Explicit multi-symplectic method for the Zakharov–Kuznetsov equation*

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We propose an explicit multi-symplectic method to solve the two-dimensional Zakharov–Kuznetsov equation. The method combines the multi-symplectic Fourier pseudospectral method for spatial discretization and the Euler method for temporal discretization. It is verified that the proposed method has corresponding discrete multi-symplectic conservation laws. Numerical simulations indicate that the proposed scheme is characterized by excellent conservation.

Keywords: multi-symplectic method, Fourier pseudospectral method, Euler method, Zakharov–Kuznetsov equation

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1. Introduction

In this paper, we develop the explicit multi-symplectic Fourier pseudospectral method to solve the Zakharov–Kuznetsov (ZK) equation

$$u_t + (3u^2)_x + u_{xxx} + u_{xyy} = 0. \tag{1}$$

The three-dimensional ZK equation was proposed by Zakharov and Kuznetsov[1] to describe nonlinear ion-acoustic waves in a plasma comprised of cold ions and hot isothermal electrons in the presence of a uniform magnetic field. A variety of physical phenomena, in purely dispersive limit, are governed by this type of equation, such as the long waves on a thin liquid film,[2] the Rossby waves in rotating atmosphere,[3] and the isolated vortex of drift waves in three-dimensional plasma.[4] In spite of such ubiquity, detailed properties of the solutions to Eq. (1) have not yet been fully explored. A cylindrically symmetric solitary wave solution was obtained numerically and its stability has been found by means of a variational estimation.[5] Some exact solitary wave solutions to the ZK equation have been discussed in Refs. [6] and [7]. There have been a few numerical methods proposed for solving the ZK equation. Xu and Shu[8] introduced a local discontinuous Galerkin method to solve the ZK equation. However, symplectic and multi-symplectic methods, especially explicit ones, have been seldom used.

The multi-symplectic methods that can preserve multi-symplectic conservation under appropriate discretization, have attracted much attention in recent years.[9−12] Bridges and Reich[13] proposed a multi-symplectic Preissman scheme by using the implicit midpoint rule. Reich[14] proposed a multi-symplectic Runge–Kutta collocation method for Hamiltonian wave equations. Bridges and Reich[15] introduced the idea of a multi-symplectic Fourier transform for multi-symplectic PDEs under periodic boundary conditions. Then, Chen and Qin[16] proposed a multi-symplectic Fourier pseudospectral method in real space. However, most of the multi-symplectic schemes are implicit, and we need to solve nonlinear algebraic equations. In Ref. [17], Moore and Reich proposed an explicit Euler box scheme for the Hamiltonian PDEs. In this paper, we investigate the multi-symplectic pseudospectral method for the ZK equation. For the temporal discretization, we choose the symplectic Euler scheme. Its advantage lies in the fact that the generated full-discrete scheme is an explicit multi-symplectic scheme, which does not need to solve nonlinear algebraic equations at every time step.

The paper is organized as follows. In Section 2, the multi-symplectic formulation of the ZK equation and some of its corresponding conservation laws are presented. In Section 3, the explicit multi-symplectic pseudospectral method for the ZK equation is constructed. Numerical simulations of the propagation and collision solutions of the ZK equation are pre-

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sented in Section 4, where the efficiency of the proposed method is demonstrated. Finally, conclusions are made in Section 5.

2. Multi-symplectic formulation of the ZK equation

Introducing new variables p, q, ϕ, v, and ω, equation (1) can be reformulated in the following first-order form:

$$\begin{cases} \phi_x = u, \\ \dfrac{1}{2}\phi_t + v_x + w_y = p - 3u^2, \\ w_x - v_y = 0, \\ \dfrac{1}{2}u_t + p_x = 0, \\ u_x - q_y = v, \\ q_x + u_y = w, \end{cases} \tag{2}$$

which can be written as a multi-symplectic Hamiltonian system

$$\boldsymbol{M}z_t + \boldsymbol{K}z_x + \boldsymbol{L}z_y = \boldsymbol{\nabla}_{\boldsymbol{z}}\boldsymbol{S}(\boldsymbol{z}), \tag{3}$$

where $\boldsymbol{z} = [p, u, q, \phi, v, w]^{\mathrm{T}}$, the three skew-symmetric matrices

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the Hamiltonian function $S(z) = up - (v^2 + w^2)/2 - u^3$.

