An extension of the modified Sawada–Kotera equation and conservation laws*

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Based on the modified Sawada–Kotera equation, we introduce a 3×3 matrix spectral problem with two potentials and derive a hierarchy of new nonlinear evolution equations. The second member in the hierarchy is a generalization of the modified Sawada–Kotera equation, by which a Lax pair of the modified Sawada–Kotera equation is obtained. With the help of the Miura transformation, explicit solutions of the Sawada–Kotera equation, the Kaup–Kupershmidt equation, and the modified Sawada–Kotera equation are given. Moreover, infinite sequences of conserved quantities of the first two nonlinear evolution equations in the hierarchy and the modified Sawada–Kotera equation are constructed with the aid of their Lax pairs.

Keywords: spectral problem, explicit solutions, conservation laws

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1. Introduction

It is well known that the two important fifth-order nonlinear evolution equations, the Sawada–Kotera (SK) equation[1−3]

$$q_t = q_{5x} + 5qq_{3x} + 5q_xq_{xx} + 5q^2q_x \tag{1}$$

and the Kaup–Kupershmidt (KK) equation[4,5]

$$r_t = r_{5x} + 10rr_{3x} + 25r_xr_{xx} + 20r^2r_x, \tag{2}$$

have been studied extensively, see Refs. [6]–[13], just to mention a few. The research has been conducted on their Hamiltonian structures, Lax pairs, Bäcklund transformations, bilinear transformations, soliton solutions, and other properties. Equations (1) and (2) are fundamentally different despite of their evident duality. Another fifth-order nonlinear evolution equation

$$v_t = v_{5x} - (5v_xv_{xx} + 5vv_x^2 + 5v^2v_{xx} - v^5)_x \tag{3}$$

possesses a close connection with Eqs. (1) and (2). If v is a solution of Eq. (3), then functions q and r determined by the Miura transformations

$$q = v_x - v^2, \quad r = -v_x - \frac{1}{2}v^2 \tag{4}$$

satisfy SK Eq. (1) and KK Eq. (2), respectively. Equation (3) is called the modified SK equation. During the recent decades, the study of soliton equations has been one of the most prominent events in the field of nonlinear science. Especially, finding explicit solutions and conservation laws of the soliton equations is an important way to investigate the properties of the soliton equations.[14−23]

In this paper, we introduce a 3×3 matrix spectral problem, from which a hierarchy of new nonlinear evolution equations is derived. The first typical member in the hierarchy is

$$\begin{aligned} u_t &= u_{xx} + 2uu_x - 2(uv)_x - 2v_{xx}, \\ v_t &= -v_{xx} + 2vv_x - 2(uv)_x + 2u_{xx}, \end{aligned} \tag{5}$$

and the second member is reduced to the modified SK Eq. (3) with $u = 2v$. Based on these results, a Lax pair of the modified SK equation is given. The explicit solutions of the SK equation, the KK equation, and the modified SK equation are obtained with the help of the Miura transformation. Moreover, infinite sequences of conserved quantities of the first two nonlinear evolution equations in the hierarchy and the modified SK equation are constructed with the aid of their Lax pairs.

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2. Nonlinear evolution equations

In this section, we derive a hierarchy of new nonlinear evolution equations. We first introduce a 3×3 matrix spectral problem

$$\boldsymbol{\phi}_x = \boldsymbol{U}\boldsymbol{\phi}, \quad \boldsymbol{\phi} = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix}u & 1 & 0\\ 0 & v & \lambda\\ 1 & 0 & 0\end{pmatrix}, \tag{6}$$

where u and v are two potentials, and λ is a constant spectral parameter. In order to derive the hierarchy of nonlinear evolution equations associated with spectral problem (6), we solve the stationary zero-curvature equation with the help of the approach introduced in Refs. [24] and [25]

$$\boldsymbol{V}_x - [\boldsymbol{U}, \boldsymbol{V}] = 0, \quad \boldsymbol{V} = (V_{ij})_{3\times 3}, \tag{7}$$

