Form invariance and approximate conserved quantity of Appell equations for a weakly nonholonomic system*

Jia Li-Qun(贾利群)†, Zhang Mei-Ling(张美玲),
Wang Xiao-Xiao(王肖肖), and Han Yue-Lin(韩月林)

School of Science, Jiangnan University, Wuxi 214122, *China*

(Received 26 November 2011; revised manuscript received 22 December 2011)

A weakly nonholonomic system is a nonholonomic system whose constraint equations contain a small parameter. The form invariance and the approximate conserved quantity of the Appell equations for a weakly nonholonomic system are studied. The Appell equations for the weakly nonholonomic system are established, and the definition and the criterion of form invariance of the system are given. The structural equation of form invariance for the weakly nonholonomic system and the approximate conserved quantity deduced from the form invariance of the system are obtained. Finally, an example is given to illustrate the application of the results.

Keywords: weakly nonholonomic system, Appell equations, form invariance, approximate conserved quantity

PACS: 02.20.Sv, 11.30.–j, 45.20.Jj, 03.50.–z **DOI:** 10.1088/1674-1056/21/7/070204

1. Introduction

Research on the dynamics of nonholonomic systems has not only important theoretical significance but also noticeable value for applications, and great progress has been made over the years.[1−8] Moreover, many results on the study of symmetry and conserved quantity for mechanical systems have been obtained.[9−30] There is a special nonholonomic system whose constraint equations contain a small parameter. Such a system is called a weakly nonholonomic system. For the weakly nonholonomic system, Refs. [31]–[33] discussed its equations of motion and their approximate solution, canonical transformation, and stability, respectively. Reference [34] studied the Noether symmetry and the corresponding conserve quantity of the Lagrange equations for a weakly nonholonomic system. However, so far, the symmetry and the corresponding approximate conserved quantity of the Appell equations for a weakly nonholonomic system have not yet been studied. This paper will study the form invariance and the corresponding conserved quantity of the Appell equations for a weakly nonholonomic system. First, we establish the Appell equations for a weakly nonholonomic system to obtain its differential equations of motion and its first degree approximation. Second, we give the definition and the criterion of the Appell equations of the system. Finally, we present the structural equation of form invariance and the expression of the conserved quantity of the Appell equations for a weakly nonholonomic system.

2. Differential equations of motion of Appell equations for a weakly nonholonomic system

Suppose that the position of a mechanical system is determined by n generalized coordinates q_s $(s = 1, 2, \ldots, n)$, and the motion is subject to g ideal bilateral linear homogeneous nonholonomic constraints

$$f_\beta = \dot{q}_{\varepsilon+\beta} - \mu B_{\varepsilon+\beta,\delta}(q_s, t)\dot{q}_\delta = 0, \quad (\beta = 1, 2, \ldots, g;\ \delta = 1, 2, \ldots, \varepsilon;\ \varepsilon = n - g;\ s = 1, 2, \ldots, n), \tag{1}$$

where μ is a small parameter. The differential equations of motion of the system can then be expressed

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11142014 and 61178032).

†Corresponding author. E-mail: jlq0000@163.com

© 2012 Chinese Physical Society and IOP Publishing Ltd

as

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{2}$$

where S is the Appell function, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are non-potential generalized forces, and λ_β are undetermined multipliers.

From Eqs. (1) and (2), we can solve λ_β as functions of $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$, and μ. Equation (2) can be written as

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \Lambda_s, \tag{3}$$

which are called the Appell equations of the holonomic system corresponding to weakly nonholonomic systems (1) and (2), where

$$\Lambda_s = \Lambda_s(\boldsymbol{q}, \dot{\boldsymbol{q}}, t, \mu) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \tag{4}$$

are generalized constraint forces. It has been proved that the solution of the Appell equations of the corresponding holonomic system (3) gives the motion of the weakly nonholonomic system if the initial conditions of motion satisfy constraint Eq. (1).

