Area and entropy spectra of black holes via an adiabatic invariant*

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By considering and using an adiabatic invariant for black holes, the area and entropy spectra of static spherically-symmetric black holes are investigated. Without using quasi-normal modes of black holes, equally-spaced area and entropy spectra are derived by only utilizing the adiabatic invariant. The spectra for non-charged and charged black holes are calculated, respectively. All these results are consistent with the original Bekenstein spectra.

Keywords: black hole, adiabatic invariant, area spectrum, entropy spectrum

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1. Introduction

The quantization of area and entropy is an important aspect of studying the quantum properties of a black hole. Initially, by studying particles emitted from a black hole or those absorbed by it, the famous Bekenstein conjecture that the entropy of a black hole is proportional to its horizon area and quantization was proposed.[1] The minimum increase of horizon area can be obtained by absorbing a test particle and is expressed as[1,2]

$$(\triangle A)_{\min} = 8\pi\hbar, \tag{1}$$

where the unit that $c = G = 1$ is used, $\hbar$ is Planck's constant, c is the speed of light, and G is Newton's constant. Bekenstein further proved that the horizon area of black holes is an adiabatic invariant.[3–5] That is, the area of black holes changes very slowly compared with the variation of external perturbations. According to the Ehrenfest principle, any classical adiabatic invariant corresponds to a quantum quantity with a discrete spectrum. Thus, black hole area and entropy can be quantized and the area quantum (1) was presented.

The quantization of black holes was also proposed via the quasi-normal modes (QNMs) of black holes. Based on Bohr's correspondence principle, Hod[6,7] argued that highly damped black hole QNMs should be identical to the quantum transition of the corresponding system and the real part of asymptotic QNMs should be equal to the quantum transition frequency. If the variation of black hole energy equals the radiation energy, the area spectrum of black holes could be obtained. Later, the important progress of relating the QNM to the quantization of black holes was made. Maggiore[8] suggested that the quantum levels corresponding the transition frequency of black holes should be identical to the absolute values of the QNMs as a set of complex frequencies. For highly-excited black holes, the imaginary part of QNMs is much larger than the real part and a proper frequency comes from the imaginary part. Moreover, the proper frequency from the imaginary part of QNMs was mentioned earlier in Ref. [9]. Combining the proposal that the horizon area of black holes is an adiabatic invariant with the method of relating the QNMs to the quantization of black holes, Kunstatter presented an adiabatic invariant formula for black holes.[10] It was showed that, for a system with energy E and vibrational frequency $\omega(E)$, there is a natural adiabatic invariant $I = \int \mathrm{d}E/\omega(E)$. As a semiclassical limit, a black hole with transition frequency can be considered as a classical system of periodic motion with the vibrational frequency from the QNMs. Then, following the Bohr–Sommerfeld quantization rule $I = n\hbar$, the area and entropy spectra of black holes were produced.[10–12] By using the adiabatic invariant and the QNMs of black holes, many other studies have been made of black-hole quantization and

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the quantum spectra in different spacetimes have been provided.[13−27]

In addition, motivated by Bekenstein's proposal that the black-hole area is an adiabatic invariant, different adiabatic invariants have been proposed to obtain the quantum spectra of black holes.[28−35] In Refs. [28]–[32], using some basic properties of Euclidean black-hole thermodynamics, a method for studying the quantization of black holes was put forward, where the mass and charge of black holes are treated as canonical conjugates and the time is Euclideanized. Then, by quantizing the action of black holes, quantum spectra of different black holes were derived without using the QNMs of black holes. In Ref. [33], an adiabatic invariant for black holes was proposed given that the outgoing particles pass through the event horizon. It can be written as[33]

$$\begin{aligned} I &= \int p_i \mathrm{d}q_i \\ &= \iint_0^{p_i} \mathrm{d}p_i' \mathrm{d}q_i \\ &= \iint_0^{p_0} \mathrm{d}p_0' \mathrm{d}q_0 + \iint_0^{p_r} \mathrm{d}p_r' \mathrm{d}q_r, \end{aligned} \tag{2}$$

where p_i is the conjugate momentum of the coordinate q_i, $q_0 = \tau$ with τ being the Euclidean time, and $q_r = r$ is the radial coordinate. Then, following the Bohr–Sommerfeld quantization rule, the quantum spectrum that is the same as area (1) for a static spherically-symmetric non-charged black hole was given. Similar to Refs. [28]–[32], the significant feature of Ref. [33] is that the quantization of black holes can be derived without using the QNMs of black holes.