which is equivalent to

$$\begin{aligned}
&V_{11,x} + V_{13} - V_{21} = 0,\\
&V_{12,x} + (v-u)V_{12} + V_{11} - V_{22} = 0,\\
&V_{13,x} + \lambda V_{12} - uV_{13} - V_{23} = 0,\\
&V_{22,x} + V_{21} - \lambda V_{32} = 0,\\
&V_{21,x} + (u-v)V_{21} + V_{23} - \lambda V_{31} = 0,\\
&V_{23,x} - vV_{23} + \lambda(V_{22} - V_{33}) = 0,\\
&V_{31,x} + uV_{31} + V_{33} - V_{11} = 0,\\
&V_{32,x} + vV_{32} + V_{31} - V_{12} = 0,\\
&V_{33,x} + \lambda V_{32} - V_{13} = 0,
\end{aligned} \tag{8}$$

where each entry $V_{ij} = V_{ij}(A, B, C, D)$ is a function of A, B, C, and D

$$\begin{aligned}
&V_{11} = (\partial + u)B - (\partial^2 + \partial v + u\partial + uv)A + \lambda C,\\
&V_{12} = B, \quad V_{13} = \lambda(A + \partial C), \quad V_{21} = \lambda D,\\
&V_{22} = (2\partial + v)B - (\partial^2 + \partial v + u\partial + uv)A + \lambda C,\\
&V_{23} = \lambda(\partial - u)A + \lambda B + \lambda(\partial^2 - u\partial)C,\\
&V_{31} = B - (\partial + v)A, \quad V_{32} = A, \quad V_{33} = \lambda C,
\end{aligned} \tag{9}$$

with $\partial = \partial_x$. Substituting Eq. (9) into Eq. (8) yields

$$\begin{aligned}
&(2\partial + v - u)A + (\partial^2 - u\partial)C + (\partial + u - v)D = 0,\\
&(\partial u + u\partial + \partial v + v\partial)A - 3\partial B\\
&\quad - (\partial^3 - \partial u\partial - v\partial^2 + uv\partial)C = 0,\\
&(\partial^2 + \partial u)B - (\partial^3 + \partial^2 v + \partial u\partial + \partial uv)A\\
&\quad + \lambda(2\partial C + A - D) = 0,\\
&(2\partial^2 + \partial v)B - (\partial^3 + \partial^2 v + \partial u\partial + \partial uv)A\\
&\quad + \lambda(\partial C - A + D) = 0.
\end{aligned} \tag{10}$$

Functions A, B, C, and D are expanded into a Laurent series of λ

$$\begin{aligned}
&A = \sum_{j\geq 0} A_j\lambda^{-j}, \quad B = \sum_{j\geq 0} B_j\lambda^{-j},\\
&C = \sum_{j\geq 0} C_j\lambda^{-j}, \quad D = \sum_{j\geq 0} D_j\lambda^{-j}.
\end{aligned} \tag{11}$$

Substituting Eq. (11) into Eq. (10) and comparing coefficients for the same power of λ yield the recursion equations ($j \geq 0$)

$$\begin{aligned}
&2\partial C_0 + A_0 - D_0 = 0, \quad \partial C_0 - A_0 + D_0 = 0,\\
&(2\partial + v - u)A_j + (\partial^2 - u\partial)C_j\\
&\quad + (\partial + u - v)D_j = 0,\\
&(\partial u + u\partial + \partial v + v\partial)A_j - 3\partial B_j\\
&\quad - (\partial^3 - \partial u\partial - v\partial^2 + uv\partial)C_j = 0,\\
&(\partial^2 + \partial u)B_j - (\partial^3 + \partial^2 v + \partial u\partial + \partial uv)A_j\\
&\quad + 2\partial C_{j+1} + A_{j+1} - D_{j+1} = 0,\\
&(2\partial^2 + \partial v)B_j - (\partial^3 + \partial^2 v + \partial u\partial + \partial uv)A_j\\
&\quad + \partial C_{j+1} - A_{j+1} + D_{j+1} = 0.
\end{aligned} \tag{12}$$