In order to discuss the approximate solution of the Appell equations for a weakly nonholonomic system, we expand generalized constraint forces Λ_s as power series of parameter μ

$$\begin{aligned} \Lambda_s = {} & \Lambda_{s0}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \mu \Lambda_{s1}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \\ & + \mu^2 \Lambda_{s2}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \cdots . \end{aligned} \tag{5}$$

Then we obtain the first degree approximation of Eq. (3)

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \Lambda_{s0} + \mu \Lambda_{s1}. \tag{6}$$

All generalized accelerations can be solved from Eq. (3) as

$$\ddot{q}_s = \alpha_s \left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t, \mu\right). \tag{7}$$

From Eq. (6), we obtain

$$\ddot{q}_s = \alpha_{s0} \left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right) + \mu \alpha_{s1} \left(\boldsymbol{q}, \dot{\boldsymbol{q}}, t\right). \tag{8}$$

3. Definition and criterion of form invariance

Introduce the infinitesimal transformations of group of time and coordinates as

$$\begin{aligned} & t^* = t + \varepsilon \xi_0 \left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right), \\ & q_s^* \left(t^*\right) = q_s \left(t\right) + \varepsilon \xi_s \left(t, q, \dot{\boldsymbol{q}}\right), \quad (s = 1, 2, \ldots, n), \end{aligned} \tag{9}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are the generating functions of the infinitesimal transformations. Introduce infinitesimal generator vector $X^{(0)}$ as well as its first and second extended infinitesimal generators $\tilde{X}^{(1)}$ and $\tilde{X}^{(2)}$ as

$$\begin{aligned} & X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s}, \\ & \tilde{X}^{(1)} = X^{(0)} + \left(\frac{\bar{\mathrm{d}} \xi_s}{\mathrm{d}t} - \dot{q}_s \frac{\bar{\mathrm{d}} \xi_0}{\mathrm{d}t} \right) \frac{\partial}{\partial \dot{q}_s}, \\ & \tilde{X}^{(2)} = \tilde{X}^{(1)} + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \left(\frac{\bar{\mathrm{d}} \xi_s}{\mathrm{d}t} - \dot{q}_s \frac{\bar{\mathrm{d}} \boldsymbol{\xi_0}}{\mathrm{d}t} \right) - \ddot{q}_s \frac{\bar{\mathrm{d}} \boldsymbol{\xi_0}}{\mathrm{d}t} \right] \frac{\partial}{\partial \ddot{q}_s}, \end{aligned}$$

where the total derivative with respect to t along the curve of a function can be written as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \alpha_s \frac{\partial}{\partial \dot{q}_s} + \dot{\alpha}_s \frac{\partial}{\partial \ddot{q}_s}.$$

Suppose that dynamical functions S, Q_s, Λ_s, and f_β become S^*, Q_s^*, Λ_s^*, and f_β^*, respectively, after going through the infinitesimal transformation (9). And then taking the Taylor expansions of S^*, Q_s^*, Λ_s^*, and f_β^* at the point of $(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$, we have

$$\begin{aligned} S^* &= S\left(\boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, \frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}, t^*\right) \\ &= S(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t) + \varepsilon \tilde{X}^{(2)}(S) + O(\varepsilon^2), \\ Q_s^* &= Q_s\left(\boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, t^*\right) \\ &= Q_s(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \varepsilon \tilde{X}^{(1)}(Q_s) + O(\varepsilon^2), \\ \Lambda_s^* &= \Lambda_s\left(\boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, t^*\right) \\ &= \Lambda_s(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \varepsilon \tilde{X}^{(1)}(\Lambda_s) + O(\varepsilon^2), \\ f_\beta^* &= f_\beta\left(\boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, t^*\right) \\ &= f_\beta(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) + \varepsilon \tilde{X}^{(1)}(f_\beta) + O(\varepsilon^2). \end{aligned} \tag{10}$$

Definition If the form equations are invariant when the dynamical functions are replaced by the corresponding transformed ones, respectively, then such an invariance is called the form invariance.

According to the definition, the form invariances of Eqs. (3) and (1) are, respectively,

$$\frac{\partial S^*}{\partial \ddot{q}_s} = Q_s^* + \Lambda_s^*, \tag{11}$$

$$f_\beta\left(\boldsymbol{q}^*, \frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}, t^*\right) = 0. \tag{12}$$

Substituting expression (10) into Eq. (11), neglecting the terms of ε^2 and the higher order terms, and using Eq. (3), we obtain

$$\frac{\partial}{\partial \ddot{q}_s}[\tilde{X}^{(2)}(S)] = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(\Lambda_s). \tag{13}$$

Similarly, substituting expression (10) into Eq. (12), neglecting the terms of ε^2 and the higher order terms, and using Eq. (1), we have

$$\tilde{X}^{(1)}(f_\beta) = 0. \tag{14}$$

For the first degree approximate Eq. (6), we have

$$\frac{\partial}{\partial \ddot{q}_s}[\tilde{X}^{(2)}(S)] = \tilde{X}^{(1)}(Q_s) + \tilde{X}^{(1)}(\Lambda_{s0}) + \mu \tilde{X}^{(1)}(\Lambda_{s1}). \tag{15}$$

Criterion 1 For the Appell equations of the holonomic system (3) corresponding to the weakly nonholonomic system (1) and (2), if infinitesimal generators ξ_0 and ξ_s satisfy Eq. (13), then the relevant invariance is a form invariance of the Appell equations of the holonomic system (3).

