Symmetry of the Lagrangians of holonomic nonconservative system in event space*

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This paper analyzes the symmetry of Lagrangians and the conserved quantity for the holonomic non-conservative system in the event space. The criterion and the definition of the symmetry are proposed first, then a quantity caused by the symmetry and its existence condition are given. An example is shown to illustrate the application of the result at the end.

Keywords: symmetry of Lagrangians, event space, holonomic nonconservative system, conserved quantity

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1. Introduction

The symmetry principle is one of the top-level principles in physics. There are mainly three kinds of symmetries, i.e., Noether symmetry, Lie symmetry, and Mei symmetry, for the mechanical system. In recent years, some important results have been obtained for symmetries and conserved quantities in dynamical systems.[1−6] It was in the starting investigation about a dynamical system in event space that Synge established the parameter equations of conservative holonomic dynamical systems in 1960[7] and Rumyantsev gave the Lagrangian with multipliers for the conservative non-holonomic dynamical system in 1982.[8] A great deal of attention has been paid to the dynamical system in event space.[9−13] And in 1997, Zhang[14] discussed the differential variational principle of a dynamical system in event space.

In 1966, Currie and Saletan studied the problem of equivalent Lagrangians for the single-degree-of-freedom mechanical system and pointed out that there exist conserved quantities in this case.[15] Reference [16] extended the result to the general multi-degree-of-freedom systems in 1981. Zhao and Mei[17] named the equivalence of Lagrianians the symmetry of the Lagrianians. In addition, many other important results on the symmetry of Lagrianians have been obtained.[18−24] This paper extends the symmetry of Lagrianians to the holonomic non-conservative mechanical system in event space.

2. Symmetry of Lagrianians

Consider a holonomic mechanical system, whose configuration space is determined by n generalized coordinates q_s $(s = 1, 2, \ldots, n)$. Transform the configuration space to the event space, in which the point is determined by generalized coordinates $\boldsymbol{q}$ and time t, and introduce signals

$$x_1 = t, \quad x_{s+1} = q_s, \quad (s = 1, 2, \ldots, n). \tag{1}$$

All variables x_α $(\alpha = 1, 2, \ldots, n+1)$ can be regarded as known functions of a certain parameter τ, namely, $x_\alpha = x_\alpha(\tau)$. Let

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau} = x'_\alpha, \ (\alpha = 1, 2, \ldots, n+1), \tag{2}$$

and they are not equal to zero at the same time, we obtain

$$\dot{x}_\alpha = \frac{\mathrm{d}x_\alpha}{\mathrm{d}t} = \frac{x'_\alpha}{x'_1}, \ (\alpha = 1, 2, \ldots, n+1). \tag{3}$$

Then the Lagrangian and the non-potential generalized forces of the system in the event space can be respectively written as[14]

$$\Lambda(\boldsymbol{x}, \boldsymbol{x}') = x'_1 L\left(\boldsymbol{x}, \frac{x'_2}{x'_1}, \ldots, \frac{x'_{n+1}}{x'_1}\right), \tag{4}$$

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$$P''_\alpha = \begin{cases} -\sum\limits_{s=1}^{n} x'_{s+1} Q''_s \left(\boldsymbol{x}, \dfrac{x'_2}{x'_1}, \ldots, \dfrac{x'_{n+1}}{x'_1}\right), \\ \qquad (\alpha = 1), \\ x'_1 Q''_s \left(\boldsymbol{x}, \dfrac{x'_2}{x'_1}, \ldots, \dfrac{x'_{n+1}}{x'_1}\right), \\ \qquad (\alpha = s+1), \quad (s = 1, \ldots, n), \end{cases} \tag{5}$$

where Q''_s is the non-potential generalized force in the configuration space. The differential equation of motion of the system is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha} - \frac{\partial \Lambda}{\partial x_\alpha} = P''_\alpha, \ (\alpha = 1, 2, \ldots, n+1). \tag{6}$$