In this paper, we investigate the area and entropy spectra of black holes by considering and rewriting an adiabatic invariant for black holes. Using some basic properties of Euclidean black-hole thermodynamics and applying the Bohr–Sommerfeld quantization rule to the rewritten adiabatic invariant, the equally-spaced area and entropy spectra of black holes are derived without using the QNMs of black holes. The quantization of static spherically-symmetric non-charged and charged black holes are discussed, respectively. The quantization of area that is the same as area (1) is obtained.

2. Invariance and quantization of black holes

In this section, we introduce an adiabatic invariance of black holes and present the area and entropy spectra of Schwarzschild black holes. By means of the radiation of black holes, the adiabatic invariant for black holes reads

$$I = \oint p_i \mathrm{d}q_i = \oint \int_0^{p_i} \mathrm{d}p_i' \mathrm{d}q_i, \tag{3}$$

where q_i is the dynamic freedom degree in Euclidean coordinates, the integral path encircles and is very close to the event horizon. It should be pointed that the adiabatic invariant is expressed by Eq. (2) and inspired by other methods and perspectives in black-hole thermodynamics. First, Hawking radiation of a black hole comes from the vacuum fluctuation near the event horizon and there are two directions for particles across the horizon.[36,37] That is, when a pair of particles is spontaneously created just inside the horizon, the positive-energy particle can tunnel out of the horizon to infinity. On the other hand, the negative-energy particle remains behind the horizon and effectively lowers the mass of the black hole. Accordingly, when a pair of particles is created just outside the horizon, the negative-energy particle can tunnel through the horizon inward and the positive-energy particle remains outside the horizon and goes to infinity. Second, it has been proposed[38,39] that in the radiation process of black holes, the quantity $\int p_r \mathrm{d}r$ is not invariant under canonical transformations, but $\oint p_r \mathrm{d}r$ is a canonical invariant. In fact, invariant (3) is also motivated by the tunneling paradigm of Hawking radiation.[40−42] In the paradigm, to obtain the emission rate of tunneling particles, the action $\int p_r \mathrm{d}r$ for outgoing particles is calculated and discussed. In addition, as will be seen in Section 3, the present adiabatic invariant can be applied to the charged black hole.

Now, let us discuss the quantization of a Schwarzschild black hole by using the adiabatic invariant (3). The Schwarzschild black hole is described by one parameter of mass M_t and the element is

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1-\frac{2M_t}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M_t}{r}\right)^{-1}\mathrm{d}r^2 \\ &+ r^2\mathrm{d}\Omega^2. \end{aligned} \tag{4}$$

By using the transformation $t \to -\mathrm{i}\tau$, the Euclidean

metric can be obtained as

$$\mathrm{d}s^2 = \left(1-\frac{2\mathrm{i}M_\tau}{r}\right)\mathrm{d}\tau^2 + \left(1-\frac{2\mathrm{i}M_\tau}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{5}$$

where M_τ is the Euclidean mass[31,32] and

$$M_t \to \mathrm{i}M_\tau. \tag{6}$$

For the non-charged static spherically-symmetric spacetime, the only dynamic freedom degree can be denoted as $q_r = r$ and the adiabatic invariant quantity (3) becomes

$$I = \oint p_r \mathrm{d}q_r = \oint \int_0^{p_r} \mathrm{d}p'_r \mathrm{d}r. \tag{7}$$

To proceed with an explicit computation on the adiabatic invariant (7), we replace the momentum p_r by energy and apply Hamilton's equation