Equation (3) satisfies the multi-symplectic conservation law

$$\omega_t + \kappa_x + \tau_y = 0, \tag{4}$$

where $\omega = \mathrm{d}z \wedge \boldsymbol{M}\mathrm{d}z/2$, $\kappa = \mathrm{d}z \wedge \boldsymbol{K}\mathrm{d}z/2$, and $\tau = \mathrm{d}z \wedge \boldsymbol{L}\mathrm{d}z/2$ are pre-symplectic forms, and $\mathrm{d}z$ satisfies the first variation form of Eq. (3).[13]

For the ZK equation, the corresponding conservation laws are given by

$$\begin{aligned} \omega &= \frac{1}{2}\mathrm{d}u \wedge \mathrm{d}\phi, \\ \kappa &= \mathrm{d}p \wedge \mathrm{d}\phi + \mathrm{d}u \wedge \mathrm{d}v + \mathrm{d}q \wedge \mathrm{d}w, \\ \tau &= \mathrm{d}u \wedge \mathrm{d}w + \mathrm{d}v \wedge \mathrm{d}q. \end{aligned}$$

3. Explicit multi-symplectic pseudospectral method for the ZK equation

We notice that the first-order differential operator ∂_x yields the Fourier spectral differentiation matrix $\boldsymbol{D}_1$. Here, $\boldsymbol{D}_1$ is an $N \times N$ skew-symmetric matrix whose elements are

$$(\boldsymbol{D}_1)_{j,s} = \begin{cases} \dfrac{1}{2}(-1)^{j+s}\mu \cot\left(\mu\dfrac{x_j - x_s}{2}\right), & s \neq j, \\ 0, & s = j, \end{cases}$$

for $j, s = 1, 2, \ldots, N$, and $\mu = 2\pi/L$. For more details, one can consult Ref. [16] and references therein.

In this paper, we only consider the periodic boundary conditions. Then, we divide the computation domain $[x_{\mathrm{L}}, x_{\mathrm{R}}] \times [y_{\mathrm{L}}, y_{\mathrm{R}}]$ into $N_x \times N_y$ equal cells, and set

$$\begin{aligned} \boldsymbol{Z} = [&z_{1,1}, z_{2,1}, \ldots, z_{N_x,1}, z_{1,2}, z_{2,2}, \ldots, z_{N_x,2}, \ldots, \\ &z_{1,N_y}, z_{2,N_y}, \ldots, z_{N_x,N_y}]^{\mathrm{T}}, \end{aligned}$$

where $z_{i,j}$ is approximate to $z(x_i, y_j, t)$, both N_x and N_y are even integers. Then, it is true that $\boldsymbol{A}_1\boldsymbol{Z}$ and $\boldsymbol{A}_2\boldsymbol{Z}$ are the Fourier pseudospectral discretization of z_x and z_y, respectively. Here $\boldsymbol{A}_1 = \boldsymbol{D}_1^x \otimes \boldsymbol{I}_{N_y}$, $\boldsymbol{A}_2 = \boldsymbol{I}_{N_x} \otimes \boldsymbol{D}_1^y$, $\boldsymbol{D}_1^x$ ($\boldsymbol{D}_1^y$) is the $N_x \times N_x$ ($N_y \times N_y$) first-order Fourier spectral differentiation matrix, and $\boldsymbol{I}_{N_x}$ ($\boldsymbol{I}_{N_y}$) is the $N_x \times N_x$ ($N_y \times N_y$) identity matrix. The $\otimes$ denotes the Kronecker inner product.