There are two groups of dynamical functions, Λ, $\boldsymbol{P}''$ and $\bar{\Lambda}$, $\bar{\boldsymbol{P}}''$. Let

$$\begin{aligned} \Lambda_\alpha &= \frac{\partial^2 \Lambda}{\partial x'_\alpha \partial x'_\beta} x''_\beta + \frac{\partial^2 \Lambda}{\partial x'_\alpha \partial x_\beta} x'_\beta - \frac{\partial \Lambda}{\partial x_\alpha} - P''_\alpha \\ &= W_{\alpha\beta} x''_\beta + \frac{\partial^2 \Lambda}{\partial x'_\alpha \partial x_\beta} x'_\beta - \frac{\partial \Lambda}{\partial x_\alpha} - P''_\alpha, \end{aligned} \tag{7}$$

$$\begin{aligned} \bar{\Lambda}_\alpha &= \frac{\partial^2 \bar{\Lambda}}{\partial x'_\alpha \partial x'_\beta} x''_\beta + \frac{\partial^2 \bar{\Lambda}}{\partial x'_\alpha \partial x_\beta} x'_\beta - \frac{\partial \bar{\Lambda}}{\partial x_\alpha} - \bar{P}''_\alpha \\ &= \bar{W}_{\alpha\beta} x''_\beta + \frac{\partial^2 \bar{\Lambda}}{\partial x'_\alpha \partial x_\beta} x'_\beta - \frac{\partial \bar{\Lambda}}{\partial x_\alpha} - \bar{P}''_\alpha, \end{aligned} \tag{8}$$

where

$$\begin{aligned} W_{\alpha\beta} &= \frac{\partial^2 \Lambda}{\partial x'_\alpha \partial x'_\beta}, \\ \bar{W}_{\alpha\beta} &= \frac{\partial^2 \bar{\Lambda}}{\partial x'_\alpha \partial x'_\beta}. \end{aligned} \tag{9}$$

Definition 1 For system (6), if we can obtain

$$\Lambda_\alpha = 0, \quad (\alpha = 1, 2, \ldots, n+1) \tag{10}$$

from

$$\bar{\Lambda}_\alpha = 0, \quad (\alpha = 1, 2, \ldots, n+1), \tag{11}$$

and vice versa, then it is called the symmetry of Lagranians.

In the light of Eq. (8), x''_β is obtained from Eq. (11) as

$$\begin{aligned} x''_\beta &= \bar{W}^{-1}_{\gamma\beta} \left(\frac{\partial \bar{\Lambda}}{\partial x_\gamma} - \frac{\partial^2 \bar{\Lambda}}{\partial x'_\gamma \partial x_\chi} x'_\chi + \bar{P}''_\gamma \right) \\ &= \bar{U}^{\beta\gamma} \left(\frac{\partial \bar{\Lambda}}{\partial x_\gamma} - \frac{\partial^2 \bar{\Lambda}}{\partial x'_\gamma \partial x_\chi} x'_\chi + \bar{P}''_\gamma \right), \end{aligned} \tag{12}$$

where

$$\bar{W}_{\alpha\beta} \bar{U}^{\beta\gamma} = \delta^\gamma_\alpha. \tag{13}$$

Substituting Eq. (12) into Eq. (10) yields

$$\begin{aligned} & W_{\alpha\beta} \bar{U}^{\beta\gamma} \left(\frac{\partial \bar{\Lambda}}{\partial x_\gamma} - \frac{\partial^2 \bar{\Lambda}}{\partial x'_\gamma \partial x_\chi} x'_\chi + \bar{P}''_\gamma \right) \\ &= \frac{\partial \Lambda}{\partial x_\alpha} - \frac{\partial^2 \Lambda}{\partial x'_\alpha \partial x_\beta} x'_\beta + P''_\alpha, \\ &\quad (\alpha = 1, 2, \ldots, n+1), \end{aligned} \tag{14}$$

then there exists the following criterion.

