Knotted pictures of the GHZ states on the surface of a trivial torus

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By means of the torus knot theory method, this paper presents the complete process of obtaining the knotted pictures of eight GHZ states on the surface of a trivial torus from the knotted pictures of eight basic three-qubit states on the surface of a trivial torus. Thus, we obtain eight knotted pictures 12_1 linkage on the ordinary plane.

Keywords: GHZ state, knotted picture, torus knot theory, torus link

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1. Introduction

The description of quantum mechanics in terms of the classical knot theory[1−3] is a newly-developed branch of quantum mechanics. Topological quantum mechanics play a significant role and have broad applications in the searching and progressing state. Recently, we have published quite a few articles about the knotted pictures of quantum states, quantum logic gates, and quantum processes,[4−10] and have successfully demonstrated the intimate relationship between knot theory and quantum entanglement. Based on the studies about the correspondence between the Bell bases and the linkage 4_1 in the knot theory,[4,5] we have shown that there exists a very simple and vivid way of obtaining the linkage 4_1 by using the torus knot theory in Ref. [11], where we firstly found the knotted picture of Bell bases ($n = 2$, n is the number of qubits) on the surface of a trivial torus and the torus link $K_{4,2}$. Then, after quitting the trivial torus and projecting the torus link on a plane, we obtained the knotted pictures of Bell bases and the linkage 4_1 on the ordinary plane. This method can be easily used to obtain the knotted pictures of entanglement states ($n > 2$) in ordinary knot theory.

1.1. Eight GHZ states

Now we discuss the case of $n = 3$. It is well known that for a system (1, 2, and 3) consisting of three spin-1/2 particles, there are eight entangled GHZ states[6]

$$\begin{aligned}
|\Phi_1^{\pm}\rangle_{123} &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle \pm |\downarrow\downarrow\downarrow\rangle),\\
|\Phi_2^{\pm}\rangle_{123} &= \frac{1}{\sqrt{2}}(|\uparrow\uparrow\downarrow\rangle \pm |\downarrow\downarrow\uparrow\rangle),\\
|\Phi_3^{\pm}\rangle_{123} &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle \pm |\downarrow\uparrow\downarrow\rangle),\\
|\Phi_4^{\pm}\rangle_{123} &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\downarrow\rangle \pm |\downarrow\uparrow\uparrow\rangle).
\end{aligned} \tag{1}$$

1.2. A unified expression $|\,\mathrm{GHZ}\rangle_{123}^{i}$

Then, we introduce a unified expression $|\,\mathrm{GHZ}\rangle_{123}^{i}$ to express the above eight GHZ states:

$$\begin{aligned}
i = 1 &\longleftrightarrow |\Phi_1^{+}\rangle_{123},\\
i = 2 &\longleftrightarrow |\Phi_1^{-}\rangle_{123},\\
i = 3 &\longleftrightarrow |\Phi_2^{+}\rangle_{123},\\
i = 4 &\longleftrightarrow |\Phi_2^{-}\rangle_{123},\\
i = 5 &\longleftrightarrow |\Phi_3^{+}\rangle_{123},\\
i = 6 &\longleftrightarrow |\Phi_3^{-}\rangle_{123},\\
i = 7 &\longleftrightarrow |\Phi_4^{+}\rangle_{123},\\
i = 8 &\longleftrightarrow |\Phi_4^{-}\rangle_{123}.
\end{aligned} \tag{2}$$

1.3. Eight basic three-qubit states

The GHZ states $|\,\mathrm{GHZ}\rangle_{123}^{i}$ are three-qubit states. In order to consider the applications of the Hadamard gate and the CNOT gate, we assume that qubit 1 corresponds to a controlled qubit denoted by c, qubit 2