Discretizing Eq. (2) with the Fourier pseudospectral method in space domain, we can obtain a semi-discrete system

$$\begin{cases} (\boldsymbol{A}_1\boldsymbol{\Phi})_{(l-1)N_x+k} = u_{k,l}, \\ \dfrac{1}{2}\dfrac{\mathrm{d}\phi_{k,l}}{\mathrm{d}t} + (\boldsymbol{A}_1\boldsymbol{V})_{(l-1)N_x+k} + (\boldsymbol{A}_2\boldsymbol{W})_{(l-1)N_x+k} = p_{k,l} - 3u_{k,l}^2, \\ (\boldsymbol{A}_1\boldsymbol{W})_{(l-1)N_x+k} - (\boldsymbol{A}_2\boldsymbol{V})_{(l-1)N_x+k} = 0, \\ \dfrac{1}{2}\dfrac{\mathrm{d}u_{k,l}}{\mathrm{d}t} + (\boldsymbol{A}_1\boldsymbol{P})_{(l-1)N_x+k} = 0, \\ (\boldsymbol{A}_1\boldsymbol{U})_{(l-1)N_x+k} - (\boldsymbol{A}_2\boldsymbol{Q})_{(l-1)N_x+k} = v_{k,l}, \\ (\boldsymbol{A}_1\boldsymbol{Q})_{(l-1)N_x+k} + (\boldsymbol{A}_2\boldsymbol{U})_{(l-1)N_x+k} = w_{k,l}, \end{cases} \tag{5}$$

where

$$\boldsymbol{U} = [u_{1,1}, u_{2,1}, \ldots, u_{N_x,1}, u_{1,2}, u_{2,2}, \ldots, u_{N_x,2}, \ldots, u_{1,N_y}, u_{2,N_y}, \ldots, u_{N_x,N_y}]^{\mathrm{T}},$$

$$\boldsymbol{\Phi} = [\phi_{1,1}, \phi_{2,1}, \ldots, \phi_{N_x,1}, \phi_{1,2}, \phi_{2,2}, \ldots, \phi_{N_x,2}, \ldots, \phi_{1,N_y}, \phi_{2,N_y}, \ldots, \phi_{N_x,N_y}]^{\mathrm{T}},$$

etc.

We can rewrite Eq. (5) in the compact form

$$\boldsymbol{M}\frac{\mathrm{d}}{\mathrm{d}t}z_{k,l} + \sum_{j=1}^{N_x}(\boldsymbol{D}_1^x)_{k,j}\boldsymbol{K}z_{j,l} + \sum_{j=1}^{N_y}(\boldsymbol{D}_1^y)_{l,j}\boldsymbol{L}z_{k,j} = \boldsymbol{\nabla}_z\boldsymbol{S}(z_{k,l}), \tag{6}$$

where $k = 1, 2, \ldots, N_x$, and $l = 1, 2, \ldots, N_y$.

Theorem 1 The Fourier pseudospectral semi-discretization (6) obeys the $N_x \times N_y$ semi-discrete multi-symplectic conservation laws[18]

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_{k,l} + \sum_{j=1}^{N_x}(\boldsymbol{D}_1^x)_{k,j}\kappa_{j,l} + \sum_{j=1}^{N_y}(\boldsymbol{D}_1^y)_{l,j}\tau_{k,j} = 0, \tag{7}$$

where $\omega_{k,l} = (\mathrm{d}z_{k,l} \wedge \boldsymbol{M}\mathrm{d}z_{k,l})/2$, $\kappa_{j,l} = (\mathrm{d}z_{k,l} \wedge \boldsymbol{K}\mathrm{d}z_{j,l} + \mathrm{d}z_{j,l} \wedge \boldsymbol{K}\mathrm{d}z_{k,l})/2$, $\tau_{k,j} = (\mathrm{d}z_{k,l} \wedge \boldsymbol{L}\mathrm{d}z_{k,j} + \mathrm{d}z_{k,j} \wedge \boldsymbol{L}\mathrm{d}z_{k,l})/2$, $k = 1, 2, \ldots, N_x$, $l = 1, 2, \ldots, N_y$.