Criterion 1 If two groups of dynamical functions Λ, $\boldsymbol{P}''$ and $\bar{\Lambda}$, $\bar{\boldsymbol{P}}''$ satisfy Eq. (14), then the corresponding symmetry is the symmetry of Lagrangians of system (6).

3. Conserved quantity deduced from the symmetry of Lagrangians

For system (6), there is a proposition in the following about the symmetry of Lagrangians.

Proposition 1 If the two groups of non-potential generalized force $\boldsymbol{P}''$ and $\bar{\boldsymbol{P}}''$ satisfy the condition

$$\frac{\partial \bar{P}''_\gamma}{\partial x'_\varepsilon} = A^\alpha_\gamma \frac{\partial P''_\alpha}{\partial x'_\varepsilon}, \tag{15}$$

then the symmetry of the Lagrangians of system (6) will lead to a conserved quantity

$$I_{\mathrm{L}} = \mathrm{Tr}(A^m) = \text{const.}, \tag{16}$$

where

$$A^\alpha_\gamma = \bar{W}_{\gamma\beta} U^{\beta\alpha}, \quad W_{\alpha\gamma} U^{\gamma\beta} = \delta^\beta_\alpha. \tag{17}$$

Proof Equating Eqs. (7) and (14) yields

$$\begin{aligned} \Lambda_\alpha &= W_{\alpha\beta} x''_\beta - W_{\alpha\beta} \bar{U}^{\beta\gamma} \left(\frac{\partial \bar{\Lambda}}{\partial x_\gamma} - \frac{\partial^2 \bar{\Lambda}}{\partial x'_\gamma \partial x_\chi} x'_\chi + \bar{P}''_\gamma \right) \\ &= W_{\alpha\beta} \bar{U}^{\beta\gamma} \left(\bar{W}_{\gamma\chi} x''_\chi + \frac{\partial^2 \bar{\Lambda}}{\partial x'_\gamma \partial x_\chi} x'_\chi - \frac{\partial \bar{\Lambda}}{\partial x_\gamma} - \bar{P}''_\gamma \right) \\ &= W_{\alpha\beta} \bar{U}^{\beta\gamma} \bar{\Lambda}_\gamma. \end{aligned} \tag{18}$$

Immediately, we obtain

$$\bar{\Lambda}_\gamma = A^\alpha_\gamma \Lambda_\alpha. \tag{19}$$

By virtue of formula (17), there is

$$\bar{W}_{\gamma\beta} = A^\alpha_\gamma W_{\alpha\beta}. \tag{20}$$

Taking the partial derivative of Eq. (20) with respect to x'_ε, we obtain

$$\begin{aligned} \frac{\partial \bar{W}_{\gamma\beta}}{\partial x'_\varepsilon} &= \frac{\partial^3 \bar{\Lambda}}{\partial x'_\gamma \partial x'_\beta \partial x'_\varepsilon} \\ &= \frac{\partial A^\alpha_\gamma}{\partial x'_\varepsilon} W_{\alpha\beta} + A^\alpha_\gamma \frac{\partial W_{\alpha\beta}}{\partial x'_\varepsilon} \end{aligned}$$

$$= \frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\varepsilon}} W_{\alpha\beta} + A^{\alpha}_{\gamma} \frac{\partial^3 \Lambda}{\partial x'_{\alpha} \partial x'_{\beta} \partial x'_{\varepsilon}}, \quad (21)$$

namely,

$$\frac{\partial \bar{W}_{\gamma\varepsilon}}{\partial x'_{\beta}} - A^{\alpha}_{\gamma} \frac{\partial W_{\alpha\varepsilon}}{\partial x'_{\beta}} = \frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\varepsilon}} W_{\alpha\beta}. \quad (22)$$

Taking the partial derivative of $A^{\alpha}_{\gamma} W_{\alpha\varepsilon}$ with respect to x'_{β} gives

$$\frac{\partial (A^{\alpha}_{\gamma} W_{\alpha\varepsilon})}{\partial x'_{\beta}} = \frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\beta}} W_{\alpha\varepsilon} + A^{\alpha}_{\gamma} \frac{\partial W_{\alpha\varepsilon}}{\partial x'_{\beta}}. \quad (23)$$

