Noether conserved quantities and Lie point symmetries for difference nonholonomic Hamiltonian systems in irregular lattices*

Xia Li-Li(夏丽莉)[a)b)] and Chen Li-Qun(陈立群)[a)c)†]

[a)] *Shanghai Institute of Applied Mathematics and Mechanics, Shanghai* 200072, *China*

[b)] *Department of Physics, Henan Institute of Education, Zhengzhou* 450046, *China*

[c)] *Department of Mechanics, Shanghai University, Shanghai* 200444, *China*

(Received 30 November 2011; revised manuscript received 28 December 2011)

The Noether conserved quantities and the Lie point symmetries for difference nonholonomic Hamiltonian systems in irregular lattices are studied. The generalized Hamiltonian equations of the systems are given on the basis of the transformation operators in the space of discrete Hamiltonians. The Lie transformations acting on the lattice, as well as the equations and the determining equations of the Lie symmetries are obtained for the nonholonomic Hamiltonian systems. The discrete analogue of the Noether conserved quantity is constructed by using the Lie point symmetries. An example is discussed to illustrate the results.

Keywords: discrete nonholonomic Hamiltonian systems, Lie point symmetry, Noether conserved quantity

PACS: 02.20.Sv, 02.20.Qs, 11.30.–j, 45.20.Jj **DOI:** 10.1088/1674-1056/21/7/070202

1. Introduction

Various forms of difference mechanical systems appear in mathematics, computer science, engineering to biology, and so on. The application of difference equations and meshes provides appropriate methods for solving some problems in these fields. In physics, the theory of continuous symmetries and conserved quantities is of great theoretical significance and application value, and has developed rapidly in recent years.[1−10] Symmetries are intrinsic and fundamental features of the differential equations in mathematical physics, and symmetries are also the basis for modeling a given mechanical system with the use of difference equations and meshes. This is also one of the reasons why we are interested in the continuous symmetries of discrete equations.[11]

In 1970, Cadzow proposed a discrete variational principle and obtained the discrete Euler–Lagrange equation.[12] Three years later, Logan constructed the first integrals of the discrete Euler–Lagrange equation by studying the invariance of the discrete Lagrangian, which is called the discrete Noether theorem.[13] Lee proposed a new discrete variational principle, and obtained not only the equations of motion of the discrete systems but also the conserved laws for the conservative systems.[14] Levi *et al.* presented a method for calculating the Lie point symmetries of the difference equations with one or several independent variables.[15] They also proposed an approach on the applications of the Lie group theory to the difference equations and the discrete mechanical systems in Refs. [16] and [17], in which the Lie group acts on the equation, but not on the lattice. Dorodnitsyn gave a different approach, in which the Lie group transformations act both on the difference equations and the lattice.[18,19] Based on those two approaches, researchers generalized those theories to many kinds of mechanical systems.[20−24] However, there are many evolution equations without appropriate or natural Lagrangian formulation. For those equations, one may prefer the Hamiltonian viewpoint. So studies on the discrete Hamiltonian system shoud be consid-

*Project supported by the National Outstanding Young Scientist Fund of China (Grant No. 10725209), the National Natural Science Foundation of China (Grant Nos. 90816001 and 11102060), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20093108110005), the Shanghai Subject Chief Scientist Project, China (Grant No. 09XD1401700), and the Shanghai Leading Academic Discipline Project, China (Grant No. S30106).

†Corresponding author. E-mail: lqchen@staff.shu.edu.cn

© 2012 Chinese Physical Society and IOP Publishing Ltd

ered. Dorodnitsyn and Kozlov considered the relation between symmetries and first integrals for both continuous canonical Hamiltonian equations and discrete ones[25] Chen *et al.* studied the total variation in the Hamiltonian formalism and the symplectic-energy integrators.[26] Zhang *et al.* obtained the discrete variational principle in the Hamiltonian formalism and the first integrals.[27] Wang *et al.* obtained the discrete analogue of Noether's identity and the forms of the Noether conserved quantity for the nonconservative and nonholonomic Hamiltonian system.[28]

The Lie group theory itself started out as a theory of transformation of solutions of the differential equations.[29] Lie point transformations have a clear geometrical interpretation, and one can construct the orbits if the group is in a finite dimensional space of independent and dependent variables. The applications of the Lie group theory to the difference equations are much more recent.[30−36] Whereas, the applications of the Lie symmetries of a discrete nonholonomic Hamiltonian with irregular lattice have not been investigated. In this paper, we apply the Lie point transformation to nonholonomic Hamiltonian systems. This paper is organized as follows. In Section 2, we give the equations of motion for the nonholonomic Hamiltonian systems. In Section 3, the Lie group transformations acting both on the difference equations and the lattice are considered, the determining equations of the Lie symmetries are obtained for the systems by using the Lie point symmetries, and the discrete analogue of the Noether conserved quantity is constructed. The conclusion is drawn in Section 5 after an illustrated example is given in Section 4.

