscr{Z}_s(\beta), \quad (13)$$

where

$$\mathscr{Z}_s(\beta) = \mathrm{Tr}\, e^{-\beta(-g^*\mu_B\hat{\sigma}_z B)} = 2\cosh\left(\frac{g^*}{2}\beta\mu_B B\right) = 2\cosh\left(\frac{\beta\hbar\omega_c g^*}{4}\right). \quad (14)$$

Similarly, we can obtain the specific heat

$$\frac{c_V}{k_B n_0} = \left(\frac{\beta\hbar\omega_+}{2}\right)^2 \frac{1}{\sinh^2(\beta\hbar\omega_+/2)} + \left(\frac{\beta\hbar\omega_-}{2}\right)^2 \frac{1}{\sinh^2(\beta\hbar\omega_-/2)} + \left(\frac{\beta\hbar\omega_c g^*}{4}\right)^2 \frac{1}{\cosh^2(\beta\hbar\omega_c g^*/4)}. \quad (15)$$

As $T \to 0$, the specific heat vanishes as

$$\frac{c_V}{k_B n_0} \approx (\beta\hbar\omega_+)^2 e^{-\beta\hbar\omega_+} + (\beta\hbar\omega_-)^2 e^{-\beta\hbar\omega_-} + (g^*\beta\hbar\omega_c/2)^2 e^{-g^*\beta\hbar\omega_c/2}. \quad (16)$$

It is evident that in the low temperature regime, the system can be treated as a combination of three harmonic oscillators with frequencies ω_+, ω_-, and ω_c respectively. A similar analysis can be carried out as that for the case without spin, but with more complicated situations, since now it is the competition among three frequencies. Again, similar to Fig. 1, a peak is observed in the low temperature regime as shown in Fig. 5, where the system absorbs the thermal energy with a local maximum efficiency.

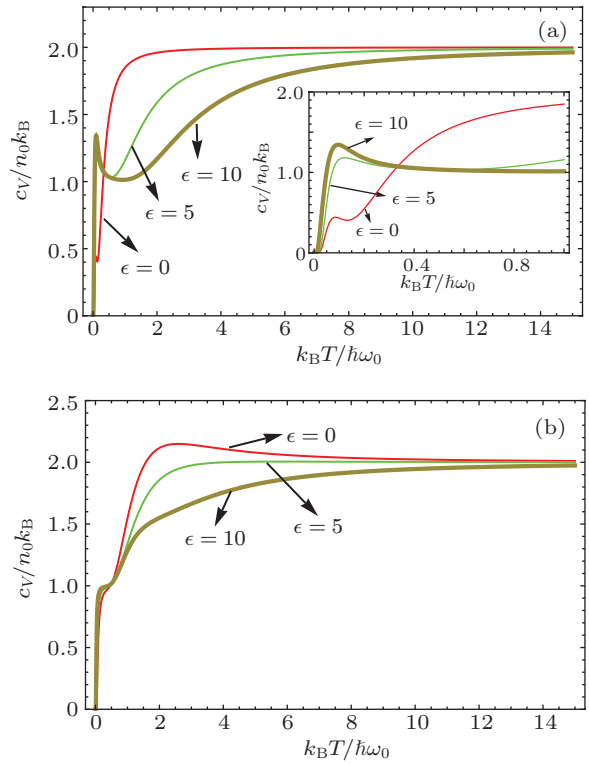


Fig. 5. (colour online) Variations of specific heat with temperature for different values of asymmetry parameter ϵ and different magnetic fields ($\omega_c = 0.2\omega_0$ in panel (a) and $\omega_c = 4.0\omega_0$ in panel (b)) with the spin considered. The insert gives the low temperature variations.

Note that when the spin is considered and $T \to \infty$, $c_V/(k_B n_0) \to 2$, which is similar to the case without spin, but usually the specific heat curve is lower. The variation of specific heat with the strength of magnetic field is presented in Fig. 6. Again when $\omega_c \to \infty$, $c_V/(k_B n_0) \to 2$. Overall, the effects of temperature, anisotropy, and magnetic field in this case are very similar to those in the case without spin. If we compare Fig. 6 with Fig. 2, it is obvious that the effect of spin is always to move the specific heat curve up. To see this point clearly, we plot the specific heat as a function of the strength of magnetic field for the cases with and without the spin in Fig. 7.