Substituting Eq. (17) into Eq. (23) yields

$$A^{\alpha}_{\gamma} \frac{\partial W_{\alpha\varepsilon}}{\partial x'_{\beta}} = \frac{\partial \bar{W}_{\gamma\varepsilon}}{\partial x'_{\beta}} - \frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\beta}} W_{\alpha\varepsilon}. \quad (24)$$

Contrasting Eqs. (22) and (24), we obtain

$$\frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\beta}} W_{\alpha\varepsilon} = \frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\varepsilon}} W_{\alpha\beta}. \quad (25)$$

Taking the partial derivative of Eq. (14) with respect to x'_{ε} gives

$$\begin{aligned} &\frac{\partial^3 \bar{\Lambda}}{\partial x'_{\gamma} \partial x_{\chi} \partial x'_{\varepsilon}} x'_{\chi} + \frac{\partial^2 \bar{\Lambda}}{\partial x'_{\gamma} \partial x_{\varepsilon}} - \frac{\partial^2 \bar{\Lambda}}{\partial x_{\gamma} \partial x'_{\varepsilon}} - \frac{\partial \bar{P}''_{\gamma}}{\partial x'_{\varepsilon}} \\ &= \frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\varepsilon}} \left(\frac{\partial^2 \Lambda}{\partial x'_{\alpha} \partial x_{\beta}} x'_{\beta} - \frac{\partial \Lambda}{\partial x_{\alpha}} - P''_{\alpha} \right) \\ &\quad + A^{\alpha}_{\gamma} \left(\frac{\partial^3 \Lambda}{\partial x'_{\alpha} \partial x_{\beta} \partial x'_{\varepsilon}} x'_{\beta} + \frac{\partial^2 \Lambda}{\partial x'_{\alpha} \partial x_{\varepsilon}} \right. \\ &\quad \left. - \frac{\partial^2 \Lambda}{\partial x_{\alpha} \partial x'_{\varepsilon}} - \frac{\partial P''_{\alpha}}{\partial x'_{\varepsilon}} \right). \end{aligned} \quad (26)$$

In view of Eqs. (7) and (10), we have

$$W_{\alpha\beta} x''_{\beta} = \frac{\partial \Lambda}{\partial x_{\alpha}} - \frac{\partial^2 \Lambda}{\partial x'_{\alpha} \partial x_{\beta}} x'_{\beta} + P''_{\alpha}. \quad (27)$$

Taking Eq. (27) into Eq. (26), we can write Eq. (26) as

$$\begin{aligned} &\frac{\partial^2 \bar{\Lambda}}{\partial x'_{\gamma} \partial x_{\varepsilon}} - \frac{\partial^2 \bar{\Lambda}}{\partial x_{\gamma} \partial x'_{\varepsilon}} - \frac{\partial \bar{P}''_{\gamma}}{\partial x'_{\varepsilon}} \\ &= -\frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\varepsilon}} W_{\alpha\beta} x''_{\beta} - \frac{\partial \bar{W}_{\gamma\varepsilon}}{\partial x_{\beta}} x'_{\beta} + A^{\alpha}_{\gamma} \frac{\partial W_{\alpha\varepsilon}}{\partial x_{\beta}} x'_{\beta} \\ &\quad + A^{\alpha}_{\gamma} \left(\frac{\partial^2 \Lambda}{\partial x'_{\alpha} \partial x_{\varepsilon}} - \frac{\partial^2 \Lambda}{\partial x_{\alpha} \partial x'_{\varepsilon}} - \frac{\partial P''_{\alpha}}{\partial x'_{\varepsilon}} \right). \end{aligned} \quad (28)$$

By virtue of formula (24), there is

$$-\frac{\partial \bar{W}_{\gamma\varepsilon}}{\partial x_{\beta}} x'_{\beta} + A^{\alpha}_{\gamma} \frac{\partial W_{\alpha\varepsilon}}{\partial x_{\beta}} x'_{\beta} = -\frac{\partial A^{\alpha}_{\gamma}}{\partial x_{\beta}} W_{\alpha\varepsilon} x'_{\beta}. \quad (29)$$