Since $\boldsymbol{D}_1^x$ and $\boldsymbol{D}_1^y$ are skew-symmetric, we can merge Eq. (7) into the global symplecticity conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{N_x}\sum_{l=1}^{N_y}\omega_{k,l} = 0, \tag{8}$$

which implies the conservation of global symplecticity over time. Therefore, we choose a symplectic integration in time domain for Eq. (6) in order to preserve the global symplecticity.

We apply the symplectic Euler scheme to Eq. (6) and obtain the following equation:

$$\boldsymbol{M}_+D_t^+z_{k,l}^i + \boldsymbol{M}_-D_t^-z_{k,l}^i + \sum_{j=1}^{N_x}(\boldsymbol{D}_1^x)_{k,j}\boldsymbol{K}z_{j,l}^i + \sum_{j=1}^{N_y}(\boldsymbol{D}_1^y)_{l,j}\boldsymbol{L}z_{k,j}^i = \boldsymbol{\nabla}_z\boldsymbol{S}(z_{k,l}^i), \tag{9}$$

where D^+ and D^- are the forward and backward difference operators, respectively. Hence we have

$$D_t^+z_{k,l}^i = \frac{z_{k,l}^{i+1} - z_{k,l}^i}{\Delta t}, \quad D_t^-z_{k,l}^i = \frac{z_{k,l}^i - z_{k,l}^{i-1}}{\Delta t}.$$

The $\boldsymbol{M}_+$ and $\boldsymbol{M}_-$ are the splitted matrices of the symplectic matrix $\boldsymbol{M}$, i.e.,

$$\boldsymbol{M} = \boldsymbol{M}_+ + \boldsymbol{M}_-, \quad \boldsymbol{M}_+^{\mathrm{T}} = -\boldsymbol{M}_-.$$

The matrix splitting of $\boldsymbol{M}$ is not unique. Different forms of splitting may produce different schemes. Here, we take $\boldsymbol{M}_+$ as the upper triangle matrix.

Then, Eq. (9) can be written as

$$\begin{cases} \boldsymbol{A}_1\boldsymbol{\Phi}^i = \boldsymbol{U}^i, \\ \dfrac{1}{2}\dfrac{\boldsymbol{\Phi}^{i+1} - \boldsymbol{\Phi}^i}{\Delta t} + \boldsymbol{A}_1\boldsymbol{V}^i + \boldsymbol{A}_2\boldsymbol{W}^i = \boldsymbol{P}^i - 3(\boldsymbol{U}^i)^2, \\ \boldsymbol{A}_1\boldsymbol{W}^i - \boldsymbol{A}_2\boldsymbol{V}^i = 0, \\ \dfrac{1}{2}\dfrac{\boldsymbol{U}^i - \boldsymbol{U}^{i-1}}{\Delta t} + \boldsymbol{A}_1\boldsymbol{P}^i = 0, \\ \boldsymbol{A}_1\boldsymbol{U}^i - \boldsymbol{A}_2\boldsymbol{Q}^i = \boldsymbol{V}^i, \\ \boldsymbol{A}_1\boldsymbol{Q}^i + \boldsymbol{A}_2\boldsymbol{U}^i = \boldsymbol{W}^i, \end{cases} \tag{10}$$

where $(\boldsymbol{U}^i)^2 = [(u_{1,1}^i)^2, (u_{2,1}^i)^2, \ldots, (u_{N_x,1}^i)^2, \ldots, (u_{1,N_y}^i)^2, (u_{2,N_y}^i)^2, \ldots, (u_{N_x,N_y}^i)^2]^{\mathrm{T}}$. Assuming $N_x = N_y$, we notice that $\boldsymbol{A}_1\boldsymbol{A}_2 = \boldsymbol{A}_2\boldsymbol{A}_1$. After eliminating the auxiliary variables p, q, ϕ, v, and w, Eq. (12) can be rewritten as

$$\boldsymbol{U}^{i+1} = \boldsymbol{U}^{i-1} - 2\Delta t(\boldsymbol{A}_1^3\boldsymbol{U}^i + \boldsymbol{A}_1\boldsymbol{A}_2^2\boldsymbol{U}^i + 3\boldsymbol{A}_1(\boldsymbol{U}^i)^2). \tag{11}$$

It is obvious that the explicit scheme (11) does not need to solve the nonlinear algebraic equation at every time step.