Then A_j, B_j, C_j, and D_j are recursively determined by using Eq. (12). The recursion Eq. (12) can be solved successively to deduce the results

$$\begin{aligned}
A_0 &= D_0 = \alpha_0, \quad B_0 = \frac{1}{3}\alpha_0(u+v) + \alpha_1, \quad C_0 = \beta_0,\\
C_1 &= \frac{1}{9}\alpha_0(-3u_x + 3v_x - u^2 - v^2 + 4uv)\\
&\quad - \frac{1}{3}\alpha_1(u+v) + \beta_1,\\
A_1 &= \frac{1}{9}\alpha_0\Big[2u_{xx} - v_{xx} + 2vv_x - 3u_xv - uv_x\\
&\quad - \frac{4}{9}(u^3 + v^3) + \frac{2}{3}(uv^2 + u^2v)\Big]\\
&\quad - \frac{1}{9}\alpha_1(-3v_x + u^2 + v^2 - uv) + \alpha_2,\\
D_1 &= \frac{1}{9}\alpha_0\Big[-u_{xx} - v_{xx} + 2uu_x - 2vv_x - u_xv + uv_x\\
&\quad - \frac{4}{9}(u^3 + v^3) + \frac{2}{3}(uv^2 + u^2v)\Big]\\
&\quad - \frac{1}{9}\alpha_1(3v_x - 3u_x + u^2 + v^2 - uv) + \alpha_2,\\
B_1 &= \frac{1}{27}\alpha_0\Big[3(u-v)_{xxx} + 3(u-v)(u-v)_{xx}\\
&\quad - (2u^2 + 2v^2 + uv)(u-v)_x + u_x^2 + v_x^2\\
&\quad - 7u_xv_x + \frac{1}{3}u^2v^2 + \frac{8}{9}uv(u^2 + v^2)\\
&\quad - \frac{7}{9}(u^4 + v^4)\Big] + \frac{1}{3}\alpha_2(u+v)\\
&\quad - \frac{1}{27}\alpha_1\Big[\frac{5}{3}(u^3 + v^3) - 3(u+v)_{xx}
\end{aligned}$$

$$+3(u+v)(u-v)_x - uv^2 - u^2v\Big] + \beta_2, \qquad (13)$$

where α_j and β_j $(j=0,1,2)$ are arbitrary constants.

We now assume that $\boldsymbol{\phi}$ satisfies spectral problem (6) and the auxiliary problem

$$\boldsymbol{\phi}_{t_m} = \boldsymbol{V}^{(m)}\boldsymbol{\phi}, \qquad (14)$$

where each entry $V_{ij}^{(m)} = V_{ij}(A^{(m)}, B^{(m)}, C^{(m)}, D^{(m)})$ in matrix $\boldsymbol{V}^{(m)}$ is a polynomial of the spectral parameter λ with

$$A^{(m)} = \sum_{j=0}^{m} A_j\lambda^{m-j}, \quad B^{(m)} = \sum_{j=0}^{m} B_j\lambda^{m-j},$$
$$C^{(m)} = \sum_{j=0}^{m} C_j\lambda^{m-j}, \quad D^{(m)} = \sum_{j=0}^{m} D_j\lambda^{m-j}. \qquad (15)$$

Then the compatibility condition of Eqs. (6) and (14) yields the zero-curvature equation, $\boldsymbol{U}_{t_m} - \boldsymbol{V}_x^{(m)} + [\boldsymbol{U}, \boldsymbol{V}^{(m)}] = 0$, which is equivalent to a hierarchy of new nonlinear evolution equations

$$\begin{aligned} u_{t_m} &= (\partial^2 + \partial u)B_m - (\partial^3 + \partial^2 v + \partial u\partial + \partial uv)A_m, \\ v_{t_m} &= (2\partial^2 + \partial v)B_m - (\partial^3 + \partial^2 v + \partial u\partial + \partial uv)A_m, \\ & m \geq 0. \end{aligned} \qquad (16)$$