Considering Eqs. (25) and (29), we can express Eq. (28) as

$$\begin{aligned} &\frac{\partial^2 \bar{\Lambda}}{\partial x'_{\gamma} \partial x_{\varepsilon}} - \frac{\partial^2 \bar{\Lambda}}{\partial x_{\gamma} \partial x'_{\varepsilon}} - \frac{\partial \bar{P}''_{\gamma}}{\partial x'_{\varepsilon}} \\ &= -\frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\beta}} W_{\alpha\varepsilon} x''_{\beta} - \frac{\partial A^{\alpha}_{\gamma}}{\partial x_{\beta}} W_{\alpha\varepsilon} x'_{\beta} \\ &\quad + A^{\alpha}_{\gamma} \left(\frac{\partial^2 \Lambda}{\partial x'_{\alpha} \partial x_{\varepsilon}} - \frac{\partial^2 \Lambda}{\partial x_{\alpha} \partial x'_{\varepsilon}} - \frac{\partial P''_{\alpha}}{\partial x'_{\varepsilon}} \right). \end{aligned} \quad (30)$$

As the independent variables of dynamical equations, Λ and $\bar{\Lambda}$ do not include τ, hence

$$\frac{\mathrm{d} A^{\alpha}_{\gamma}}{\mathrm{d}\tau} = \frac{\partial A^{\alpha}_{\gamma}}{\partial x'_{\beta}} x''_{\beta} + \frac{\partial A^{\alpha}_{\gamma}}{\partial x_{\beta}} x'_{\beta}. \quad (31)$$

From Eqs. (30) and (31), we can obtain

$$\begin{aligned} &\frac{\partial^2 \bar{\Lambda}}{\partial x'_{\gamma} \partial x_{\varepsilon}} - \frac{\partial^2 \bar{\Lambda}}{\partial x_{\gamma} \partial x'_{\varepsilon}} - \frac{\partial \bar{P}''_{\gamma}}{\partial x'_{\varepsilon}} \\ &= A^{\alpha}_{\gamma} \left(\frac{\partial^2 \Lambda}{\partial x'_{\alpha} \partial x_{\varepsilon}} - \frac{\partial^2 \Lambda}{\partial x_{\alpha} \partial x'_{\varepsilon}} - \frac{\partial P''_{\alpha}}{\partial x'_{\varepsilon}} \right) \\ &\quad - (A^{\alpha}_{\gamma})' W_{\alpha\varepsilon}. \end{aligned} \quad (32)$$

Introduce seven $n \times n$ matrices $\boldsymbol{A}$, $\boldsymbol{T}$, $\bar{\boldsymbol{T}}$, $\boldsymbol{W}$, $\bar{\boldsymbol{W}}$, $\boldsymbol{U}$, and $\bar{\boldsymbol{U}}$ whose elements are respectively

$$\begin{aligned} &\boldsymbol{A} = (A^{j}_{i}), \ \boldsymbol{T} = \left(\frac{\partial^2 \Lambda}{\partial x'_{i} \partial x_{j}} - \frac{\partial^2 \Lambda}{\partial x_{i} \partial x'_{j}} \right), \\ &\bar{\boldsymbol{T}} = \left(\frac{\partial^2 \bar{\Lambda}}{\partial x'_{i} \partial x_{j}} - \frac{\partial^2 \bar{\Lambda}}{\partial x_{i} \partial x'_{j}} \right), \\ &\boldsymbol{W} = (W_{ij}), \quad \bar{\boldsymbol{W}} = (\bar{W}_{ij}), \\ &\boldsymbol{U} = (U^{ij}), \quad \bar{\boldsymbol{U}} = (\bar{U}^{ij}). \end{aligned} \quad (33)$$