2. Generalized Hamiltonian equations of discrete nonholonomic systems

2.1. Transformation operators in the space of discrete Hamiltonians

We consider space z of sequence $(t, \boldsymbol{q}, \boldsymbol{p})$, and denote the first-order linear differential operator by

$$\mathrm{D} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}^i \frac{\partial}{\partial p^i} + \cdots . \tag{1}$$

We fix an arbitrary parameter value $h > 0$, and form a pair of operators of discrete transformation to the right and the left

$$\begin{aligned} \mathrm{S}_{+h} &= \mathrm{e}^{h\mathrm{D}} = \sum_{i=0}^{\infty} \frac{h^i}{s!} \mathrm{D}^i, \\ \mathrm{S}_{-h} &= \mathrm{e}^{-h\mathrm{D}} = \sum_{i=0}^{\infty} \frac{(-h)^i}{s!} \mathrm{D}^i, \quad (s = 1, \ldots, n), \end{aligned} \tag{2}$$

where D is the derivation in z operators, S_{+h} and S_{-h} commute with each other and satisfy $\mathrm{S}_{+h}\mathrm{S}_{-h} = \mathrm{S}_{-h}\mathrm{S}_{+h} = 1$. Consider difference Hamiltonian equations at some lattice point $(t, \boldsymbol{q}, \boldsymbol{p})$. Generally, the lattice is irregular. The notations are given in Fig. 1.

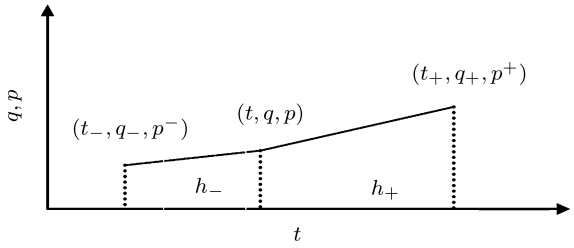


Fig. 1. Elementary stencil for a three-point difference equation.

Using S_{+h} and S_{-h}, we define a pair of right and left discrete (finite-difference) differentiation operators by setting

$$\begin{aligned} \mathrm{D}_{-h} &= \frac{1 - \mathrm{S}_{-h}}{h} = \sum_{i=1}^{\infty} \frac{h^{i-1}}{s!} \mathrm{D}^i, \\ \mathrm{D}_{+h} &= \frac{\mathrm{S}_{+h} - 1}{h} = \sum_{i=1}^{\infty} \frac{(-h)^{i-1}}{s!} \mathrm{D}^i. \end{aligned} \tag{3}$$

Operators S_{+h}, S_{-h}, D_{+h} and D_{-h} commute in any combination, while $\mathrm{D}_{+h} = \mathrm{D}_{-h}\mathrm{S}_{+h}$, $\mathrm{D}_{-h} = \mathrm{D}_{+h}\mathrm{S}_{-h}$.

By definition, we extend the action of discrete shift S_{+h} to functions in the Euclidean space A

$$\mathrm{S}_{+h}\left(F(z)\right) = F\left(\mathrm{S}_{+h}(z)\right). \tag{4}$$

This permits finding the difference derivatives of $F \in A$ in the form

$$\mathrm{D}_{\pm h}\left(F(z)\right) = \pm \frac{F\left(\mathrm{S}_{\pm h}(z)\right) - F(z)}{h}. \tag{5}$$

Starting from this definition, it is easy to introduce the discrete Leibniz rule for the operators of right and left discrete differentiation

$$\begin{aligned} \mathrm{D}_{+h}(FG) &= \mathrm{D}_{+h}(F)G + F\mathrm{D}_{+h}(G) \\ &\quad + h\mathrm{D}_{+h}(F)\mathrm{D}_{+h}(G), \end{aligned} \tag{6}$$

where F, $G \in A$. We simply give the proof of the equation of the Leibniz rule by using Eqs. (4) and (5)

$$\begin{aligned} &\mathrm{D}_{+h}(FG) \\ &= \frac{\mathrm{S}_{+h}\left(FG\right) - FG}{h} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{S}_{+h}(F)\,G - FG}{h} + \frac{F\mathrm{S}_{+h}(G) - FG}{h} \\
&\quad + h\frac{(\mathrm{S}_{+h}(F) - F)\,(\mathrm{S}_{+h}(G) - G)}{h^2} \\
&= \mathrm{D}_{+h}(F)G + F\mathrm{D}_{+h}(G) + h\mathrm{D}_{+h}(F)\mathrm{D}_{+h}(G).
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
\mathrm{D}_{-h}(FG) &= \mathrm{D}_{-h}(F)G + F\mathrm{D}_{-h}(G) \\
&\quad - h\mathrm{D}_{-h}(F)\mathrm{D}_{-h}(G). \qquad (7)
\end{aligned}
$$