The first nontrivial member in hierarchy (16) is

$$\begin{aligned} u_{t_0} &= \frac{\alpha_0}{3}[u_{xx} + 2uu_x - 2(uv)_x - 2v_{xx}] + \alpha_1 u_x, \\ v_{t_0} &= \frac{\alpha_0}{3}[-v_{xx} + 2vv_x - 2(uv)_x + 2u_{xx}] + \alpha_1 v_x, \end{aligned} \qquad (17)$$

which is just Eq. (5) with $\alpha_0 = 3$ and $\alpha_1 = 0$. And the second one is

$$\begin{aligned} u_{t_1} &= \frac{1}{243}\alpha_0[-7u^5 + 20u^4v - 15u^3v^2 - 10u^2v^3 \\ &\quad + 5uv^4 - 5(2u^3 + 2v^3 - 3u^2v - 3uv^2 \\ &\quad + 9vu_x - 9v_{xx})(u-2v)_x \\ &\quad + 45(u^2 + v^2 - uv + u_x + v_x)u_{xx} \\ &\quad + 45u(u_x^2 - v_x^2) - 27u_{xxxx}]_x, \\ v_{t_1} &= \frac{1}{243}\alpha_0[-7v^5 + 20v^4u - 15v^3u^2 - 10v^2u^3 \\ &\quad + 5vu^4 - 5(2u^3 + 2v^3 - 3u^2v - 3uv^2 \\ &\quad - 9uv_x - 9u_{xx})(2u-v)_x \\ &\quad + 45(u^2 + v^2 - uv - u_x - v_x)v_{xx} \\ &\quad - 45v(u_x^2 - v_x^2) - 27v_{xxxx}]_x, \end{aligned} \qquad (18)$$

with $\alpha_1 = \alpha_2 = \beta_2 = 0$ for the sake of simplicity. It is easy to find that equation (18) is exactly reduced to the modified SK Eq. (3) if we choose $u = 2v$ and $\alpha_0 = -9$.

3. The Lax pair of the modified Sawada–Kotera equation and conservation laws

In this section, we construct a Lax pair for the modified SK Eq. (3) and discuss infinite sequences of conserved quantities of Eqs. (3), (5), and (18). Based on the above result, we deduce the Lax pair given by spectral problems

$$\mathrm{L}\psi = \lambda\psi, \quad \psi_t = \mathrm{B}\psi, \qquad (19)$$

where L and B are two differential operators given by

$$\begin{aligned} \mathrm{L} &= \partial^3 - 3v\partial^2 + 2(v^2 - v_x)\partial, \\ \mathrm{B} &= -9\partial^5 + 45v\partial^4 + 75(v_x - v^2)\partial^3 \\ &\quad + (60v_{xx} - 165vv_x + 45v^3)\partial^2 \\ &\quad + (20v_{xxx} - 45v_x^2 - 70vv_{xx} \\ &\quad + 70v^2v_x - 5v^4)\partial. \end{aligned} \qquad (20)$$

A direct calculation shows that the Lax equation, $\mathrm{L}_t = [\mathrm{B}, \mathrm{L}]$, gives rise to the modified SK Eq. (3). Assume that v is a solution of the modified SK Eq. (3), then functions q and r determined by the Miura transformation (4) (see Ref. [26]) satisfy the SK Eq. (1) and the KK Eq. (2), respectively. In fact, a direct calculation gives

$$\begin{aligned} & q_t - (q_{5x} + 5qq_{3x} + 5q_xq_{xx} + 5q^2q_x) \\ &= (\partial - 2v)[v_t - v_{5x} + 5(v_xv_{xx} \\ &\quad + vv_x^2 + v^2v_{xx})_x - 5v^4v_x], \end{aligned} \qquad (21)$$

$$\begin{aligned} & r_t - (r_{5x} + 10rr_{3x} + 25r_xr_{xx} + 20r^2r_x) \\ &= -(\partial + v)[v_t - v_{5x} + 5(v_xv_{xx} \\ &\quad + vv_x^2 + v^2v_{xx})_x - 5v^4v_x]. \end{aligned} \qquad (22)$$