Substituting Eqs. (15) and (33) into Eq. (32) yields

$$\boldsymbol{A}' = \boldsymbol{A}\boldsymbol{T}\boldsymbol{U} - \bar{\boldsymbol{T}}\boldsymbol{U}. \quad (34)$$

Considering the facts that $\boldsymbol{T}$ and $\bar{\boldsymbol{T}}$ are antisymmetric, $\boldsymbol{U}$ and $\bar{\boldsymbol{W}}$ are symmetric, and $\boldsymbol{A} = \bar{\boldsymbol{W}}\boldsymbol{U}$, then for any positive integer m, we have

$$\begin{aligned} \boldsymbol{A}'\boldsymbol{A}^{m-1} &= (\boldsymbol{A}\boldsymbol{T}\boldsymbol{U} - \bar{\boldsymbol{T}}\boldsymbol{U})\boldsymbol{A}^{m-1} \\ &= \bar{\boldsymbol{W}}\boldsymbol{U}\boldsymbol{T}\boldsymbol{U}(\bar{\boldsymbol{W}}\boldsymbol{U})^{m-1} \\ &\quad - \bar{\boldsymbol{T}}\boldsymbol{U}(\bar{\boldsymbol{W}}\boldsymbol{U})^{m-1}. \end{aligned} \quad (35)$$

Due to the properties of the trace of matrix, $\mathrm{Tr}[\bar{\boldsymbol{W}}\boldsymbol{U}\boldsymbol{T}\boldsymbol{U}(\bar{\boldsymbol{W}}\boldsymbol{U})^{m-1}] = 0$ and $\mathrm{Tr}[\bar{\boldsymbol{T}}\boldsymbol{U}(\bar{\boldsymbol{W}}\boldsymbol{U})^{m-1}] = 0$, namely,

$$\mathrm{Tr}(\boldsymbol{A}'\boldsymbol{A}^{m-1}) = 0, \quad (36)$$

at the same time

$$\frac{\mathrm{d}\boldsymbol{A}^{m}}{\mathrm{d}\tau} = m\boldsymbol{A}'\boldsymbol{A}^{m-1}. \quad (37)$$

Equating Eqs. (36) and (37), we have $\mathrm{Tr}[\mathrm{d}\boldsymbol{A}^{m}/\mathrm{d}\tau] = 0$. Then equation (16) is obtained, and Proposition 1 is verified.

4. Illustrative example

One example is shown to illustrate the above result. In the configuration space, the Lagrangian of a system is

$$L = \frac{1}{2}\dot{q}^2 \exp(-\gamma t), \quad (\gamma = \text{const.}). \tag{38}$$

The corresponding Lagrangian of the system in the event space can be expressed as

$$\Lambda = \frac{1}{2}x_1' \left(\frac{x_2'}{x_1'}\right)^2 \exp(-\gamma x_1). \tag{39}$$

Let two kinds of non-potential generalized forces be

$$\begin{aligned} P_1'' &= -\frac{\gamma}{2}\left(\frac{x_2'}{x_1'}\right) x_2' \,\mathrm{e}^{-\gamma x_1} - \frac{3}{4}\left(\frac{x_2'}{x_1'}\right)^2 \frac{x_1''}{x_1'} \,\mathrm{e}^{-\gamma x_1}, \\ P_2'' &= 0. \end{aligned} \tag{40}$$

Considering Eq. (7), we can obtain

$$\begin{aligned} \Lambda_1 &= \frac{1}{4}\left(\frac{x_2'}{x_1'}\right)^2 \frac{x_1''}{x_1'} \,\mathrm{e}^{-\gamma x_1} - \frac{x_2'}{(x_1')^2} x_2'' \,\mathrm{e}^{-\gamma x_1} - P_1'', \\ \Lambda_2 &= -\frac{x_2'}{(x_1')^2} x_1'' \,\mathrm{e}^{-\gamma x_1} + \frac{x_2''}{x_1'} \,\mathrm{e}^{-\gamma x_1} \\ &\quad - \gamma x_2' \,\mathrm{e}^{-\gamma x_1} - P_2''. \end{aligned} \tag{41}$$