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}H'}{\mathrm{d}p'_r} = \frac{\mathrm{d}M'_\tau}{\mathrm{d}p'_r}, \tag{8}$$

where the radial path $\dot{r}$ is determined by the $(\tau - r)$ sector of Eq. (5) with $\mathrm{d}\Omega^2 = 0$, $H' = M'_\tau$ is the energy of the black hole after the radiation with energy $\omega = M_\tau - M'_\tau$. Then, substituting Eq. (8) into Eq. (7) yields

$$I = \oint \int_0^H \frac{\mathrm{d}H'}{\mathrm{d}\dot{r}}\mathrm{d}r = \oint \int_0^{M_\tau} \mathrm{d}M'_\tau \mathrm{d}\tau. \tag{9}$$

Next, in order to perform the integral over time, the basic characteristics of the event horizon can be used. It has been pointed out that the period of Euclidean time of a loop about the event horizon is[31−33,37]

$$\oint \mathrm{d}\tau = \frac{2\pi}{\kappa_\tau}, \tag{10}$$

where κ_τ is the surface gravity of black holes. Based on Eq. (10), equation (9) becomes

$$I = 2\pi \int_0^{M_\tau} \frac{\mathrm{d}M'_\tau}{\kappa'_\tau}. \tag{11}$$

Since the Hawking temperature of the black hole (5) is

$$\begin{aligned} T_\tau &= \frac{\hbar\kappa_\tau}{2\pi} \\ &= -\frac{\hbar}{4\pi}\lim_{r-r_+}\sqrt{\frac{-g^{rr}}{g^{\tau\tau}}}\frac{1}{g^{\tau\tau}}\frac{\partial g^{\tau\tau}}{\partial r} \\ &= -\frac{\hbar}{8\pi M_\tau}, \end{aligned} \tag{12}$$

where g^{rr} and $g^{\tau\tau}$ are the inverter metric components corresponding to r and τ, respectively. Then the adiabatic invariant reads

$$\begin{aligned} I &= \hbar\int_0^{M_\tau} \frac{\mathrm{d}M'_\tau}{T'_\tau} \\ &= \int_0^{M_\tau} -8\pi M'_\tau \mathrm{d}M'_\tau \\ &= -4\pi M_\tau^2. \end{aligned} \tag{13}$$

The event horizon area is

$$A = -16\pi M_\tau^2, \tag{14}$$

and then the adiabatic invariant of the black hole is written as

$$I = \frac{A}{4}. \tag{15}$$

As the present canonical invariant corresponds to one dynamic freedom degree q_r, the Bohr–Sommerfeld quantization rule can be used as

$$I = 2\pi n\hbar, \tag{16}$$

where $n = 1, 2, \ldots$. Thus, from Eqs. (13) and (16), we obtain the area spectrum for the Schwarzschild black hole

$$A_n = 4I = 8\pi n\hbar. \tag{17}$$

Accordingly, the area quantum is

$$\triangle A = A_{n+1} - A_n = 8\pi\hbar. \tag{18}$$

It is seen that the present area spectrum is equally spaced and is the same as Bekenstein's original result (1).

Next, we use the following Bekenstein–Hawking formula for the Schwarzschild black hole in Einstein theory:

$$S_{\mathrm{bh}} = \frac{A}{4\hbar}. \tag{19}$$

Then, from Eqs. (17) and (18), we obtain the entropy spectra

$$S_{\mathrm{bh}(n)} = \frac{A_n}{4\hbar} = 2\pi n, \tag{20}$$

$$\triangle S_{\mathrm{bh}} = \frac{\triangle A}{4\hbar} = 2\pi. \tag{21}$$

Similar to the area spectrum, the obtained entropy spectrum is also equally spaced.

In addition, for the black hole (5), we consider the first law of black hole thermodynamics

$$\frac{\mathrm{d}H'}{T'_\tau} = \mathrm{d}S'_{\mathrm{bh}}, \tag{22}$$

the adiabatic invariant (13) can be written as

$$I = \hbar \int_0^{S_{\rm bh}} {\rm d}S'_{\rm bh} = \hbar S_{\rm bh}. \tag{23}$$

Then, according to the Bohr–Sommerfeld quantization formula (16), the entropy spectrum of black holes can be obtained as

$$S_{{\rm bh}(n)} = \frac{I_n}{\hbar} = 2\pi n, \tag{24}$$

$$\triangle S_{\rm bh} = S_{bh(n+1)} - S_{{\rm bh}(n)} = 2\pi. \tag{25}$$

Thus, the entropy spectrum formulae (20) and (21) can be reproduced more directly.