2.2. Discrete Hamiltonian equations and energy equations with nonholonomic constraints

In this paper, the Noether conserved quantities and the Lie point symmetries for the difference nonholonomic Hamiltonian systems are studied. The results will be built upon those for the continuous canonical Hamiltonian equations with nonholonomic constraints[5]

$$
\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} + Q_s + \Lambda_s, \qquad (8)
$$

where Q_s and Λ_s are the nonpotential generalized forces and the generalized nonholonomic reactive constraint forces, respectively, and they are functions of t, $\boldsymbol{q}$, $\boldsymbol{p}$. Now we consider the difference nonholonomic Hamiltonian equation at some lattice point $(t, \boldsymbol{q}, \boldsymbol{p})$. Suppose an n-dimensional configuration manifold $\boldsymbol{q} = \{q^1, \ldots, q^n\}$, $\boldsymbol{p} = \{p_1, \ldots, p_n\}$ and discrete Hamiltonian function $H = H(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)$ subjected to $g\,(g > 0)$ ideal bilateral discrete constraints of the form[37]

$$
\begin{aligned}
f_\beta &= f_\beta(t, t_+, \boldsymbol{q}, \boldsymbol{p}, \mathrm{D}_{+h}(q^i), \mathrm{D}_{+h}(p_i)), \\
&\quad (\beta = 1, \ldots, g,\ i = 1, \ldots, n), \qquad (9)
\end{aligned}
$$

which satisfy the Appell–Chetaev's condition

$$
\frac{\partial f_\beta}{\partial \mathrm{D}_{+h}(q^i)}\delta q^i = 0, \quad (i = 1, \ldots, n), \qquad (10)
$$

where $\delta\tilde{q}^i$ is the discrete virtual placement satisfying

$$
\delta q^i = \Delta q^i - \frac{q_+^i - q^i}{h_+}\Delta t = \varepsilon\left(\eta^i - \frac{q_+^i - q^i}{h_+}\xi\right), \qquad (11)
$$

with η^i and ξ being the generators of the one-parameter infinitesimal transformation group. The Hamiltonian equations can be obtained by using the variational principle from the finite-difference functional Hamiltonian action

$$
H_h = \sum_{\Omega}[p_i^+(q_+^i - q^i) - H(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)h_+] \qquad (12)
$$

and

$$
\begin{aligned}
\Delta H_h &= -\sum_{\Omega}\left[Q^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+) + \Lambda^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)\right] \\
&\quad \times h_+\delta q^i, \qquad (13)
\end{aligned}
$$

where $Q^i = Q^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)$ is the discrete force, λ_β is the constrained multiplier, $\Lambda^i = \Lambda^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+) = \lambda_\beta\partial f_\beta/[\partial \mathrm{D}_{+h}(q^i)]$ is the discrete generalized nonholonomic reactive constraint force. The variation of function H_h along a curve $q^i = \varphi_i(t)$, $p_i = \psi_i(t)$ at some point $(t, \boldsymbol{q}, \boldsymbol{p})$ will affect only two terms of the sum in Eq. (12)

$$
\begin{aligned}
H_h &= \cdots + p_i(q^i - q_-^i) - H(t_-, t, \boldsymbol{q}_-, \boldsymbol{p})h_- \\
&\quad + p_i^+(q_+^i - q^i) - H(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)h_+ \\
&\quad + \cdots . \qquad (14)
\end{aligned}
$$

So

$$
\begin{aligned}
&\Delta H_h \\
&= \frac{\partial H_h}{\partial p_i}\Delta p_i + \frac{\partial H_h}{\partial q^i}\Delta q^i + \frac{\partial H_h}{\partial t}\Delta t \\
&= -\left[Q^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+) + \Lambda^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)\right]h_+\delta q^i. \qquad (15)
\end{aligned}
$$

Substituting Eq. (11) into Eq. (15), we obtain

$$
\begin{aligned}
&\frac{\partial H_h}{\partial p_i}\Delta p_i + \left[\frac{\partial H_h}{\partial q^i} + (Q^i + \Lambda^i)h_+\right]\Delta q^i \\
&+ \left[\frac{\partial H_h}{\partial t} - (q_+^i - q^i)(Q^i + \Lambda^i)\right]\Delta t = 0. \qquad (16)
\end{aligned}
$$

With the help of Eq. (14), we obtain the expression

$$
\begin{aligned}
&\left[q^i - q_-^i - h_-\frac{\partial H^-}{\partial p_i}\right]\Delta p_i \\
&- \left[p_i^+ - p_i + h_+\frac{\partial H}{\partial q^i} - (Q^i + \Lambda^i)h_+\right]\Delta q^i \\
&- \left[h_+\frac{\partial H}{\partial t} - H + h_-\frac{\partial H^-}{\partial t} + H^-\right. \\
&\left. + (q_+^i - q^i)(Q^i + \Lambda^i)\right]\Delta t \\
&= 0, \qquad (17)
\end{aligned}
$$

where $H = H(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)$ and $H^- = H^-(t_-, t, \boldsymbol{q}_-, \boldsymbol{p})$.