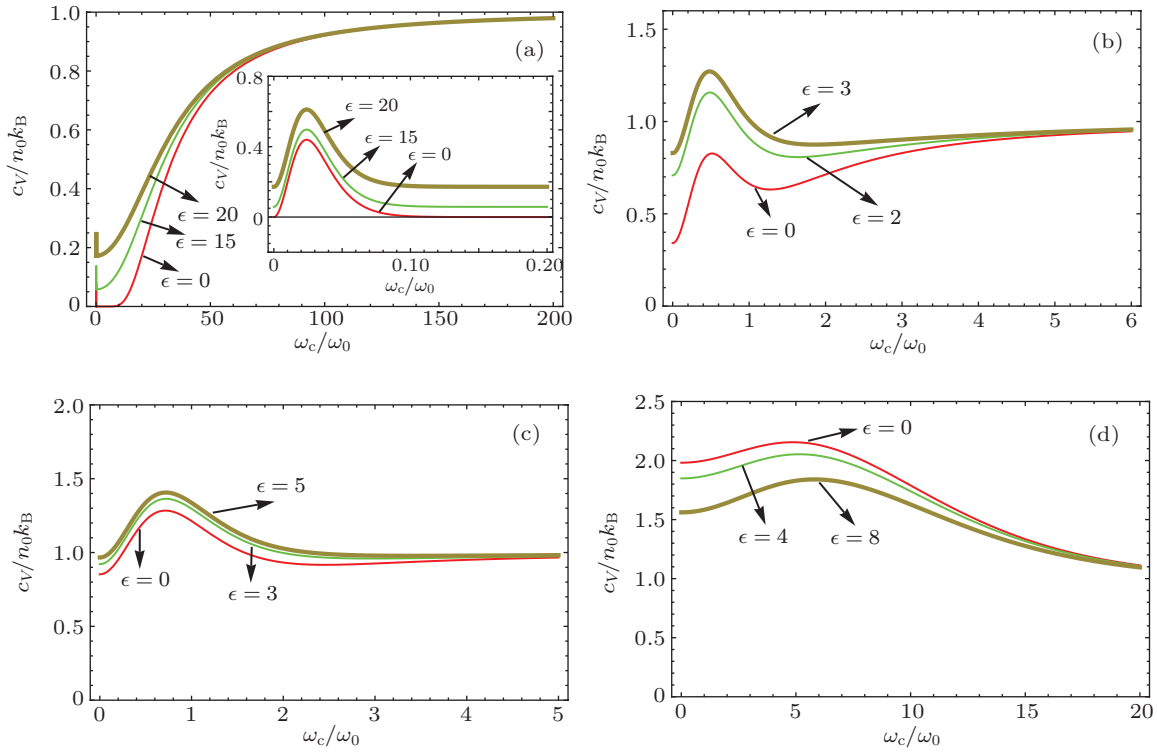


Fig. 6. (colour online) Variations of specific heat with magnetic field for different values of asymmetry parameter ϵ at the temperatures of (a) $k_B T = 0.01\omega_0$, (b) $k_B T = 0.2\omega_0$, (c) $k_B T = 0.3\omega_0$, and (d) $k_B T = 3\omega_0$. The insert gives the low temperature variations.

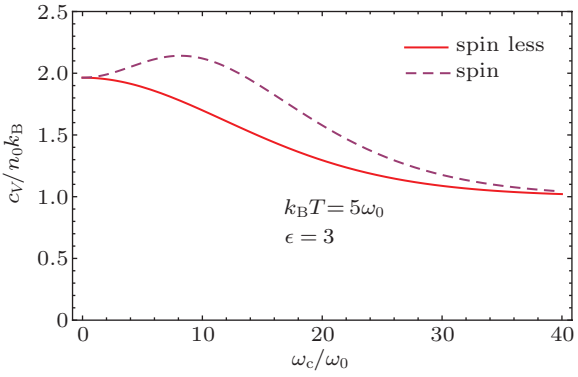


Fig. 7. (colour online) Variations of specific heat with magnetic field when $k_B T = 5.0\omega_0$ and $\epsilon = 3$.

4. Conclusion

We have investigated the specific heat for a quasi-two-dimensional Boltzmann gas in an asymmetric quantum dot. We have discussed in detail the effects due to anisotropy, temperature, and magnetic field interaction. The cases without and with the inclusion of spin Zeeman interaction are considered. In both cases, we find that when the temperature $T \to 0$, the specific heat vanishes. In the low temperature regime, a peak (where the system absorbs the thermal energy with a local maximum efficiency) is observed, which arises from the competition (adjusted by the parameter ϵ and the value of the magnetic field) between the oscillators to absorb the thermal energy. Hence, the anisotropy plays an important role in the low temperature regime. When the temperature increases, the specific heat approaches a constant value $2k_B n_0$. And when the magnetic field increases, the specific heat approaches another constant value $k_B n_0$. The effect of spin is to move the specific heat curve up, while the

effect of anisotropy is quite complicated as analyzed in the text. These results are expected to be observed in experiments under appropriate conditions.

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