Again, we use the Bekenstein–Hawking area law (19). Then, from the entropy spectrum formulae (24) and (25), the area spectra of black holes can be obtained as

$$A_n = 4\hbar S_{{\rm bh}(n)} = 8\pi n\hbar, \tag{26}$$

$$\triangle A = 4\hbar \triangle S_{\rm bh} = 8\pi\hbar. \tag{27}$$

Thus, the area spectrum formulae (17) and (18) are derived again.

3. The area and entropy spectra of a charged black hole

In this section, using the adiabatic invariant formula (3), we discuss the quantization of a static spherically-symmetric black hole with two dynamic degrees of freedom. In Einstein–Maxwell theory, the static charged black hole is described as the Reissner–Nordstrom metric

$$\begin{aligned} {\rm d}s^2 = & -\left(1-\frac{2M_t}{r}+\frac{Q_t^2}{r^2}\right){\rm d}t^2 \\ & +\left(1-\frac{2M_t}{r}+\frac{Q_t^2}{r^2}\right)^{-1}{\rm d}r^2 + r^2{\rm d}\Omega^2, \end{aligned} \tag{28}$$

where the dynamic parameters M_t and Q_t are the mass and electric charge of a black hole, respectively. The event horizon r_+ and the inner horizon r_- of the black hole are

$$r_\pm = M_t \pm \sqrt{M_t^2 - Q_t^2}. \tag{29}$$

The area of the event horizon is

$$A = 4\pi r_+^2. \tag{30}$$

For the electromagnetic field, the electromagnetic potential is

$$A_\mu = \left(-\frac{Q_t}{r}, 0, 0, 0\right). \tag{31}$$

Implementing the transformation $t \to -{\rm i}\tau$, we obtain the Euclidean metric of Reissner–Nordstrom black hole as

$$\begin{aligned} {\rm d}s^2 = & \left(1-\frac{2{\rm i}M_\tau}{r}-\frac{Q_\tau^2}{r^2}\right){\rm d}\tau^2 \\ & +\left(1-\frac{2{\rm i}M_\tau}{r}-\frac{Q_\tau^2}{r^2}\right)^{-1}{\rm d}r^2 + r^2{\rm d}\Omega^2, \end{aligned} \tag{32}$$

where M_τ and Q_τ are the Euclidean mass and electric charge, respectively, $M_t \to {\rm i}M_\tau$, and $Q_t \to {\rm i}Q_\tau$.

It should be noted that the metrics of Eqs. (28) and (32) describe the outside geometry but not the interior metric of the Reissner–Nordstrom black hole. However, the integral path of Eq. (3) encircles the event horizon and we should consider both the inside and outside regions of the black hole. In addition, the black hole radiation is a quantum effect arising from the region very close to the event horizon and the integral path is very close to the horizon. Furthermore, if the interior metric is smoothly connected to the exterior metric through the event horizon, the interior metric near the horizon may be identified to be the same as the exterior metric near the horizon.[43] Then, the interior metric near the horizon can be given by Eq. (32).

In addition, when we calculate the adiabatic invariant (3) in charged black holes, the effect of electromagnetic field should be taken into account. That is, we should consider the matter–gravity system consisting of the black hole and the electromagnetic field. For the electromagnetic field in Reissner–Nordstrom spacetime, from Eqs. (31) and (32), it is seen that the non-zero component of electromagnetic potential is

$$A_\tau = -\frac{Q_\tau}{r}. \tag{33}$$

The Lagrangian function for the matter–gravity system is

$$L = L_{\rm m} + L_{\rm e}, \tag{34}$$

where $L_{\rm m}$ is the Lagrangian function of the matter field, and $L_{\rm e}$ is the Lagrangian function of the electromagnetic field corresponding to the generalized coordinates A_τ.[44] As $L_{\rm e}$ described by A_μ is $-F_{\mu\nu}F^{\mu\nu}/4$, it is seen that the generalized coordinate A_τ in Eq. (33) is an ignorable coordinate. To eliminate this degree of freedom completely, the action of the matter–gravity system can be written as