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corresponds to the first worked qubit denoted by w_1, and qubit 3 corresponds to the second worked qubit denoted by w_2. The three-qubit state $|1,2,3\rangle$ is represented by $|c,w_1,w_2\rangle$. Hence we rewrite $|\,\mathrm{GHZ}\rangle^i_{123}$ as $|\,\mathrm{GHZ}\rangle^i_{cw_1w_2}$. The value of a qubit is taken to be 0 and 1 for $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively . For simplicity, we shall use the symbol $|j\rangle$ $(j=1,2,\ldots,8)$ to represent these eight basic three-qubit states, and thus we have

$$\begin{aligned}
&|1\rangle=|\uparrow\uparrow\uparrow\rangle=|0,0,0\rangle, |2\rangle=|\uparrow\uparrow\downarrow\rangle=|0,0,1\rangle,\\
&|3\rangle=|\uparrow\downarrow\uparrow\rangle=|0,1,0\rangle, |4\rangle=|\uparrow\downarrow\downarrow\rangle=|0,1,1\rangle,\\
&|5\rangle=|\downarrow\uparrow\uparrow\rangle=|1,0,0\rangle, |6\rangle=|\downarrow\uparrow\downarrow\rangle=|1,0,1\rangle,\\
&|7\rangle=|\downarrow\downarrow\uparrow\rangle=|1,1,0\rangle, |8\rangle=|\downarrow\downarrow\downarrow\rangle=|1,1,1\rangle.
\end{aligned} \tag{3}$$

1.4. Knotted pictures of eight basic three-qubit states

In Ref. [10], we have pointed out that for single-qubit states, the knotted pictures of two basic states $|\uparrow\rangle$ and $|\downarrow\rangle$ are the counter-clockwise and clockwise circles, respectively. Naturally, we shall use three disconnected circles to represent eight basic three-qubit states $|c,w_1,w_2\rangle$. We assume that the inner circle represents the controlled qubit c, the outer circle represents the first worked qubit w_1, and the outermost circle represents the second worked qubit w_2. The counter-clockwise orientation corresponds to the qubit with a value of zero, i.e., $|\uparrow\rangle$, while the clockwise orientation corresponds to the qubit with an value of one, i.e., $|\downarrow\rangle$. Here we use the superscript "+" to denote the counter-clockwise direction, and the superscript "−" to denote the clockwise direction. Hence, we readily obtain the knotted pictures of the eight basic three-qubit states as shown in Fig. 1.

1.5. Knotted pictures of the eight GHZ states $|\,\mathrm{GHZ}\rangle^i_{cw_1w_2}$

In Ref. [7] we have already obtained the knotted pictures of eight GHZ states $|\,\mathrm{GHZ}\rangle^i_{cw_1w_2}$, which correspond to the linkage $12^3_1(c,w_1,w_2)$ in the knot theory, they are shown in Fig. 2.[6]

From Figs. 1 and 2, we can clearly see that each single-qubit state corresponds to a trivial knot (orientated circle), different entangled three-qubit states correspond to the resulted links of combined trivial knots with different kinds of twist. In other words, entangled states correspond to links with different kinds of twist. For three-qubit states, the corresponding links have 3 components.

1.6. Method of torus knot theory

All the above-mentioned knotted pictures are supported on a plane. In this paper, we shall use the method of torus knot theory[12] to represent all the knotted pictures on the surface of a trivial torus. The knotted pictures of eight basic qubit states are simply represented as eight kinds of three oriented parallel lines on the surface of a cylinder with a circular cross

