Exact solutions of the Klein–Gordon equation with ring-shaped oscillator potential by using the Laplace integral transform

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We present exact solutions for the Klein–Gordon equation with a ring-shaped oscillator potential. The energy eigenvalues and the normalized wave functions are obtained for a particle in the presence of non-central oscillator potential. The angular functions are expressed in terms of the hypergeometric functions. The radial eigenfunctions have been obtained by using the Laplace integral transform. By means of the Laplace transform method, which is efficient and simple, the radial Klein–Gordon equation is reduced to a first-order differential equation.

Keywords: ring-shaped oscillator, Klein–Gordon equation, Laplace integral transform, bound states

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1. Introduction

The solutions of the Klein–Gordon, the Dirac, and the Duffin–Kemmer–Petiau equations with various potentials have great importance in nuclear physics and high energy physics. These equations are the most frequently used wave equations for the description of particle dynamics in the relativistic quantum mechanics.[1] Many studies have focused on solving the equations with physical potentials. An increasing interest in the study of the Klein–Gordon equation with various potentials, such as five-parameter exponential-type potentials,[2] pseudoharmoic oscillator,[3] and ring-shaped oscillator,[4] has occured in recent years. Different methods, such as supersymmetry,[5] the Nikiforov–Uvarov (NU) method,[6] and the asymptotic iteration method (AIM),[7] have been used to solve the Klein–Gordon equation.

In this paper, we study relativistic solutions for a particle in the presence of noncentral potential. In the spherical coordinates, the noncentral oscillator potential is given by

$$V(r,\theta)=\frac{1}{2}\mu\omega^2r^2+V'(r,\theta), \tag{1}$$

where μ is the rest mass, and ω is the angular frequency of the oscillator. The first term is the oscillator potential, and the second term is the noncentral term. A ring-shaped oscillator potential is defined by[8]

$$V(r,\theta)=\frac{1}{2}\mu\omega^2r^2+\frac{a}{r^2\sin^2\theta}, \tag{2}$$

where a is a constant. Recently, the Klein–Gordon equation with the double ring-shaped oscillator potential has been solved by using the AIM.[7,9]

In this paper, we introduce the Laplace integral transform method to solve the radial Klein–Gordon equation. The Laplace transform method has been used to obtain exact solutions of the Schrödinger equation with the Morse potential,[10] the N-dimensional harmonic oscillator,[11] and the two-dimensional Coulomb potential.[12,13] A very powerful technique for solving differential equations is the Laplace transform, which literally transforms the original differential equation into an elementary algebraic equation.[14]

The rest of this paper is organized as follows. In the next section, the Klein–Gordon equation with equal scalar and vector potentials is shown. Furthermore, the separation of the Klein–Gordon equation with the ring-shaped potential is presented. In the following sections, the angular and the radial parts of the Klein–Gordon equation have been solved. Namely, the normalized wave functions and the energy eigenvalues have been obtained in the next sections. The last section is devoted to a summary and conclusion.

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2. The Klein–Gordon equation with equal scalar and vector potentials

The Klein–Gordon equation with vector and scalar potentials can be written as

$$\left[-\hbar^2c^2\nabla^2+\left(\mu c^2+\frac{1}{2}S(r,\theta,\varphi)\right)^2 - \left(E-\frac{1}{2}V(r,\theta,\varphi)\right)^2\right]\psi(r,\theta,\varphi)=0, \quad (3)$$

where μ and E are the rest mass and the energy of the particle, respectively, $\hbar$ is the reduced Planck constant, c is the velocity of light, $S(r,\theta,\varphi)$ and $V(r,\theta,\varphi)$ are scalar and vector potentials, respectively. For the case of equal scalar and vector potentials, the exact solutions of the Klein–Gordon equation are possible to be obtained only for certain potentials. We assume that the scalar and the vector potentials are equal to each other.[1,7] For the case of $S(r,\theta,\varphi)=V(r,\theta,\varphi)$, in the spherical coordinates, the Klein–Gordon equation for a particle in the presence of a general non-central potential is given by

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi^2}-\left(\frac{E+\mu c^2}{\hbar^2c^2}\right)V(r,\theta) + \left(\frac{E^2-\mu^2c^4}{\hbar^2c^2}\right)\right]\psi(r,\theta,\varphi)=0. \quad (4)$$