With the help of the transformation introduced in Ref. [26], we obtain two explicit solutions of the modified SK Eq. (3)

$$v = -a\frac{c_1\sinh(ax + a^5t + \delta_0) + c_2\cosh(ax + a^5t + \delta_0)}{c_1\cosh(ax + a^5t + \delta_0) + c_2\sinh(ax + a^5t + \delta_0)}, \qquad (23)$$

and

$$v = a\frac{c_1\sin(ax+a^5t+\delta_0)+c_2\cos(ax+a^5t+\delta_0)}{c_1\cos(ax+a^5t+\delta_0)-c_2\sin(ax+a^5t+\delta_0)}, \tag{24}$$

where a, c_1, c_2, and δ_0 are arbitrary constants, and $a>0$. Under the Miura transformation (4), the functions given by Eqs. (23) and (24) correspond to trivial solutions $q=\pm a^2$ of the SK Eq. (1) and two nontrivial solutions of the KK Eq. (2)

$$r = \frac{2a^2(c_1^2-c_2^2)-a^2[c_1\sinh(ax+a^5t+\delta_0)+c_2\cosh(ax+a^5t+\delta_0)]^2}{2[c_1\cosh(ax+a^5t+\delta_0)+c_2\sinh(ax+a^5t+\delta_0)]^2}, \tag{25}$$

and

$$r = -\frac{2a^2(c_1^2+c_2^2)+a^2[c_1\sin(ax+a^5t+\delta_0)+c_2\cos(ax+a^5t+\delta_0)]^2}{2[c_1\cos(ax+a^5t+\delta_0)-c_2\sin(ax+a^5t+\delta_0)]^2}. \tag{26}$$

In the following, we derive infinitely many conservation laws for Eqs. (3), (5), and (18). We first eliminate ϕ_1 and ϕ_2 from Eq. (6) by writing the column vector $\boldsymbol{\phi}$ in components as $\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3)^{\mathrm{T}}$ to get a single equation for $\psi=\phi_3$, namely,

$$\psi_{xxx}-(u+v)\psi_{xx}+(uv-u_x)\psi_x-\lambda\psi=0. \tag{27}$$

If we set $\rho=(\ln\psi)_x$ in Eq. (27), it is clear that ρ satisfies

$$\begin{aligned}&\rho_{xx}+3\rho\rho_x+\rho^3-(u+v)(\rho_x+\rho^2)\\&\quad+(uv-u_x)\rho-\lambda=0.\end{aligned} \tag{28}$$

Then the corresponding t evolution of ψ implies a conservation law for ρ, which is

$$\begin{aligned}&\rho_t=\theta_x,\\&\theta=V_{31}^{(m)}\rho+V_{32}^{(m)}(\rho_x+\rho^2-u\rho)+V_{33}^{(m)}.\end{aligned} \tag{29}$$

Case 1 In order to get conservation laws for Eq. (5), we choose $V_{31}^{(0)}=u-2v$, $V_{32}^{(0)}=3$, and $V_{33}^{(0)}=0$. In this case, θ can be written as

$$\theta=-2(u+v)\rho+3(\rho_x+\rho^2). \tag{30}$$

In Eqs. (28) and (30), functions ρ and θ each can be expanded as a series in powers of λ with the coefficients called conserved densities ($\lambda=\eta^3$)

$$\begin{aligned}&\rho=\eta+\sum_{j=0}^{\infty}\rho_j\eta^{-j},\\&\theta=3\eta^2+\sum_{j=0}^{\infty}\theta_j\eta^{-j}.\end{aligned} \tag{31}$$