Criterion 2 For the weakly nonholonomic system (1) and (2), if infinitesimal generators ξ_0 and ξ_s satisfy Eqs. (13) and (14), then the relevant invariance is a form invariance of weakly nonholonomic systems (1) and (2).

Criterion 3 For the Appell equations of the first degree approximate holonomic system (6) corresponding to weakly nonholonomic systems (1) and (2), if infinitesimal generators ξ_0 and ξ_s satisfy Eq. (15), then the relevant invariance is a form invariance of the Appell equations of the first degree approximate holonomic system (6).

Criterion 4 For the Appell equations of the first degree approximate system (6) of weakly nonholonomic systems (1) and (2), if infinitesimal generators ξ_0 and ξ_s satisfy Eqs. (15) and (14), then the relevant invariance is a form invariance of the Appell equations of the first degree approximate holonomic system (6).

4. Conserved quantity deduced from the form invariance

According to the theory of form invariance for the Appell equations of general nonholonomic systems,[35] the form invariance for the Appell equations of the weakly nonholonomic systems can also deduce a conserved quantity.

Proposition 1 If infinitesimal generators ξ_0 and ξ_s are the form invariance for the Appell equations of the corresponding holonomic system (3) and there exists a gauge function $G_{\rm F} = G_{\rm F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t, \mu)$ satisfying the structural equation

$$\begin{aligned}&\tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right] \\ &+ (\xi_s - \dot{q}_s\xi_0)\tilde{E}_s\left[\tilde{X}^{(2)}(S)\right] \\ &+ \xi_0\left[\tilde{X}^{(1)}(Q_s + \Lambda_s)\right]\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} + \frac{\bar{\mathrm{d}}G_{\rm F}}{\mathrm{d}t} = 0, \end{aligned}\tag{16}$$

where

$$\tilde{E}_s = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s},$$

then the form invariance for the Appell equations of the corresponding holonomic system (3) can lead to the conserved quantity

$$\begin{aligned}I_{\rm F} &= \xi_0\tilde{X}^{(2)}(S) + \frac{\partial \tilde{X}^{(2)}(S)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\rm F} \\ &= \text{const.}\end{aligned}\tag{17}$$

Proposition 2 If infinitesimal generators ξ_0 and ξ_s are ones of form invariance for the Appell equations of weakly nonholonomic systems (1) and (2), and there exists a gauge function $G_{\rm F} = G_{\rm F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t, \mu)$ satisfying the structural Eq. (16), then the form invariance for the Appell equations of weakly nonholonomic systems (1) and (2) can lead to conserved quantity (17).

Proposition 3 If infinitesimal generators ξ_0 and ξ_s are ones of form invariance for the Appell equations of the first degree approximate holonomic system (6) and there exists a gauge function $G_{\rm F} = G_{\rm F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t, \mu)$ satisfying the structural equation

$$\begin{aligned}&\tilde{X}^{(2)}(S)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S)\right] \\ &+ (\xi_s - \dot{q}_s\xi_0)\tilde{E}_s\left[\tilde{X}^{(2)}(S)\right] \\ &+ \xi_0\left[\tilde{X}^{(1)}(Q_s + \Lambda_{s0} + \mu\Lambda_{s1})\right]\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} + \frac{\bar{\mathrm{d}}G_{\rm F}}{\mathrm{d}t} = 0, \end{aligned}\tag{18}$$

then the form invariance for the Appell equations of the first degree approximate holonomic system (6) can lead to conserved quantity (17).

Proposition 4 If infinitesimal generators ξ_0 and ξ_s are the form invariance for the first degree approximate system of the weakly nonholonomic system and there exists a gauge function $G_{\rm F} = G_{\rm F}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t, \mu)$ satisfying the structural Eq. (18), then the form invariance can lead to conserved quantity (17).