To separate the angular and the radial parts of the wave function $\psi(r,\theta,\varphi)$, the wave function is written as

$$\psi(r,\theta,\varphi)=u(r)H(\theta)\Phi(\varphi). \quad (5)$$

The above wave function and the ring shaped oscillator potential are inserted into Eq. (4), the separated equations of the Klein–Gordon equation are

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}u(r)}{\mathrm{d}r}+\left[\left(\frac{E^2-\mu^2c^4}{\hbar^2c^2}\right) - \left(\frac{E+\mu c^2}{\hbar^2c^2}\right)\frac{1}{2}\mu\omega^2r^2-\frac{\ell(\ell+1)}{r^2}\right]u(r)=0, \quad (6)$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}H(\theta)}{\mathrm{d}\theta}\right)-\left[\left(\frac{E+\mu c^2}{\hbar^2c^2}\right)\frac{a}{\sin^2\theta} - \ell(\ell+1)+\frac{m^2}{\sin^2\theta}\right]H(\theta)=0, \quad (7)$$

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2}=-m^2\Phi(\varphi), \quad (8)$$

where m^2 and $\ell(\ell+1)$ are separation constants. It is well known that Eq. (8) leads to

$$\Phi_m(\varphi)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\varphi}, \quad (m=0,\pm1,\pm2,\pm3,\ldots). \quad (9)$$

The solutions of Eqs. (7) and (6) will be obtained in the next sections.

3. Solutions of the angular equation

If we define a variable $x=\cos\theta+1/2$, equation (7) becomes

$$\left[x^2(1-x)^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x(1-x)(1-2x)\frac{\mathrm{d}}{\mathrm{d}x} + \ell(\ell+1)x(1-x)-A^2\right]H(x)=0, \quad (10)$$

where

$$A=\frac{1}{2}\left[m^2+\frac{(E+\mu c^2)a}{\hbar^2c^2}\right]^{1/2}. \quad (11)$$

Considering the boundary conditions at $\theta=0$ ($x=1$) and $\theta=\pi$ ($x=0$), we can write the wave function as

$$H(x)=[x(1-x)]^A f(x). \quad (12)$$

When this wave function is inserted into Eq. (10), the second-order differential equation becomes

$$\left[x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}+[(2A+1)-2(2A+1)x]\frac{\mathrm{d}}{\mathrm{d}x} + \ell(\ell+1)-2A(2A+1)\right]f(x)=0. \quad (13)$$

This equation is known as the hypergeometric differential equation. The solution of Eq. (13) is the hypergeometric function

$$f(x)={}_2\mathrm{F}_1(\alpha,\beta;\gamma;x), \quad (14)$$

where

$$\alpha=2A-\ell, \quad \beta=2A+\ell+1, \quad \gamma=2A+1. \quad (15)$$

Since $\mathrm{Re}(\alpha+\beta-\gamma)=2A>0$, the hypergeometric function diverges at the point $x=1$ except zero and negative integer values [15,16]

$$\alpha=2A-\ell=-n', \quad (n'=0,1,2,\ldots). \quad (16)$$

Thus, equations (11) and (16) give

$$\ell=n'+\sqrt{m^2+\frac{(E+\mu c^2)a}{\hbar^2c^2}}. \quad (17)$$

The angular wave function can be written as

$$\begin{aligned} &H(\theta) \\ &= C_{n'm}\left(\frac{\cos\theta+1}{2}\right)^{A}\left(\frac{1-\cos\theta}{2}\right)^{A} \\ &\times {}_2\mathrm{F}_1\left(-n', 4A+1+n'; 2A+1; \frac{\cos\theta+1}{2}\right), \end{aligned} \tag{18}$$

where $C_{n'm}$ is determined from the orthogonality condition of the hypergeometric functions. The normalization condition of the angular wave function is

$$\int_0^{\pi}[H(\theta)]^2\sin\theta\mathrm{d}\theta = 1. \tag{19}$$

As the orthogonality condition of the hypergeometric functions is[16]