Substituting the above expansions into Eqs. (28) and (30) and comparing the same powers of λ yield two infinite sequences of conserved quantities for Eq. (5). The first densities and currents are given by

$$\begin{aligned}
\rho_0 &= \frac{1}{3}(u+v),\\
\rho_1 &= \frac{1}{9}(u^2+v^2-uv-3v_x),\\
\rho_2 &= \frac{1}{81}[9(u_x-v_x)(2v-u)+9(2v-u)_{xx}\\
&\quad -3uv(u+v)+2(u^3+v^3)],\\
\rho_3 &= \frac{1}{27}[u_x^2+4v_x^2-7u_xv_x+(v^2-2u^2+2uv)u_x\\
&\quad +(u^2-2v^2+2uv)v_x+(u-5v)u_{xx}\\
&\quad +(4v-2u)v_{xx}+3(u-v)_{xxx})],\\
\rho_4 &= \frac{1}{729}[-2(u^5+v^5)+(u^3+v^3)(6v_x-9u_x+5uv)\\
&\quad -(u+v)(2u^2v^2+9uvv_x+45u_xv_x)\\
&\quad -9v(4u_x^2-8v_x^2-9u_{xxx}+6v_{xxx})\\
&\quad +9u(8u_x^2-4v_x^2+3v_{xxx})+27(v-2u)_{xxxx}\\
&\quad +9(5u^2-v^2-5uv+18v_x)u_{xx}\\
&\quad +9(5v^2-u^2-5uv-21v_x+15u_x)v_{xx}],\\
\theta_0 &= \frac{1}{3}(u^2+v^2-4uv+3u_x-3v_x),\\
\theta_1 &= \frac{1}{27}[4(u^3+v^3)-6uv(u+v)+27vu_x\\
&\quad +9(u-2v)v_x+9(v-2u)_{xx}].
\end{aligned} \tag{32}$$

The recursion relations for ρ_j and θ_j ($j\geq 0$) are

$$\begin{aligned}
3\rho_{j+2} &= -3\rho_{j+1,x}+2(u+v)\rho_{j+1}-\rho_{j,xx}\\
&\quad +(u+v)\rho_{j,x}-(uv-u_x)\rho_j\\
&\quad -3\sum_{\substack{i+k=j+1,\\ i,k\geq 0}}\rho_i\rho_k-\sum_{\substack{i+k=j,\\ i,k\geq 0}}[3\rho_i\rho_{k,x}\\
&\quad -(u+v)\rho_i\rho_k]-\sum_{\substack{i+k+l=j,\\ i,k,l\geq 0}}\rho_i\rho_k\rho_l,
\end{aligned} \tag{33}$$

and

$$\theta_j = 6\rho_{j+1} + 3\rho_{j,x} - 2(u+v)\rho_j + 3\sum_{\substack{i+k=j,\\ i,k\geq 0}} \rho_i\rho_k. \quad (34)$$

Then the first two conservation laws for Eq. (5) are

$$\begin{aligned}&(u+v)_t = (u^2+v^2-4uv+3u_x-3v_x)_x,\\ &(u^2+v^2-uv-3v_x)_t\\ =\ &\frac{1}{3}[4(u^3+v^3)-6uv(u+v)\\ &+27vu_x+9(u-2v)v_x+9(v-2u)_{xx}]_x. \end{aligned} \quad (35)$$

Case 2 Next we will give the conservation laws for Eq. (18) by choosing

$$\begin{aligned}\theta =\ &\Big[-\frac{2}{3}\alpha_0(u+v)\rho+\alpha_0(\rho_x+\rho^2)+C_1\Big]\lambda\\ &+[B_1-(\partial+v+u)A_1]\rho+A_1(\rho_x+\rho^2), \end{aligned} \quad (36)$$

where A_1, B_1, C_1, and α_0 have appeared in Eq. (13), and functions ρ and θ are expanded as ($\lambda=\eta^3$)

$$\begin{aligned}\rho &= \eta+\sum_{j=0}^{\infty}\rho_j\eta^{-j},\\ \theta &= \alpha_0\eta^5+\sum_{j=0}^{\infty}\theta_j\eta^{-j}. \end{aligned} \quad (37)$$