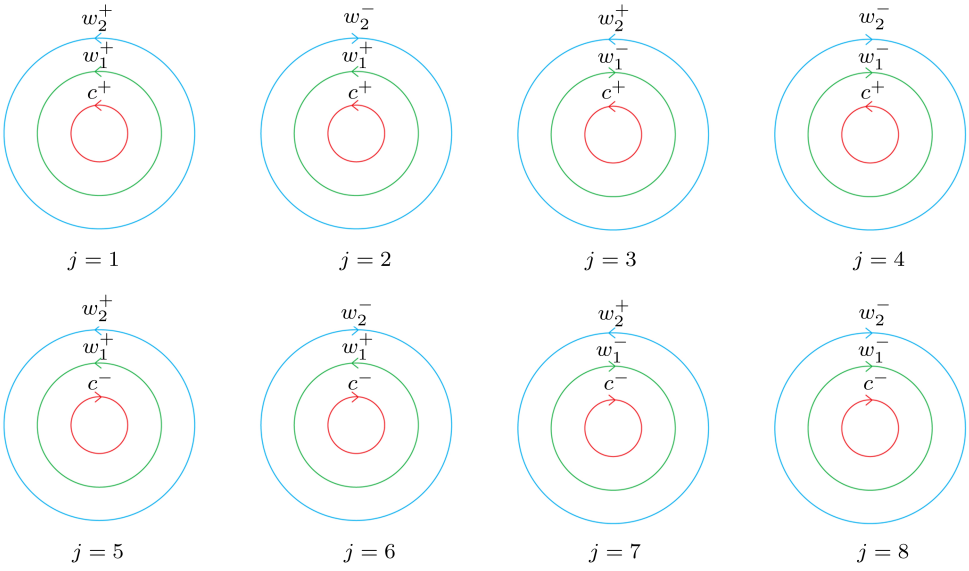


Fig. 1. (colour online) Knotted pictures of the eight basic three-qubit states $|j\rangle$.

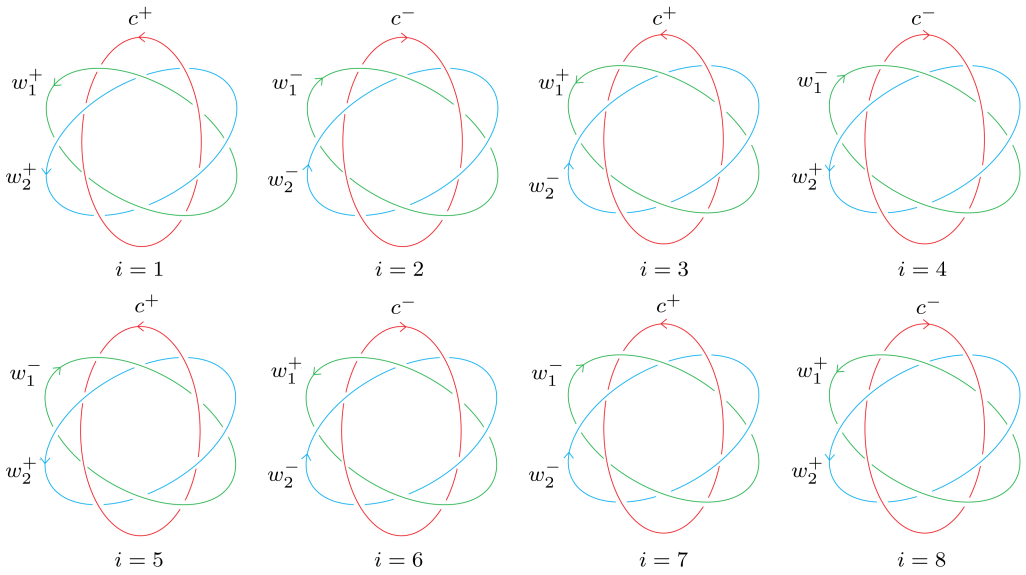


Fig. 2. (colour online) Knotted pictures of the eight GHZ states $|\,\mathrm{GHZ}\rangle^i_{cw_1w_2}$.

section. By twisting the cylinder at an angle of 4π, and using the method of constructing a trivial torus, we can easily obtain eight corresponding oriented links of the eight GHZ states on the trivial torus, i.e., the torus links $\mathrm{K}_{6,3}$ composed of three entangled orientated knots. After quitting the trivial torus and projecting the torus links $\mathrm{K}_{6,3}$ on a plane, we obtain the linkage $12^3_1(c, w_1, w_2)$.