Theorem 2 The discrete scheme (9) obeys the $N_x \times N_y$ full-discrete multi-symplectic conservation law

$$\frac{\omega_{k,l}^i - \omega_{k,l}^{i-1}}{\Delta t}$$

$$+\sum_{j=1}^{N_x}(\boldsymbol{D}_1^x)_{k,j}\kappa_{j,l}^i+\sum_{j=1}^{N_y}(\boldsymbol{D}_1^y)_{l,j}\tau_{k,j}^i=0, \quad (12)$$

where $\omega_{k,l}^i = (\mathrm{d}z_{k,l}^i \wedge \boldsymbol{M}_+\mathrm{d}z_{k,l}^{i+1})/2$, $\kappa_{j,l}^i = (\mathrm{d}z_{k,l}^i \wedge \boldsymbol{K}\mathrm{d}z_{j,l}^i + \mathrm{d}z_{j,l}^i \wedge \boldsymbol{K}\mathrm{d}z_{k,l}^i)/2$, $\tau_{k,j}^i = (\mathrm{d}z_{k,l}^i \wedge \boldsymbol{L}\mathrm{d}z_{k,j}^i + \mathrm{d}z_{k,j}^i \wedge \boldsymbol{L}\mathrm{d}z_{k,l}^i)/2$, $k = 1,2,\ldots,N_x$, and $l = 1,2,\ldots,N_y$.

Proof The variational equation associated with Eq. (9) is

$$\begin{aligned}&\boldsymbol{M}_+\frac{\mathrm{d}z_{k,l}^{i+1}-\mathrm{d}z_{k,l}^i}{\Delta t}+\boldsymbol{M}_-\frac{\mathrm{d}z_{k,l}^i-\mathrm{d}z_{k,l}^{i-1}}{\Delta t}\\&+\sum_{j=1}^{N_x}(\boldsymbol{D}_1^x)_{k,j}\boldsymbol{K}\mathrm{d}z_{j,l}^i+\sum_{j=1}^{N_y}(\boldsymbol{D}_1^y)_{l,j}\boldsymbol{L}\mathrm{d}z_{k,j}^i\\&=S_{zz}(z_{k,l}^i)\mathrm{d}z_{k,l}^i. \quad (13)\end{aligned}$$

Taking the wedge product with $\mathrm{d}z_{k,l}^i$ on both sides of Eq. (13) and noting

$$\mathrm{d}z_{k,l}^i \wedge S_{zz}(z_{k,l}^i)\mathrm{d}z_{k,l}^i = 0,$$

we obtain the $N_x \times N_y$ full-discrete multi-symplectic conservation law (12).

4. Numerical simulations

Next, we conduct some typical numerical examples[8] to examine the efficiency of the new explicit scheme (11).

Example 1 An accuracy test is made for the ZK equation

$$u_t + uu_x + \varepsilon(u_{xxx} + u_{xyy}) = 0, \quad (14)$$

with the steady progressive wave solutions

$$\begin{aligned}u(x,y,t) = 3c\,\mathrm{sech}^2\Big\{0.5\sqrt{\frac{c}{\varepsilon}}[(x-ct)\\\times\cos\theta + y\sin\theta]\Big\}, \quad (15)\end{aligned}$$

where θ is the inclined angle relative to the x axis, and c and η are constants. Table 1 shows the numerical errors and CPU times of the explicit scheme as well as those of the implicit multi-symplectic midpoint scheme used in Ref. 18]. Numerical results show that the explicit method has similar accuracy as the implicit method, but takes much less CPU time.

Table 1. Spatial accuracy of the explicit and implicit schemes with initial condition (16) in a computation domain of $[0,8]\times[0,8]$. $c = 0.1$, $\varepsilon = 0.01$, $\theta = 0$, $x_0 = 4$, $\Delta t = 0.001$, and $t = 1$).