For the stationary value of discrete Hamiltonian action (13), the system of $2n + 1$ equations can be derived as

$$
q^i - q_-^i - h_-\frac{\partial H^-}{\partial p_i} = 0, \qquad (18)
$$

$$
p_i^+ - p_i + h_+\frac{\partial H}{\partial q^i} - (Q^i + \Lambda^i)h_+ = 0, \qquad (19)
$$

$$
h_+\frac{\partial H}{\partial t} - H + h_-\frac{\partial H^-}{\partial t}
$$

$$+ H^- + (q^i_+ - q^i)(Q^i + \Lambda^i) = 0. \quad (20)$$

According to the total left shift and right shift operators and discrete derivation operators, expressions (18)–(20) can be written as

$$\mathrm{D}_{+h}(q^i) = \frac{q^i_+ - q^i}{h_+} = \frac{\partial H}{\partial p^+_i}, \quad (21)$$

$$\mathrm{D}_{+h}(p_i) = \frac{p^+_i - p_i}{h_+} = -\frac{\partial H}{\partial q^i} + Q^i + \Lambda^i, \quad (22)$$

$$h_+\frac{\partial H}{\partial t} - H + h_-\frac{\partial H^-}{\partial t} + H^- + \mathrm{D}_{+h}(q^i)(Q^i + \Lambda^i)h_+ = 0. \quad (23)$$

Note that the first $2n$ equations (Eqs. (21) and (22)) are first-order difference equations, which are the generalized discrete nonholonomic canonical Hamiltonian equations. Equation (23) is of the second order, it defines the lattice on which the canonical Hamiltonian equations with nonholonomic constraints are discretized, which is the energy equation. In the continuous limit, the lattice equation disappears. Equations (21)–(23) require the discrete equations of the nonholonomic Hamiltonian systems.

3. Noether conserved quantities and Lie point symmetries

3.1. Lie point symmetries of difference equations of nonholonomic mechanical systems

The first steps in the construction of the Lie symmetries for difference equations were taken by Maeda in 1980.[30] Later it was extended by many researchers.[31−35] On the base of these, we generalize the theory of the Lie symmetry to the discrete nonholonomic Hamiltonian equations.

We introduce the infinitesimal transformations with respect to time, generalized coordinates, and generalized momenta as

$$\begin{aligned}\Delta t &= t + \varepsilon\xi(t, \boldsymbol{q}, \boldsymbol{p}),\\ \Delta q^i &= q^i + \varepsilon\eta^i(t, \boldsymbol{q}, \boldsymbol{p}),\\ \Delta p_i &= p_i + \varepsilon\zeta_i(t, \boldsymbol{q}, \boldsymbol{p}),\end{aligned} \quad (24)$$

where ε is a small parameter, and infinitesimal generators ξ, η^i, ζ_i constitute the Lie group of the infinitesimal transformations. The Lie group is completely defined by the operator

$$\mathrm{X} = \xi(t, \boldsymbol{q}, \boldsymbol{p})\frac{\partial}{\partial t} + \eta^i(t, \boldsymbol{q}, \boldsymbol{p})\frac{\partial}{\partial q^i} + \zeta_i(t, \boldsymbol{q}, \boldsymbol{p})\frac{\partial}{\partial p_i}, \quad (25)$$

where $\boldsymbol{q} = \{q^1, \ldots, q^n\}$, and $\boldsymbol{p} = \{p_1, \ldots, p_n\}$. To consider the difference equations, we need three points of a lattice. The prolongation of the Lie group operator (25) for points $(t_-, \boldsymbol{q}_-, \boldsymbol{p}^-)$ and $(t_+, \boldsymbol{q}_+, \boldsymbol{p}^+)$ is[38]

$$\begin{aligned}\mathrm{Pr}\bar{\mathrm{X}} = {}& \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial q^i} + \zeta_i\frac{\partial}{\partial p_i} + \xi_-\frac{\partial}{\partial t_-} + \eta^i_-\frac{\partial}{\partial q^i_-}\\ &+ \zeta^-_i\frac{\partial}{\partial p^-_i} + \xi_+\frac{\partial}{\partial t_+} + \eta^i_+\frac{\partial}{\partial q^i_+}\\ &+ \zeta^+_i\frac{\partial}{\partial p^+_i} + (\xi_+ - \xi)\frac{\partial}{\partial h_+}\\ &+ (\xi - \xi_-)\frac{\partial}{\partial h_-},\end{aligned} \quad (26)$$