$$\begin{aligned} &\int_0^1 x^{\gamma-1}(1-x)^{\alpha-\gamma}{}_2\mathrm{F}_1(-n,\alpha+n;\gamma;x)^2\mathrm{d}x \\ &= \frac{n!}{(2n+\alpha)}\cdot\frac{\Gamma(\gamma)^2\Gamma(n+\alpha-\gamma+1)}{\Gamma(n+\alpha)\Gamma(n+\gamma)}, \end{aligned} \tag{20}$$

the normalization constant is determined to be

$$C_{n'm} = \sqrt{\frac{(2n'+4A+1)\Gamma(n'+4A+1)}{2n'!\Gamma(4A+1)^2}}. \tag{21}$$

Consequently, we have obtained the normalized angular wave functions

$$\begin{aligned} &H_{n'm}(\theta) \\ &= \sqrt{\frac{(2n'+4A+1)\Gamma(n'+4A+1)}{2n'!\Gamma(4A+1)^2}}\left(\frac{1-\cos^2\theta}{4}\right)^{A} \\ &\times {}_2\mathrm{F}_1\left(-n', 4A+1+n'; 2A+1; \frac{\cos\theta+1}{2}\right). \end{aligned} \tag{22}$$

4. Solutions of the radial equation

If we use the following abbreviations:

$$\begin{aligned} \kappa &= \frac{E^2-\mu^2c^4}{\hbar c}\left[\frac{2}{\mu\omega^2(E+\mu c^2)}\right]^{1/2}, \\ \xi^2 &= \left[\frac{\mu\omega^2(E+\mu c^2)}{2\hbar^2c^2}\right]^{1/2}, \end{aligned} \tag{23}$$

and we define a new variable $\rho = \xi^2 r^2$, equation (6) can be arranged as

$$\left[\rho\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{3}{2}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{1}{4}\left(\kappa-\rho-\frac{\ell(\ell+1)}{\rho}\right)\right]u(\rho) = 0. \tag{24}$$

Considering the asymptotic behavior of the radial wave function at $\rho\to 0$ and $\rho\to\infty$, physically we should have

$$u(\rho) = \rho^{\ell/2}\mathrm{e}^{-\rho/2}f(\rho). \tag{25}$$

If we use this physically acceptable solution, then Eq. (24) can not be reduced to a first-order differential equation by the means of Laplace transform. Therefore, instead of this physically acceptable solution,we take the wave function of the form

$$u(\rho) = \rho^{\ell'}\phi(\rho), \tag{26}$$

where ℓ' is a constant, and $\phi(\rho) = \rho^{\ell/2-\ell'}\mathrm{e}^{-\rho/2}f(\rho)$. Inserting Eq. (26) into Eq. (24), we obtain

$$\begin{aligned} &\left[\rho\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \left(\frac{3}{2}+2\ell'\right)\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{1}{4}\left(\kappa-\rho-\frac{\ell(\ell+1)}{\rho}\right.\right. \\ &\left.\left.+\frac{4\ell'(\ell'-1)+6\ell'}{\rho}\right)\right]\phi(\rho) = 0. \end{aligned} \tag{27}$$

If we eliminate the terms with $\frac{\ell(\ell+1)}{\rho}$ and $\frac{4\ell'(\ell'-1)+6\ell'}{\rho}$, then the Laplace integral transform can be carried out. So we should have

$$\ell' = -\frac{\ell+1}{2}, \tag{28}$$

and consequently Eq. (27) leads to

$$\left[\rho\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \left(\frac{3}{2}+2\ell'\right)\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{\kappa-\rho}{4}\right]\phi(\rho) = 0. \tag{29}$$

4.1. Energy eigenvalues

The Laplace transform of function $\phi(\rho)$ is[14]

$$\mathscr{L}\big[\phi(\rho)\big] = y(s) = \int_0^{\infty}\mathrm{e}^{-s\rho}\phi(\rho)\mathrm{d}\rho. \tag{30}$$

Applying the Laplace transform to Eq. (29), we obtain the first-order differential equation as

$$\left(s^2-\frac{1}{4}\right)y'(s) = \left[\frac{\kappa}{4} + \left(2\ell'-\frac{1}{2}\right)s\right]y(s). \tag{31}$$