$$S = \oint \left(L{\rm d}t - p_{A_\tau}{\rm d}A_\tau\right), \tag{35}$$

where p_{A_τ} is the canonical momentum conjugate to A_τ. Then, we write the adiabatic invariant quantity for Reissner–Nordstrom black hole as

$$\begin{aligned} I &= \oint (p_r \,\mathrm{d}q_r - p_{A_\tau}\,\mathrm{d}A_\tau) \\ &= \oint \int_0^{p_i} \left(\mathrm{d}p'_r \mathrm{d}r - \mathrm{d}p'_{A_\tau}\,\mathrm{d}A_\tau\right) \\ &= \oint \int_0^{p_i} \left(\mathrm{d}p'_r \dot{r} - \mathrm{d}p'_{A_\tau}\dot{A}_\tau\right) \mathrm{d}\tau, \end{aligned} \tag{36}$$

where $\dot{r}= \mathrm{d}r/\mathrm{d}\tau$ and $\dot{A_\tau}= \mathrm{d}A_\tau/\mathrm{d}\tau$.

For an explicit computation of Eq. (36), we apply Hamilton's equation

$$\dot{r}= \left(\frac{\mathrm{d}H'}{\mathrm{d}p'_r}\right)_{(r;A_\tau,p_{A_\tau})} = \frac{\mathrm{d}{M_\tau}'}{\mathrm{d}p'_r}, \tag{37}$$

$$\dot{A_\tau}= \left(\frac{\mathrm{d}H'}{\mathrm{d}p'_{A_\tau}}\right)_{(A_\tau;r,p_r)} = \frac{\mathrm{d}E'_{Q_\tau}}{\mathrm{d}p'_{A_\tau}}, \tag{38}$$

where E'_Q represents the energy of the electromagnetic field corresponding to black hole with electric charge Q'_τ and

$$\mathrm{d}E'_{Q_\tau} = -\mathrm{i}\,\mathrm{d}E'_{Q_t} = -\mathrm{i}\frac{Q'_t}{r_+}\mathrm{d}Q'_t. \tag{39}$$

Substituting Eqs. (38) and (39) into Eq. (36), we have

$$I = \oint \int_{(0,0)}^{(M_\tau,Q_\tau)} \left(\mathrm{d}M'_\tau - \mathrm{d}E'_{Q_\tau}\right) \mathrm{d}\tau. \tag{40}$$

By switching the order of integral and finishing the integral over time by using Eq. (10), Eq. (40) becomes

$$I = 2\pi \int_{(0,0)}^{(M_\tau,Q_\tau)} \frac{\mathrm{d}M'_\tau - \mathrm{d}E'_{Q_\tau}}{\kappa'_\tau}. \tag{41}$$

For spacetime (32), the Hawking temperature is

$$\begin{aligned} T_\tau &= \frac{\hbar\kappa_\tau}{2\pi} \\ &= -\frac{\hbar}{4\pi}\lim_{r-r_+}\sqrt{\frac{-g^{rr}}{g^{\tau\tau}}}\frac{1}{g^{\tau\tau}}\frac{\partial g^{\tau\tau}}{\partial r} \\ &= -\mathrm{i}\frac{\hbar}{4\pi}\frac{r_+ - r_-}{r_+^2}. \end{aligned} \tag{42}$$

Then, substituting Eqs. (42), (39), and (6) into Eq. (41), we have

$$\begin{aligned} I &= \hbar \int_{(0,0)}^{(M_\tau,Q_\tau)} \frac{\mathrm{d}M'_\tau - \mathrm{d}E'_{Q_\tau}}{T'_\tau} \\ &= \int_{(0,0)}^{(M_t,Q_t)} \frac{4\pi r'^2_+}{r'_+ - r'_-}\left(\mathrm{d}M'_t - \frac{Q'_t}{r'_+}\mathrm{d}Q'_t\right). \end{aligned} \tag{43}$$