where $\xi_- = \xi(t_-, \boldsymbol{q}_-, \boldsymbol{p}^-)$, $\eta^i_- = \eta^i(t_-, \boldsymbol{q}_-, \boldsymbol{p}^-)$, $\zeta^-_i = \zeta_i(t_-, \boldsymbol{q}_-, \boldsymbol{p}^-)$, $\xi_+ = \xi(t_+, \boldsymbol{q}_+, \boldsymbol{p}^+)$, $\eta^i_+ = \eta^i(t_+, \boldsymbol{q}_+, \boldsymbol{p}^+)$, and $\zeta^+_i = \zeta_i(t_+, \boldsymbol{q}_+, \boldsymbol{p}^+)$.

In Eqs. (7), (24), and (25),

$$\begin{aligned}\mathrm{D}_{+h}(q^i) &= \frac{q^i_+ - q^i}{t^+ - t} = \frac{q^i_+ - q^i}{h_+},\\ \mathrm{D}_{-h}(q^i) &= \frac{q^i - q^i_-}{t - t_-} = \frac{q^i - q^i_-}{h_-},\\ \mathrm{D}_{+h}(p^i) &= \frac{p^i_+ - p^i}{t^+ - t} = \frac{p^i_+ - p^i}{h_+},\\ \mathrm{D}_{-h}(p^i) &= \frac{p^i - p^i_-}{t - t_-} = \frac{p^i - p^i_-}{h_-}.\end{aligned} \quad (27)$$

Equations (21)–(23) can be discretized to

$$\begin{aligned}&E_1(t, t_+, t_-, \boldsymbol{q}, \boldsymbol{p}, \boldsymbol{q}_-, \boldsymbol{p}_-, \boldsymbol{q}_+, \boldsymbol{p}_+)\\ &= \mathrm{D}_{+h}(q^i) - \frac{\partial H}{\partial p^+_i}\\ &= 0,\end{aligned} \quad (28)$$

$$\begin{aligned}&E_2(t, t_+, t_-, \boldsymbol{q}, \boldsymbol{p}, \boldsymbol{q}_-, \boldsymbol{p}_-, \boldsymbol{q}_+, \boldsymbol{p}_+)\\ &= \mathrm{D}_{+h}(p_i) + \frac{\partial H}{\partial q^i} - Q^i - \Lambda^i = 0,\end{aligned} \quad (29)$$

$$\begin{aligned}&\Omega(t, t_+, t_-, \boldsymbol{q}, \boldsymbol{p}, \boldsymbol{q}_-, \boldsymbol{p}_-, \boldsymbol{q}_+, \boldsymbol{p}_+)\\ &= h_+\frac{\partial H}{\partial t} - H + h_-\frac{\partial H^-}{\partial t} + H^-\\ &\quad + \mathrm{D}_{+h}(q^i)(Q^i + \Lambda^i)h_+\\ &= 0.\end{aligned} \quad (30)$$

The requirement that Eqs. (28)–(30) should be invariant under the considered group translates into

$$\mathrm{Pr}\bar{\mathrm{X}}E_1\big|_{E_1=0, E_2=0, \Omega=0} = 0, \quad (31)$$

$$\mathrm{Pr}\bar{\mathrm{X}}E_2|_{E_1=0, E_2=0, \Omega=0} = 0, \quad (32)$$

$$\mathrm{Pr}\bar{\mathrm{X}}\Omega|_{E_1=0,E_2=0,\Omega=0}=0. \tag{33}$$

Equations (31)–(33) are the determining equations of the Lie symmetries for the nonholonomic Hamiltonian systems.

3.2. Noether conserved quantities associated with the Lie point symmetries

The Lie symmetries can be carried over from the continuous case to the discrete case, so we can use the Lie symmetries of discrete equations to find conservation laws. Just as in the continuous case, the invariance of difference nonholonomic Hamiltonian on a specified mesh yields the first integrals of the discrete system. Based on Ref. [25], Dorodnitsyn and Kozlov gave the discrete Hamiltonian identity for the discrete Hamiltonian equations as

$$\begin{aligned}
&\zeta_i^+\mathrm{D}_{+h}(q^i)+p_i^+\mathrm{D}_{+h}(\eta^i)\\
&-\mathrm{X}(H)-H\mathrm{D}_{+h}(\xi)+\mathrm{D}_{+h}(G_i)\\
\equiv\ &\xi\left(\mathrm{D}_{+h}(H^-)-\frac{\partial H}{\partial t}-\frac{h_-}{h_+}\frac{\partial H^-}{\partial t}\right)\\
&-\eta^i\left(\mathrm{D}_{+h}(p_i)+\frac{\partial H}{\partial q^i}\right)\\
&+\zeta_i^+\left(\mathrm{D}_{+h}(q^i)-\frac{\partial H}{\partial p_i^+}\right)\\
&+\mathrm{D}_{+h}\left(\eta^i p_i-\xi\left(H^-+h_-\frac{\partial H^-}{\partial t}\right)+G_i\right).
\end{aligned} \tag{34}$$