This equation can be easily solved to obtain

$$y(s) = C\left(s-\frac{1}{2}\right)^{\kappa/4+\ell'-1/4}\left(s+\frac{1}{2}\right)^{-\kappa/4+\ell'-1/4} \tag{32}$$

where C is an integration constant. If $\kappa/4+\ell'-1/4$ is a positive integer, then $s=1/2$, and the wave function is square-integrable, since its asymptotic form contains $\mathrm{e}^{-\rho/2}$, namely, $y(s)$ has no singularity at $s=1/2$. So we must have

$$\frac{\kappa}{4}+\ell'-\frac{1}{4} = n, \quad (n=0,1,2,3,\ldots), \tag{33}$$

where n is the radial quantum number. This condition gives the energy spectrum

$$\kappa_n = 4n-4\ell'+1. \tag{34}$$

If Eqs. (23) and (28) are inserted into Eq. (34), the energy spectrum becomes

$$\sqrt{\frac{(E_{nm}^2-\mu^2c^4)(E_{nm}-\mu c^2)}{2\mu c^2}}=\left(N+\frac{3}{2}\right)\hbar\omega, \quad (35)$$

where

$$N=2n+n'+\sqrt{m^2+\frac{(E_{nm}+\mu c^2)a}{\hbar^2c^2}},$$
$$(n,n',|m|=0,1,2,\ldots). \quad (36)$$

4.1.1. The nonrelativistic limit

For the case $a=0$, the ring-shaped oscillator potential reduces to the 3D harmonic oscillator potential. In the case of 3D harmonic oscillator, N in Eq. (36) are integers

$$N=2n+n'+|m|, \quad (N=0,1,2,\ldots). \quad (37)$$

In the nonrelativistic limit, applying the following transformation:

$$E_{nm}-\mu c^2\to E_N, \quad E_{nm}+\mu c^2\to 2\mu c^2,$$

and inserting this transformation into Eq. (35) yield

$$E_N=\left(N+\frac{3}{2}\right)\hbar\omega, \quad (N=0,1,2,\ldots). \quad (38)$$

It is well known that Eq. (38) represents the energy spectrum of the 3D harmonic oscillator.

4.2. Radial eigenfunctions

The corresponding radial eigenfunctions can be determined by using the Laplace transform. We use the following abbreviations:

$$\Lambda=-n, \quad \zeta=n-2\ell'+\frac{1}{2}\cdot \quad (39)$$

Inserting Eqs. (33) and (39) into Eq. (32), we can write function $y(s)$ as

$$y(s)=C\left(s-\frac{1}{2}\right)^{-\Lambda}\left(s+\frac{1}{2}\right)^{-\zeta}=Cy_1(s)y_2(s), \quad (40)$$

where

$$y_1(s)=\left(s-\frac{1}{2}\right)^{-\Lambda}, \quad y_2(s)=\left(s+\frac{1}{2}\right)^{-\zeta}.$$

The inverse Laplace transform determines function $\phi(\rho)$. The inverse Laplace transform of function $\phi(\rho)$ is found by using the convolution theorem.[14] We consider that

$$\mathscr{L}^{-1}\big[y_1(s)\big]=y_1(t), \quad \mathscr{L}^{-1}\big[y_2(s)\big]=y_2(t) \quad (41)$$

in this instance, the convolution integral is

$$\phi(\rho)=(-1)^nC_1\mathrm{e}^{-\rho/2}\int_0^\rho t^{\Lambda-1}(\rho-t)^{\zeta-1}\mathrm{e}^t\mathrm{d}t, \quad (42)$$

where C_1 is a constant. The integral representation is expressed by

$$\int_0^u x^{\nu-1}(u-x)^{q-1}\mathrm{e}^{\beta x}\mathrm{d}x$$
$$=\frac{\Gamma(q)\Gamma(\nu)}{\Gamma(q+\nu)}u^{q+\nu-1}{}_1\mathrm{F}_1(\nu,q+\nu,\beta u), \quad (43)$$

where ${}_1\mathrm{F}_1(\nu,q+\nu,\beta u)$ is the confluent hypergeometric function.[17] So Eq. (42) becomes

$$\phi(\rho)=(-1)^nC_2\mathrm{e}^{-\rho/2}\rho^{-2\ell'-\frac{1}{2}}\times{}_1\mathrm{F}_1\left(-n,-2\ell'+\frac{1}{2},\rho\right). \quad (44)$$