By means of the identity

$$r_+\,\mathrm{d}M_t - Q_t\,\mathrm{d}Q_t = \frac{1}{2}\left(r_+ - r_-\right)\mathrm{d}r_+, \tag{44}$$

equation (43) becomes

$$I = \int_0^{r_+} \pi\,\mathrm{d}r'^2_+. \tag{45}$$

In view of Eq. (30), the adiabatic invariant is obtained as

$$I = \pi r_+^2 = \frac{A}{4}. \tag{46}$$

By applying the Bohr–Sommerfeld quantization formula $I = 2\pi n\hbar$ to the adiabatic invariant (46), the area spectrum and area quantum of the Reissner–Nordstrom black hole are obtained, i.e.,

$$A_n = 4\hbar I = 8\pi n\hbar, \tag{47}$$

$$\triangle A = A_{n+1} - A_n = 8\pi\hbar. \tag{48}$$

According to the Bekenstein–Hawking formula (19) and Eqs. (47) and (48), the entropy spectra of a Reissner–Nordstrom black hole are derived as

$$S_{\mathrm{bh}(n)} = \frac{A_n}{4\hbar} = 2\pi n, \tag{49}$$

$$\triangle S_{\mathrm{bh}} = \frac{\triangle A}{4\hbar} = 2\pi. \tag{50}$$

It is the same as the Schwarzschild black hole in that the area and entropy spectra of a Reissner–Nordstrom black hole are equally spaced and consistent with Bekenstein's original spectrum.

In addition, for the Reissner–Nordstrom black hole (32), we can apply the first law of black hole thermodynamics

$$\mathrm{d}M'_\tau - \mathrm{d}E'_{Q_\tau} = T'_\tau\,\mathrm{d}S'. \tag{51}$$

From Eq. (41), the adiabatic invariant is written as

$$I = \hbar\int_0^{S_{\mathrm{bh}}} \mathrm{d}S' = \hbar S_{\mathrm{bh}}. \tag{52}$$

Thus, according to the Bohr–Sommerfeld quantization formula $I = 2\pi n\hbar$, the entropy spectra of the black hole can be obtained as

$$S_{\mathrm{bh}(n)} = \frac{I_n}{\hbar} = 2\pi n, \tag{53}$$

$$\triangle S_{\mathrm{bh}} = S_{\mathrm{bh}(n+1)} - S_{\mathrm{bh}(n)} = 2\pi. \tag{54}$$

That is, the entropy spectrum formulae (49) and (50) can be reproduced more directly.

Again, we use the Bekenstein–Hawking area law (19). Then, from the entropy spectrum formulae (53) and (54), the area spectrum of the black hole can be derived as

$$A_n = 4\hbar S_{\mathrm{bh}(n)} = 8\pi n\hbar, \tag{55}$$

$$\triangle A = 4\hbar \triangle S_{\mathrm{bh}} = 8\pi\hbar. \tag{56}$$

Thus, the area spectrum formulae (47) and (48) are obtained again.

4. Discussion and conclusion

To investigate the quantization of black holes, we discuss and rewrite an adiabatic invariance of black holes by considering black-hole radiation. Two directions of radiation particles going out and into black holes are considered and the rewritten adiabatic invariant has canonical invariance. By using some basic properties of Euclidean black hole thermodynamics and applying the Bohr–Sommerfeld quantization rule to the adiabatic invariant, the quantum spectra for static spherically-symmetric black holes are discussed. Without using the QNMs of black holes, the equally spaced area and entropy spectra are derived by only taking advantage of the adiabatic invariant. The spectra for a Schwarzschild black hole and a Reissner–Nordstrom black hole are investigated, respectively. All these results are consistent with Bekenstein's original spectrum (1). For supporting the universality of black hole spectroscopy, the present discussion provides an illustration via considering and using the adiabatic invariance of black holes. In addition, the present adiabatic invariant can be applied to not only non-charged but also charged black holes. Furthermore, the present discussion on black-hole quantization can be extended to other types of black hole.

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