The present paper is organized as follows. In Section 2, we shall give a method to obtain eight GHZ states from the corresponding eight basic three-qubit states $|\,j\rangle$ $(j = 1, 2, \ldots, 8)$ under the combined action of the Hadamard gate and the CNOT gate. In Section 3, we shall introduce the fundamental concepts of torus knot (or link), and the basic properties of (q, r) torus link $\mathrm{K}_{q,r}$. In Section 4, we shall give the complete process of obtaining the linkage 12_1 from the knotted pictures of eight basic three-qubit states. In Section 5, we shall give a summary of this paper.

2. Combined action of the Hadamard gate and the CNOT gate on the basic three-qubit states

2.1. Action of the Hadamard gate

In Ref. [11], we have discussed how to obtain the four Bell bases $|\,\mathrm{Bell}\rangle^i_{cw}$ by applying both the Hadamard gate and the CNOT gate on the four basic two-qubit states $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$. In this paper, we shall further obtain the eight GHZ states $|\,\mathrm{GHZ}\rangle^i_{cw_1w_2}$ by the combined application of the Hadamard gate and the CNOT gate on the eight basic three-qubit states $|\,j\rangle$ $(j = 1, 2, \ldots, 8)$.

The matrix expression of the Hadamard gate is

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{4}$$

The matrix expressions of basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ are

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5}$$

Applying H to $|\uparrow\rangle$ and $|\downarrow\rangle$, we obtain

$$\begin{aligned} H|\uparrow\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle), \\ H|\downarrow\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle). \end{aligned} \tag{6}$$

Since we only apply the Hadamard gate to the controlled qubit, hereafter we use the symbol H_c instead of H to represent the Hadamard gate.

2.2. Action of the CNOT gate

The action of the CNOT gate on a two-qubit state $|\,c,w\rangle$ is

$$|\,c,w\rangle \longrightarrow |\,c,c\oplus w\rangle, \tag{7}$$

where the operation $\oplus$ means mod 2 addition, i.e., $1\oplus 1 = 0$. Hence, when the value of the controlled qubit equals zero, the action of the CNOT gate does not change the value of worked qubit, i.e., $|\,0,w\rangle \longrightarrow |\,0,0\oplus w\rangle = |\,0,w\rangle$. Thus we have

$$\begin{aligned} \mathrm{CNOT}|\uparrow\uparrow\rangle &= \mathrm{CNOT}|\,0,0\rangle = |\,0,0\oplus 0\rangle \\ &= |\,0,0\rangle = |\uparrow\uparrow\rangle, \\ \mathrm{CNOT}|\uparrow\downarrow\rangle &= \mathrm{CNOT}|\,0,1\rangle = |\,0,0\oplus 1\rangle \\ &= |\,0,1\rangle = |\uparrow\downarrow\rangle. \end{aligned} \tag{8}$$

Hence, when the controlled qubit is $|\uparrow\rangle$, the CNOT gate has no effect on the two-qubit state $|\,c,w\rangle$. On the contrary, when the value of the controlled qubit equals one, the action of the CNOT gate does change the value of the worked qubit, i.e., $|\,1,w\rangle \longrightarrow |\,1,1\oplus w\rangle$. That is, the value of the worked qubit changes from zero to one or from one to zero. Therefore, we have

$$\begin{aligned} \mathrm{CNOT}|\downarrow\uparrow\rangle &= \mathrm{CNOT}|\,1,0\rangle = |\,1,1\oplus 0\rangle \\ &= |\,1,1\rangle = |\downarrow\downarrow\rangle, \\ \mathrm{CNOT}|\downarrow\downarrow\rangle &= \mathrm{CNOT}|\,1,1\rangle = |\,1,1\oplus 1\rangle \\ &= |\,1,0\rangle = |\downarrow\uparrow\rangle. \end{aligned} \tag{9}$$

The action of the CNOT gate can be schematically shown by Fig. 3.

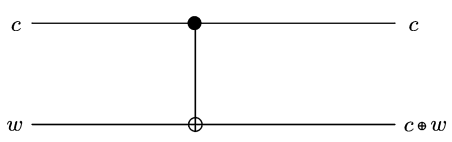


Fig. 3. Schematic diagram of the action of the CNOT gate.