Similar to Case 1, we can obtain the first densities and currents, which are

$$\begin{aligned}\rho_0 =\ &\frac{1}{3}(u+v),\\ \rho_1 =\ &\frac{1}{9}(u^2+v^2-uv-3v_x),\\ \rho_2 =\ &\frac{1}{81}[9(u_x-v_x)(2v-u)+9(2v-u)_{xx}\\ &-3uv(u+v)+2(u^3+v^3)],\\ \rho_3 =\ &\frac{1}{27}[u_x^2+4v_x^2-7u_xv_x+(v^2-2u^2+2uv)u_x\\ &+(u^2-2v^2+2uv)v_x+(u-5v)u_{xx}\\ &+(4v-2u)v_{xx}+3(u-v)_{xxx}],\\ \rho_4 =\ &\frac{1}{729}[-2(u^5+v^5)+(u^3+v^3)(6v_x-9u_x+5uv)\\ &-(u+v)(2u^2v^2+9uvv_x+45u_xv_x)\\ &-9v(4u_x^2-8v_x^2-9u_{xxx}+6v_{xxx})\\ &+9u(8u_x^2-4v_x^2+3v_{xxx})+27(v-2u)_{xxxx}\\ &+9(5u^2-v^2-5uv+18v_x)u_{xx}\\ &+9(5v^2-u^2-5uv-21v_x+15u_x)v_{xx}],\\ \theta_0 =\ &\frac{1}{729}\alpha_0[-7(u^5+v^5)+25uv(u^3+v^3-u^2v-uv^2)\\ &+45(u^2+v^2-uv)(u+v)_{xx}\\ &+15(3uv^2+3u^2v-2u^3-2v^3)(u-v)_x\\ &+45(u-2v)u_x^2+90(u+v)u_xv_x\\ &+45(v-2u)v_x^2+27(5u_xu_{xx}-5v_xv_{xx}\\ &-u_{xxxx}-v_{xxxx})]. \end{aligned} \quad (38)$$

In this case, the recursion relations for ρ_j and θ_j ($j\geq 0$) are

$$\begin{aligned}3\rho_{j+2} =\ &-3\rho_{j+1,x}+2(u+v)\rho_{j+1}-\rho_{j,xx}\\ &+(u+v)\rho_{j,x}-(uv-u_x)\rho_j\\ &-3\sum_{\substack{i+k=j+1,\\ i,k\geq 0}}\rho_i\rho_k-\sum_{\substack{i+k=j,\\ i,k\geq 0}}[3\rho_i\rho_{k,x}\\ &-(u+v)\rho_i\rho_k]-\sum_{\substack{i+k+l=j,\\ i,k,l\geq 0}}\rho_i\rho_k\rho_l, \end{aligned} \quad (39)$$

and

$$\begin{aligned}\theta_j =\ &2\alpha_0\rho_{j+4}+\alpha_0\rho_{j+3,x}\\ &-\frac{2}{3}\alpha_0(u+v)\rho_{j+3}+2A_1\rho_{j+1}\\ &+[B_1-(\partial+u+v)A_1]\rho_j+A_1\rho_{j,x}\\ &+A_1\sum_{\substack{i+k=j,\\ i,k\geq 0}}\rho_i\rho_k+\alpha_0\sum_{\substack{i+k=j+3,\\ i,k\geq 0}}\rho_i\rho_k. \end{aligned} \quad (40)$$