When this function is inserted into Eq. (26), the physically wave function is obtained as

$$u_{nn'm}(r)=(-1)^nC_2\xi^{-\ell}r^\ell\mathrm{e}^{-\xi^2r^2/2}\times{}_1\mathrm{F}_1\left(-n,\ell+\frac{3}{2},\xi^2r^2\right), \quad (45)$$

where C_2 is the normalization constant determined from $\int_0^\infty r^2|u(r)|^2\mathrm{d}r=1$. The confluent hypergeometric function can be written in terms of the Laguerre polynomial[18]

$${}_1\mathrm{F}_1(-n,\alpha+1,z)=\frac{n!\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)}\mathrm{L}_n^\alpha(z). \quad (46)$$

By applying the orthogonality condition of Laguerre polynomials, the normalization constant is determined as

$$C_2=\left[\frac{2\xi^3(n+\ell+1/2)!}{n!(\ell+1/2)!^2}\right]^{1/2}, \quad (47)$$

and therefore the radial wave function is obtained as

$$u_{nn'm}(r)=(-1)^n\left[\frac{2\xi^{2\ell+3}(n+\ell+1/2)!}{n!(\ell+1/2)!^2}\right]^{1/2}\times r^\ell\mathrm{e}^{-\xi^2r^2/2}\mathrm{L}_n^{\ell+1/2}(\xi^2r^2). \quad (48)$$

5. Conclusion

In this study, we have investigated the solutions of the Klein–Gordon equation for a ring-shaped oscillator potential under the condition of equal vector and scalar potentials. The exact solutions of both angular and radial equations have been obtained. For the angular equation, we have presented normalized wave functions in terms of the hypergeometric functions.

Likewise, for the radial equation, we have obtained energy eigenvalues and normalized wave functions by using the Laplace integral transform, which is a useful and simple method. When $a = 0$, the ring-shaped oscillator potential is reduced to the 3D harmonic oscillator potential, therefore, we have derived the energy spectrum for the 3D harmonic oscillator in the nonrelativistic limit. By means of a Laplace integral transform, the second-order differential equation can be reduced to the first-order differential equation.

References

[1] Alhaidari A, Bahlouli B and Al-Hasan A 2006 *Phys. Lett.* A **349** 87

[2] Diao Y F, Yi L Z and Jia C S 2004 *Phys. Lett.* A **332** 157

[3] Chen G, Chen Z and Lou Z M 2004 *Chin. Phys.* **13** 279

[4] Qiang W C 2003 *Chin. Phys.* **12** 136

[5] Levai G 1992 *J. Phys.* A: *Math. Gen.* **25** L521

[6] Yasuk F, Berkdemir C, Berkdemir A and Önem C 2005 *Phys. Scr.* **71** 340

[7] Yasuk F and Durmus A 2008 *Phys. Scr.* **77** 015005

[8] Quesne C A 1988 *J. Phys.* A: *Math. Gen.* **21** 3093

[9] Durmus A and Yasuk F 2007 *J. Chem. Phys.* **126** 074108

[10] Chen G 2004 *Phys. Lett.* A **326** 55

[11] Chen G 2004 *Phys. Lett.* A **328** 123

[12] Chen G 2005 *Phys. Scr.* **71** 233

[13] Ran Y, Xue L, Hu S and Su R K 2000 *J. Phys.* A: *Math. Gen.* **33** 9265

[14] Schiff J L 1999 *The Laplace Transform*: *Theory and Applications* (New York: Springer) p. 233

[15] Chen C Y, Liu C L and Lu F L 2010 *Phys. Lett.* A **374** 1346

[16] Gradshteyn I S and Ryzhik I M 2007 *Table of Integrals, Series, and Products* (New York: Academic Press) p. 1171

[17] Abramowitz M and Stegun I 1972 *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables* (Wa: Dover Publications) p. 1058

[18] Lebedev N N 1965 *Special Functions and Their Applications* (New York: Prentice-Hall) p. 308