N	L_∞ error		CPU time/s	
	Explicit scheme	Implicit scheme	Explicit scheme	Implicit scheme
10	2.17×10^{-2}	1.95×10^{-2}	0.257	1.05
20	2.99×10^{-3}	3.01×10^{-3}	3.41	12.38
40	7.34×10^{-5}	7.17×10^{-5}	59.12	220.47

Example 2 We show the collision of line solitons of the ZK equation (14). The double soliton collision case of the steady progressive wave has an initial condition that

$$\begin{aligned}u(x,y,0) = \sum_{j=1}^{2}3c_j\mathrm{sech}^2\Big\{0.5\sqrt{\frac{c_j}{\varepsilon}}[(x-x_j)\\\times\cos\theta + (y-y_j)\sin\theta]\Big\}, \quad (16)\end{aligned}$$

where $c_1 = 0.45$, $c_2 = 0.25$, $\varepsilon = 0.01$, $\theta = 0$, $x_1 = 2.5$, $y_1 = 0$, $x_2 = 3.3$, and $y_2 = 0$. The numerical results in periodic boundary conditions in a computation domain of $[0,8]\times[0,8]$ are shown in Fig. 1.

Example 3 Equation (1) has a cylindrically symmetric solitary solution

$$\begin{aligned}u(x,y,t) = \frac{c}{3}\sum_{n=1}^{10}a_{2n}\Big\{\cos\Big[2n\,\mathrm{arccot}\\\times\Big(\frac{\sqrt{c}}{2}r\Big)\Big]-1\Big\}, \quad (17)\end{aligned}$$

where c is the velocity of the solitary wave solution and $r = \sqrt{(x-ct)^2+y^2}$. The coefficients are cited in Ref. [8]. As an example, we proceed to show the collision of two pulses under the initial condition

$$\begin{aligned}u(x,y,0) = \sum_{j=1}^{2}\frac{c_j}{3}\sum_{n=1}^{10}a_{2n}\Big\{\cos\Big[2n\,\mathrm{arccot}\\\times\Big(\frac{\sqrt{c_j}}{2}r_j\Big)\Big]-1\Big\}, \quad (18)\end{aligned}$$

where c_1 and c_2 are the velocities of the solitary wave solutions, $r_1 = \sqrt{(x-x_1)^2+(y-y_1)^2}$, and $r_2 = \sqrt{(x-x_2)^2+(y-y_2)^2}$. The direct collision of two dissimilar pulses is shown in Fig. 2 with $c_1 = 4$, $c_2 = 1$, $x_1 = 8$, $y_1 = 16$, $x_2 = 16$, and $y_2 = 16$.