Then the first conservation law for Eq. (18) is

$$\begin{aligned}(u+v)_t =\ &\frac{1}{81}[-7(u^5+v^5)\\ &+25uv(u^3+v^3-u^2v-uv^2)\\ &+45(u^2+v^2-uv)(u+v)_{xx}\\ &+15(3uv^2+3u^2v-2u^3-2v^3)(u-v)_x\\ &+45(u-2v)u_x^2+90(u+v)u_xv_x\\ &+45(v-2u)v_x^2+27(5u_xu_{xx}-5v_xv_{xx}\\ &-u_{xxxx}-v_{xxxx})]_x. \end{aligned} \quad (41)$$

Specially, if we choose $u=2v$ and $\alpha_0=-9$, we can obtain an infinite sequence of conserved quantities for the modified SK Eq. (3)

$$\begin{aligned}\rho_0 =\ &v,\\ \rho_1 =\ &\frac{1}{3}(v^2-v_x),\\ \rho_2 =\ &0,\\ \rho_3 =\ &\frac{1}{9}(-2v_x^2-2vv_{xx}+v_{xxx}),\\ \rho_4 =\ &\frac{1}{9}(-2v^3v_x+v^2v_{xx}+5v_xv_{xx}\\ &+2vv_x^2+2vv_{xxx}-v_{xxxx}),\\ \rho_5 =\ &\frac{1}{81}[-v^6+3v^4v_x+(57v^2-23v_x)v_x^2\\ &+(24v^3-60vv_x-27v_{xx})v_{xx}\\ &-(12v^2+36v_x)v_{xxx}-12vv_{4x}+6v_{5x}],\\ \theta_0 =\ &v^5-5vv_x^2-5v^2v_{xx}-5v_xv_{xx}+v_{xxxx},\\ \theta_1 =\ &\frac{1}{9}[5v^6-15v^4v_x+(25v_x-15v^2)v_x^2 \end{aligned}$$

$$+(30vv_x+18v_{xx}-30v^3)v_{xx}$$
$$+(15v^2+9v_x)v_{xxx}+6vv_{4x}-3v_{5x}]. \quad (42)$$

The recursion relations for ρ_j and θ_j $(j\geq 0)$ are

$$\begin{aligned}3\rho_{j+2} =& -3\rho_{j+1,x}+6v\rho_{j+1}-\rho_{j,xx}\\ &+3v\rho_{j,x}-2(v^2-v_x)\rho_j\\ &-3\sum_{\substack{i+k=j+1,\\ i,k\geq 0}}\rho_i\rho_k-\sum_{\substack{i+k=j,\\ i,k\geq 0}}3(\rho_i\rho_{k,x}-v\rho_i\rho_k)\\ &-\sum_{\substack{i+k+l=j,\\ i,k,l\geq 0}}\rho_i\rho_k\rho_l, \end{aligned} \quad (43)$$

and

$$\begin{aligned}\theta_j =& -18\rho_{j+4}-9\rho_{j+3,x}+18v\rho_{j+3}\\ &+(v^4-3v_x^2-14v^2v_x+2vv_{xx}+2v_{xxx})\rho_j\\ &+3(2vv_x-v_{xx})\Bigg(2\rho_{j+1}+\rho_{j,x}\\ &+\sum_{\substack{i+k=j,\\ i,k\geq 0}}\rho_i\rho_k\Bigg)-9\sum_{\substack{i+k=j+3,\\ i,k\geq 0}}\rho_i\rho_k. \end{aligned} \quad (44)$$

The first two conservation laws of the modified SK Eq. (3) are

$$\begin{aligned}v_t =& (v^5-5vv_x^2-5v^2v_{xx}-5v_xv_{xx}+v_{xxxx})_x,\\ (v^2-v_x)_t =& \frac{1}{3}[5v^6-15v^4v_x+(25v_x-15v^2)v_x^2\\ &+(30vv_x+18v_{xx}-30v^3)v_{xx}\\ &+(15v^2+9v_x)v_{xxx}\\ &+6vv_{4x}-3v_{5x}]_x. \end{aligned} \quad (45)$$

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