Using the above propositions, we can find the conserved quantities deduced from the form invariance for weakly nonholonomic systems.

5. Form invariance and conserved quantity of Appell equations for a two-degree-freedom weakly nonholonomic system

Suppose that a ball with unit mass does a two-degree-freedom movement, and its weakly nonholonomic constraint equation and dynamical function of relative motion are, respectively,

$$f = \dot{q}_2 - \mu t \dot{q}_1 = 0, \tag{19}$$

$$S = \frac{1}{2}(\ddot{q}_1^2 + \ddot{q}_2^2), \tag{20}$$

$$Q_1 = Q_2 = 0. \tag{21}$$

Now we study its form invariances and conserved quantities. From Eqs. (2), (4), and (19), we obtain

$$\lambda = \frac{\mu \dot{q}_1}{1+\mu^2 t^2}, \tag{22}$$

$$\ddot{q}_1 = \Lambda_1 = -\frac{\mu^2 t}{1+\mu^2 t^2}\dot{q}_1,$$

$$\ddot{q}_2 = \Lambda_2 = \frac{\mu}{1+\mu^2 t^2}\dot{q}_1. \tag{23}$$

Equations (23) are differential equations of motion obtained from Appell Eq. (3) for the holonomic system corresponding to Appell Eqs. (1) and (2) for the weakly nonholonomic system.

The criterion Eq. (13) for form invariance shows

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - \dot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - 2\ddot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} \\
&= -\xi_0\dot{q}_1\frac{\partial}{\partial t}\left(\frac{\mu^2 t}{1+\mu^2 t^2}\right) - \frac{\mu^2 t}{1+\mu^2 t^2}\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - \dot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right), \\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t} - \dot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - 2\ddot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} \\
&= \xi_0\dot{q}_1\frac{\partial}{\partial t}\left(\frac{\mu}{1+\mu^2 t^2}\right) + \frac{\mu}{1+\mu^2 t^2}\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - \dot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right).
\end{aligned}$$

Then we can find its solution

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = \mu q_2 - \dot{q}_1 - \mu t \dot{q}_2 + t. \tag{24}$$

From Eq. (14), we obtain

$$\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t} - \dot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - \mu\xi_0\dot{q}_1 - \mu t\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - \dot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) = 0. \tag{25}$$

It is easy to see that generators (24) do not satisfy Eq. (25). From Criteria 1 and 2, we can see that the system has form invariance of the holonomic equations (23) corresponding to the weakly nonholonomic system. By calculation, we have

$$\left[\tilde{X}^{(1)}\left(Q_s + \Lambda_s\right)\right] = 0, \quad s = 1, 2. \tag{26}$$

Using structural Eq. (16), we obtain

$$G_{\mathrm{F}} = \mu q_2 - \dot{q}_1 - \mu t \dot{q}_2.$$

The conserved quantity (17) then gives

$$I_{\mathrm{F}} = \mu q_2 - \dot{q}_1 - \mu t \dot{q}_2 = \text{const}. \tag{27}$$

Similarly, from Eq. (8), we have

$$\ddot{q}_1 = 0, \quad \ddot{q}_2 = \mu\dot{q}_1. \tag{28}$$

For the first degree approximate system (28), criterion equations (15) show

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - \dot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - 2\ddot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = 0, \\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t} - \dot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - 2\ddot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \mu\left(\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t} - \dot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right).
\end{aligned}$$

We can find the following solutions

$$\xi_0 = 0, \quad \xi_1 = 1, \quad \xi_2 = \dot{q}_2 - \mu\dot{q}_1 + t, \tag{29}$$

which are generators of form invariance for the first degree approximate holonomic system. Using generators (29), through the calculation, we obtain

$$\begin{aligned}
&\tilde{X}^{(2)}\left(S\right) = 0, \\
&\left[\tilde{X}^{(1)}\left(Q_s + \Lambda_{s0} + \mu\Lambda_{s1}\right)\right] = 0, \quad s = 1, 2.
\end{aligned} \tag{30}$$

Substituting Eqs. (29) and (30) into the structural Eq. (18), we then obtain

$$G_{\mathrm{F}} = \dot{q}_1 + \dot{q}_2 - \mu q_1.$$

The conserved quantity (17) then gives

$$I_{\mathrm{F}} = \dot{q}_1 + \dot{q}_2 - \mu q_1 = \text{const}. \tag{31}$$

which is a conserved quantity for the first degree approximate system (28). From Eqs. (27) and (31), the physical significance of the two conserved quantities is not very clear.[36] This is still a problem to be resolved when we use the symmetry theory to look for conserved quantities of the constrained mechanical system.