Similarly, we obtain the discrete Hamiltonian identity for the discrete nonholonomic Hamiltonian

$$\begin{aligned}
&\zeta_i^+\mathrm{D}_{+h}(q^i)+p_i^+\mathrm{D}_{+h}(\eta^i)-\mathrm{X}(H)-H\mathrm{D}_{+h}(\xi)\\
&+(Q^i+\Lambda^i)[\eta^i-\mathrm{D}_{+h}(q^i)\xi]+\mathrm{D}_{+h}(G_i)\\
\equiv\ &\xi\Bigg(\mathrm{D}_{+h}(H^-)-\frac{\partial H}{\partial t}-\frac{h_-}{h_+}\frac{\partial H^-}{\partial t}\\
&-(Q^i+\Lambda^i)\mathrm{D}_{+h}(q^i)\Bigg)\\
&-\eta^i\left(\mathrm{D}_{+h}(p_i)+\frac{\partial H}{\partial q^i}-(Q^i+\Lambda^i)\right)\\
&+\zeta_i^+\left(\mathrm{D}_{+h}(q^i)-\frac{\partial H}{\partial p_i^+}\right)\\
&+\mathrm{D}_{+h}\left(\eta^i p_i-\xi\left(H^-+h_-\frac{\partial H^-}{\partial t}\right)+G_i\right).
\end{aligned} \tag{35}$$

If the left-hand side of Eq. (35) equals zero, i.e.,

$$\begin{aligned}
&\zeta_i^+\mathrm{D}_{+h}(q^i)+p_i^+\mathrm{D}_{+h}(\eta^i)-\mathrm{X}(H)-H\mathrm{D}_{+h}(\xi)\\
&+(Q^i+\Lambda^i)[\eta^i-\mathrm{D}_{+h}(q^i)\xi]+\mathrm{D}_{+h}(G_i)=0,
\end{aligned} \tag{36}$$

it is called the discrete analogue of the generalized Noether-type identity for the system. The right-hand side of Eq. (35) equals zero when

$$\mathrm{D}_{+h}(q^i)-\frac{\partial H}{\partial p_i^+}=0, \tag{37}$$

$$\mathrm{D}_{+h}(p_i)+\frac{\partial H}{\partial q^i}+(Q^i+\Lambda^i)=0, \tag{38}$$

$$\begin{aligned}
&\mathrm{D}_{+h}(H^-)-\frac{\partial H}{\partial t}-\frac{h_-}{h_+}\frac{\partial H^-}{\partial t}-(Q^i+\Lambda^i)\mathrm{D}_{+h}(q^i)\\
&=0,
\end{aligned} \tag{39}$$

which are called the generalized quasi-extremal equations for the nonholonomic Hamiltonian system. Then the systems possess the discrete analogue of the conservation law

$$\mathrm{D}_{+h}\left(\eta^i p_i-\xi\left(H^-+h_-\frac{\partial H^-}{\partial t}\right)+G^i\right)=0, \tag{40}$$

namely

$$I=\eta^i p_i-\xi\left(H^-+h_-\frac{\partial H^-}{\partial t}\right)+G^i=0. \tag{41}$$

The difference Eq. (41) is called the difference analogue of the Noether conservation laws associated with the discrete nonholonomic Hamiltonian system.

Theorem 1 If a discrete regular function $G^i=G^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)$ exists such that the infinitesimal transformation generators ξ, η^i, ζ_i or the symmetry operators X satisfy the discrete Noether identity (36) and the discrete analogue of the Appell–Chetaev condition (10), then the nonholonomic Hamiltonian system (Eqs. (9), (21)–(23)) possesses the discrete conserved quantity (41).

This theorem is the discrete analogue of the generalized Noether theorem associated with the nonholonomic Hamiltonian system.

One of our main interests is to find the Noether conserved quantity through the Lie symmetry of such systems. In the Hamiltonian formulation with nonholonomic constraints, under certain conditions, the Lie point symmetry can lead to the Noether conserved quantity. We state the discrete analogues of the Lie theorem associated with the Lie point symmetry for the nonholonomic mechanical systems.