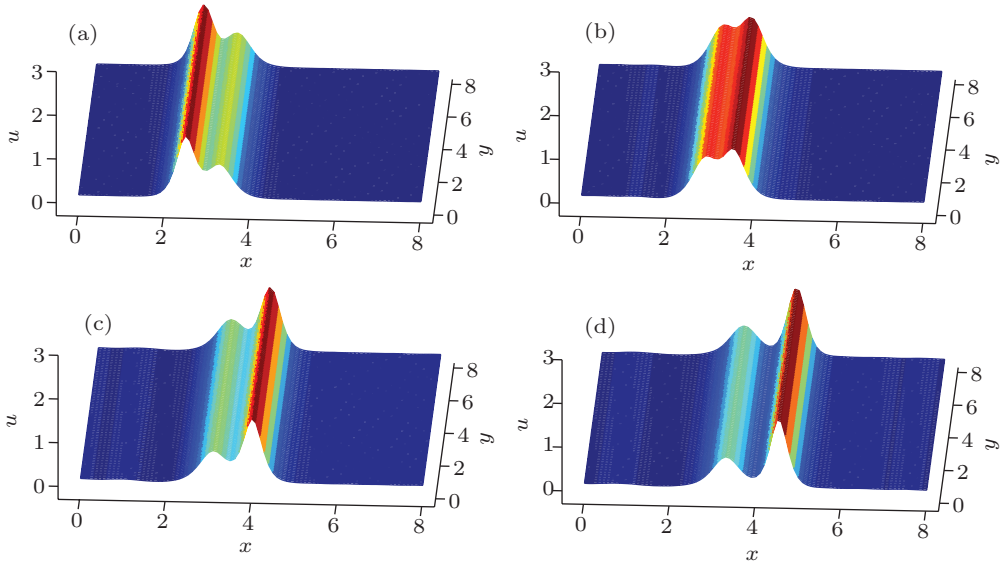


Fig. 1. (colour online) Profiles of double soliton collision under the periodic boundary conditions in a computation domain of $[0,8] \times [0,8]$ using 80×80 uniform cells at (a) $t = 0$, (b) 1, (c) 2, and (d) 3. The time step $\Delta t = 0.001$.

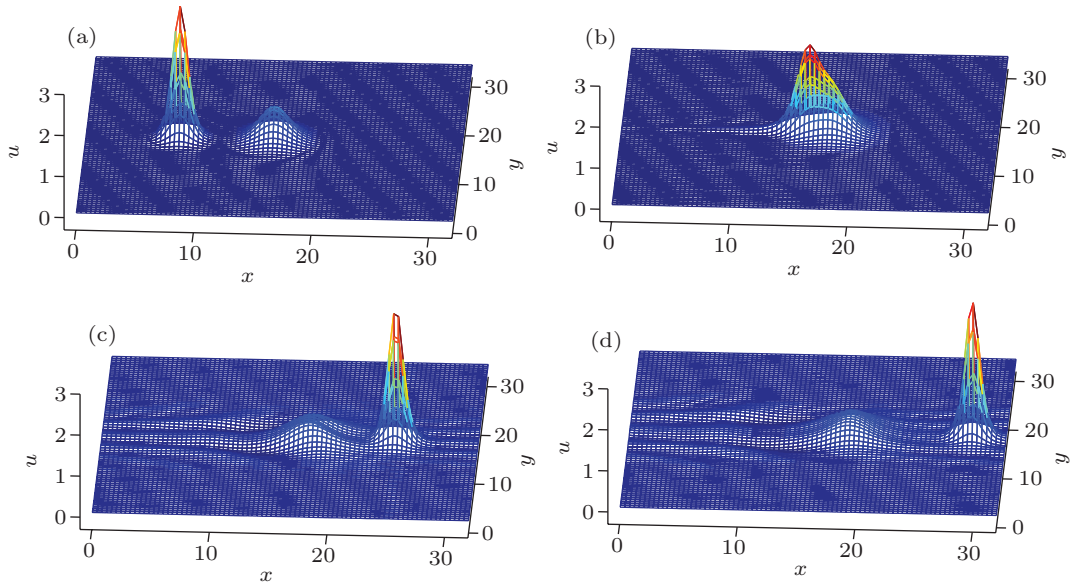


Fig. 2. (colour online) Profiles of direct collision of two dissimilar pulses under the periodic boundary conditions in a computation domain of $[0,32] \times [0,32]$ at (a) $t = 0$, (b) 2, (c) 4, and (d) 5, using 80×80 uniform cells. The time step $\Delta t = 0.001$.

Numerical results in Examples 2 and 3 are similar to those in Ref. [8], which are obtained by using the discontinuous Galerkin method. We find that the new scheme can take effect very well at a time step of $\Delta t \leq 0.003$ and the numerical results do not show nonlinear instability and un-physical oscillations in Examples 2 and 3, which are similar to those via the implicit multi-symplectic midpoint scheme in Ref. [8]. Meanwhile, the CPU time is 60%–80% less than that via the implicit multi-symplectic midpoint scheme. These results indicate the effectiveness and stability of the proposed method for the ZK equation.

5. Conclusions

In this paper, we construct a new explicit multi-symplectic scheme for the ZK equation based on the Fourier pseudospectral method and the symplectic Euler scheme. It is easier in practice and more effective than the traditional symplectic and multi-symplectic methods, because the cost of solving nonlinear algebraic equations at every time step are saved. Furthermore, this explicit multi-symplectic scheme provides new choices for simulating the Hamiltonian equations especially with high spatial dimension.

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