6. Conclusion

The paper studies the form invariance and the conserved quantity of the Appell equations for the weakly nonholonomic system. The method, in which the approximate conserved quantity can be obtained through expanding the generalized constraint forces Λ_s as power series of small parameter μ, can be spread and applied to other mechanical systems and physical systems with small parameters.

References

[1] Neimark J I and Fufaev N A 1972 *Dynamics of Nonholonomic Systems* (Providence: AMS)

[2] Mei F X 1985 *Foundations of Mechanics of Nonholonomic Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[3] Bloch A M, Krishnaprasad P S, Marsden J E and Murray R M 1996 *Arch. Rat. Mech. Anal.* **136** 21

[4] Papastavridis J G 1998 *Appl. Mech. Rev.* **51** 239

[5] Ostrovskaya S and Angels J 1998 *Appl. Mech. Rev.* **51** 415

[6] Mei F X 2000 *Appl. Mech. Rev.* **53** 283

[7] Guo Y X, Luo S K and Mei F X 2004 *Advances in Mechanics* **34** 477 (in Chinese)

[8] Zegzhda S A, Soltakhanov S K and Yushkov M P 2005 *Equations of Motion of Nonholonomic Systems and Variational Principles of Mechanics. New Kind of Control Problems* (Moscow: FIMATLIT) (in Russian)

[9] Mei F X 2000 *J. Beijing Institute of Technology* **9** 120

[10] Wang S Y and Mei F X 2002 *Chin. Phys.* **11** 5

[11] Zhang H B 2002 *Chin. Phys.* **11** 765

[12] Zhang Y and Mei F X 2003 *Chin. Phys.* **12** 1058

[13] Luo S K 2003 *Acta Phys. Sin.* **52** 2941 (in Chinese)

[14] Fu J L, Chen L Q and Xie F P 2004 *Chin. Phys.* **13** 1611

[15] Wu H B 2005 *Chin. Phys.* **14** 452

[16] Lou Z M 2005 *Chin. Phys.* **14** 660

[17] Fang J H, Peng Y and Liao Y P 2005 *Acta Phys. Sin.* **54** 496 (in Chinese)

[18] Chen X W, Liu C M and Li Y M 2006 *Chin. Phys.* **15** 470

[19] Liu H J, Fu J L and Tang Y F 2007 *Chin. Phys.* **16** 599

[20] Luo S K, Chen X W and Guo Y X 2007 *Chin. Phys.* **16** 3176

[21] Zheng S W, Xie J F and Chen W C 2008 *Chin. Phys. Lett.* **25** 809

[22] Cai J L, Luo S K and Mei F X 2008 *Chin. Phys.* B **17** 3170

[23] Cai J L and Mei F X 2008 *Acta Phys. Sin.* **57** 5369 (in Chinese)

[24] Cai J L 2009 *Acta Phys. Sin.* **58** 22 (in Chinese)

[25] Li Y C, Xia L L and Wang X M 2009 *Acta Phys. Sin.* **58** 6732 (in Chinese)

[26] Zheng S W, Xie J F, Chen X W and Du X L 2010 *Acta Phys. Sin.* **59** 5209 (in Chinese)

[27] Jia L Q, Xie Y L, Zhang Y Y, Cui J C and Yang X F 2010 *Acta Phys. Sin.* **59** 7552 (in Chinese)

[28] Jiang W A and Luo S K 2011 *Acta Phys. Sin.* **60** 060201 (in Chinese)

[29] Zhang M L, Sun X T, Wang X X, Xie Y L and Jia L Q 2011 *Chin. Phys.* B **20** 110202

[30] Jiang W A and Luo S K 2011 *Nonlinear Dyn.* **67** 475

[31] Mei F X 1989 *J. Beijing Institute of Technology* **9** 10 (in Chinese)

[32] Mei F X 1993 *Chin. Sci. Bull.* **38** 281 (in Chinese)

[33] Mei F X 1995 *J. Beijing Institute of Technology* **15** 237 (in Chinese)

[34] Ge W K 2009 *Acta Phys. Sin.* **58** 6729 (in Chinese)

[35] Jia L Q, Xie J F and Zheng S W 2008 *Chin. Phys.* B **17** 17

[36] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)