Substituting Eq. (40) into Eq. (41), we have

$$\begin{aligned} \Lambda_1 &= \left(\frac{x_2'}{x_1'}\right)^2 \frac{x_1''}{x_1'} \,\mathrm{e}^{-\gamma x_1} - \frac{x_2'}{(x_1')^2} x_2'' \,\mathrm{e}^{-\gamma x_1} \\ &\quad + \frac{\gamma}{2}\left(\frac{x_2'}{x_1'}\right) x_2' \,\mathrm{e}^{-\gamma x_1}, \\ \Lambda_2 &= -\frac{x_2'}{(x_1')^2} x_1'' \,\mathrm{e}^{-\gamma x_1} + \frac{x_2''}{x_1'} \,\mathrm{e}^{-\gamma x_1} \\ &\quad - \gamma x_2' \,\mathrm{e}^{-\gamma x_1}. \end{aligned} \tag{42}$$

Choose the other Lagrangian and non-potential generalized forces

$$\bar{\Lambda} = \frac{1}{2}x_1' \left(\frac{x_2'}{x_1'}\right)^2 \exp(-\gamma x_2), \tag{43}$$

$$\bar{P''}_1 = 0,$$

$$\bar{P''}_2 = \gamma x_2' \,\mathrm{e}^{-\gamma x_2} - \frac{3}{2}\frac{(x_2')^2}{x_1'}\gamma \,\mathrm{e}^{-\gamma x_2}, \tag{44}$$

taking Eq. (8) into account, from Eqs. (42)–(44), we have

$$\begin{aligned} \bar{\Lambda}_1 &= \left(\frac{x_2'}{x_1'}\right)^2 \frac{x_1''}{x_1'} \,\mathrm{e}^{-\gamma x_2} - \frac{x_2'}{(x_1')^2} x_2'' \,\mathrm{e}^{-\gamma x_2} \\ &\quad + \frac{\gamma}{2}\left(\frac{x_2'}{x_1'}\right)^2 x_2' \,\mathrm{e}^{-\gamma x_2} \\ &= \frac{\mathrm{e}^{-\gamma x_2}}{\mathrm{e}^{-\gamma x_1}} \Lambda_1, \end{aligned}$$

$$\begin{aligned} \bar{\Lambda}_2 &= -\frac{x_2'}{(x_1')^2} x_1'' \,\mathrm{e}^{-\gamma x_2} + \frac{x_2''}{x_1'} \,\mathrm{e}^{-\gamma x_2} - \gamma x_2' \,\mathrm{e}^{-\gamma x_2} \\ &= \frac{\mathrm{e}^{-\gamma x_2}}{\mathrm{e}^{-\gamma x_1}} \Lambda_2. \end{aligned} \tag{45}$$

Then from Eq. (45), we immediately obtain the following formula:

$$A_1^1 = A_2^2 = \frac{\exp(-\gamma x_2)}{\exp(-\gamma x_1)} = \exp(\gamma x_1 - \gamma x_2). \tag{46}$$

As the non-potential generalized forces (40) and (44) satisfy Eq. (15), from Proposition 1, we can obtain the conserved quantity

$$\begin{aligned} I_{\mathrm{L}} &= A_1^1 + A_2^2 \\ &= 2\frac{\exp(-\gamma x_2)}{\exp(-\gamma x_1)} \\ &= 2\exp(\gamma x_1 - \gamma x_2) \\ &= \text{const.} \end{aligned} \tag{47}$$

5. Conclusion

In this paper, we discuss the symmetry of Lagrangians for the holonomic non-conservative system in the event space. The criterion, the conserved quantity, and the condition under which the symmetry leads to the conserved quantity are given. The generalized coordinates and the time are equivalent in the event space, so the parameter can be chosen flexibly to establish relatively simple mechanical equations. If x_α is treated as the generalized coordinate in the configuration space and $\tau = t$, the results in the event space will become the ones in the configuration space.

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