Theorem 2 If the infinitesimal transformation generators ξ, η^i, ζ_i satisfy the determining Eqs. (31)–(33), the discrete regular function $G^i=G^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)$ exists, the discrete Noether identity (36) is satisfied, then the Lie point symmetry of the corresponding holonomic Hamiltonian system possesses the discrete conserved quantity (41).

Theorem 3 If the infinitesimal transformation generators ξ, η^i, ζ_i satisfy the determining Eq. (31)–(33), the regular function $G^i = G^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)$ exists, the discrete Noether identity (36) and the Appell–Chetaev condition (10) are satisfied, then the Lie point symmetry of the nonholonomic Hamiltonian system possesses the finite-difference conserved quantity (41).

Theorem 3 is the generalized Lie theorem for the nonholonomic Hamiltonian system.

4. An example

In this section, we present an example, which demonstrates how the Lie symmetry of a nonholonomic Hamiltonian system can be reduced and how to obtain the conservation laws for this system. We consider a dynamical system, the Hamiltonian and the nonholonomic constraint are, respectively,

$$H = \frac{1}{2}p_+^2 + q, \tag{42}$$

$$f = \frac{h_+}{\mathrm{D}_{+h}(q)} - \mathrm{D}_{+h}(q) - h_+ + p = 0. \tag{43}$$

The Hamiltonian equations and the lattice can be rewritten as

$$\mathrm{D}_{+h}(q) = p_+, \quad \mathrm{D}_{+h}(p) = -1 + \frac{1}{p_+}, \tag{44}$$

$$\frac{1}{2}p_+^2 - \frac{1}{2}p^2 + q - q_- - h_+ = 0. \tag{45}$$

We have

$$E_1 = \mathrm{D}_{+h}(q) - p_+ = 0, \tag{46}$$

$$E_2 = \mathrm{D}_{+h}(p) + 1 - \frac{1}{p_+} = 0, \tag{47}$$

$$\Omega = \frac{1}{2}p_+^2 - \frac{1}{2}p^2 + q - q_- - h_+ = 0. \tag{48}$$

Using the determining equations of Lie point symmetry (31)–(33), we can obtain the following equations:

$$\eta_+\frac{1}{h_+} - \eta\frac{1}{h_+} - \zeta_+ = 0, \tag{49}$$

$$\zeta_+\frac{1}{h_+} - \zeta\frac{1}{h_+} + \zeta_+\frac{1}{p_+^2} = 0, \tag{50}$$

$$\zeta_+ p_+ - \zeta p + \eta - \eta_- - (\xi_+ - \xi_-) = 0. \tag{51}$$

We will check the invariance of Eqs. (49)–(51) with the operators

$$\mathrm{X}_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial p}, \tag{52}$$

$$\mathrm{X}_2 = \frac{\partial}{\partial t}. \tag{53}$$

Operators (52) and (53) satisfy the Lie symmetries of the nonholonomic Hamiltonian.

The Noether identity (36) has the following expression:

$$\begin{aligned}&\zeta_+\mathrm{D}_{+h}(q) + p_+\mathrm{D}_{+h}(\eta) - \eta\frac{\partial H}{\partial q} - \zeta_+\frac{\partial H}{\partial p_+}\\ &- H\mathrm{D}_{+h}(\xi) + \Lambda(\eta - \mathrm{D}_{+h}(q)\xi)\\ &+ \mathrm{D}_{+h}(G) = 0.\end{aligned} \tag{54}$$

The equation of the discrete Appell–Chetaev's condition (10) can be rewritten as

$$\frac{\partial f_\beta}{\partial \mathrm{D}_{+h}(q^i)}\varepsilon\left(\eta^i - \frac{q_+^i - q^i}{h_+}\xi\right) = 0. \tag{55}$$

For the symmetry operator X_1, we have the divergence invariance condition

$$-1 + \frac{1}{p_+}\left(1 - \mathrm{D}_{+h}(q)\right) + \mathrm{D}_{+h}(G^1) = 0 \tag{56}$$

fulfilled on Eqs. (49)–(51), (54), and (55) with the function $G^1 = -p + t$. Using Eq. (41), we can obtain the discrete analogue of the Noether conserved quantity for the system

$$I^1 = -\frac{1}{2}p^2 - q_- + t = \text{const.} \tag{57}$$

For the symmetry operators X_2, we have

$$-1 + \mathrm{D}_{+h}(G^2) = 0 \tag{58}$$

satisfied Eqs. (49)–(51), and (54), not satisfied Eq. (55), with the function $G^2 = t + h_+/p_+ - p_+ - h_+ + p$. Therefore, we can obtain the discrete first integral of the corresponding holonomic Hamiltonian system

$$\begin{aligned}I^2 =& -\frac{1}{2}p^2 - q_- + t + \frac{h_+}{p_+} - p_+ - h_+ + p\\ =& \text{ const.}\end{aligned} \tag{59}$$

5. Conclusion

In this paper, we present the Lie symmetries and the Noether's theorem for discrete nonholonomic Hamiltonian systems. Similar to the continuous case, the approach to getting the Lie symmetries of the continuous systems is applied to the difference ones. The determining equations of the Lie symmetries on the discretization of Hamiltonian equations of nonholonomic mechanical systems are obtained. Using the connection between symmetries and conservation laws, we can obtain the condition under which the Lie symmetries can lead to the conserved quantities

for nonholonomic mechanical systems: the infinitesimal generators and the regular functions satisfy the Appell–Chetaev's condition, the determining equations of the Lie symmetries, and the discrete Noether identity. The results of this paper can be extended on the Hamiltonian systems with variable mass. The work is in progress on the Hojman conserved quantities for discrete nonholonomic Hamiltonian systems constructed by the continuous symmetries. We also plan to study the relations between the symmetries of the discrete systems and the corresponding continuous ones, and discuss whether all the continuous conservation laws remain under the discretization.

References

[1] Noether A E 1918 *Math. Phys.* **2** 235

[2] Bahar L Y and Kwatny H G 1987 *Int. J. Non-Linear Mech.* **22** 125

[3] Lutzky M 1979 *J. Phys.* A: *Math. Gen.* **12** 973

[4] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120

[5] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)

[6] Luo S K, Zhang Y F, *et al.* 2008 *Advances in the Study of Dynamics of Constrained Systems* (Beijing: Science Press) (in Chinese)

[7] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology) (in Chinese)

[8] Cai J L 2008 *Chin. Phys. Lett.* **25** 1523

[9] Xia L L and Cai J L 2010 *Chin. Phys.* B **19** 040302

[10] Xia L L 2011 *Chin. Phys. Lett.* **28** 040201

[11] Dorodnitsyn V 2011 *Applications of Lie Groups to Difference Equations* (Boca Raton: Chapman & Hall/CRC)

[12] Cadzow J A 1970 *Int. J. Control.* **11** 393

[13] Logan J D 1973 *Aequat. Math.* **9** 210

[14] Lee T D 1983 *Phys. Lett.* B **122** 217

[15] Levi D, Tremblay S and Winternitz P 2001 *Czech. J. Phys.* **51** 349

[16] Levi D and Winternitz P 1991 *Phys. Lett.* A **352** 335

[17] Levi D and Winternitz P 1996 *J. Math. Phys.* **37** 5551

[18] Dorodnitsyn V 1994 *Int. J. Mod. Phys.* C (*Phys. Comp.*) **5** 723

[19] Dorodnitsyn V 2000 *Group Properties of Difference Equations* (Moscow: Max Press) (in Russian)

[20] Levi D, Tremblay S and Winternitz P 2001 *J. Phys.* A: *Math. Gen.* **34** 9507

[21] Budd C and Dorodnitsyn V 2001 *J. Phys.* A: *Math Gen.* **34** 10387

[22] Dorodnitsyn V 2001 *Appl. Num. Math.* **39** 307

[23] Shi S Y, Fu J L and Chen L Q 2008 *Chin. Phys.* B **17** 385

[24] Fu J L, Dai G D, Jiménez S and Tang Y F 2007 *Chin. Phys.* **16** 570

[25] Dorodnitsyn V and Kozlov R 2010 *J. Eng. Math.* **66** 253

[26] Chen J B, Guo H Y and Wu K 2003 *J. Math. Phys.* **44** 1688

[27] Zhang H B, Chen L Q and Liu R W 2005 *Chin. Phys.* **14** 1063

[28] Wang X Z, Fu J L and Li C R 2012 *Appl. Mech. Mater.* **117–119** 167

[29] Lie S 1888 *Math. Ann.* **32** 213

[30] Maeda S 1980 *Math. Japan* **25** 405

[31] Levi D and Winternitz P 1991 *Phys. Lett.* A **152** 335

[32] Dorodnitsyn V 1991 *J. Sov. Math.* **55** 1490

[33] Shi S Y, Huang X H, Zhang X B and Jin L 2009 *Acta Phys. Sin.* **58** 3625 (in Chinese)

[34] Fu J L, Chen L Q and Chen B Y 2009 *Sci. China* **39** 1320 (in Chinese)

[35] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 225

[36] Fu J L, Fu H and Liu R W 2010 *Phys. Lett.* A **374** 1812

[37] Fu J L, Chen B Y and Chen L Q 2010 *Phys. Lett.* A **373** 409

[38] Dorodnitsyn V 2001 *The Group Properties of Difference Equations* (Moscow: Fizmatlit) (in Russian)