Since the CNOT gate only changes the value of the worked qubit and the value of the worked qubit depends on the value of the controlled qubit, hereafter we use the notation $\mathrm{CNOT}_{w(c)}$ instead of CNOT.

2.3. Combined action of the Hadamard gate and the CNOT gate

Now we discuss the combined action of the Hadamard gate and the CNOT gate on the three-qubit state $|\,c,w_1,w_2\rangle$. Firstly, we apply H_c. Then, we apply the $\mathrm{CNOT}_{w_1(c)}$ gate on the first worked state and apply the $\mathrm{CNOT}_{w_2(c)}$ gate on the second worked state. Finally, we obtain $\mathrm{CNOT}_{w_2(c)}\mathrm{CNOT}_{w_1(c)}$ $|\,H_c c,w_1,w_2\rangle$. the combined action of the Hadamard gate H_c and the CNOT gates $\mathrm{CNOT}_{w_1(c)}$ and $\mathrm{CNOT}_{w_2(c)}$ on the three-qubit state $|\,c,w_1,w_2\rangle$ can be schematically represented by Fig. 4.

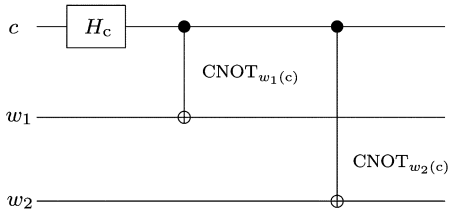


Fig. 4. Combined action of the Hadamard gate and the CNOT gate on the three-qubit state $|\,c,w_1,w_2\rangle$.

From Figs. (6) and (7), we readily obtain

$$\begin{aligned} &\mathrm{CNOT}_{w_2(c)}\mathrm{CNOT}_{w_1(c)}H_c|\uparrow\uparrow\uparrow\rangle \\ &= \mathrm{CNOT}_{w_2(c)}\mathrm{CNOT}_{w_1(c)}\frac{1}{\sqrt{2}}(|\uparrow\rangle+|\downarrow\rangle)_c|\uparrow\uparrow\rangle_{w_1,w_2} \\ &= \frac{1}{\sqrt{2}}\mathrm{CNOT}_{w_2(c)}(\mathrm{CNOT}_{w_1(c)}|\uparrow\rangle_c|\uparrow\rangle_{w_1}|\uparrow\rangle_{w_2} \\ &\quad + \mathrm{CNOT}_{w_1(c)}|\downarrow\rangle_c|\uparrow\rangle_{w_1}\uparrow\rangle_{w_2}) \\ &= \frac{1}{\sqrt{2}}\mathrm{CNOT}_{w_2(c)}(|\uparrow\rangle_c|\uparrow\rangle_{w_1}|\uparrow\rangle_{w_2}+|\downarrow\rangle_c|\downarrow\rangle_{w_1}|\uparrow\rangle_{w_2}) \\ &= \frac{1}{\sqrt{2}}(|\uparrow\rangle_c|\uparrow\rangle_{w_1}|\uparrow\rangle_{w_2}+|\downarrow\rangle_c|\downarrow\rangle_{w_1}|\downarrow\rangle_{w_2}) \\ &= |\,\mathrm{GHZ}\rangle^1_{cw_1w_2}. \end{aligned} \tag{10}$$

Then, we find that

$$\mathrm{CNOT}_{w_2(c)}\mathrm{CNOT}_{w_1(c)}H_c|\,j\rangle = |\,\mathrm{GHZ}\rangle^i_{cw_1w_2}, \tag{11}$$

where there is a correspondence between i and j:

$$\begin{aligned} &i=1 \longleftrightarrow j=1;\ i=2 \longleftrightarrow j=5; \\ &i=3 \longleftrightarrow j=2;\ i=4 \longleftrightarrow j=6; \\ &i=5 \longleftrightarrow j=3;\ i=6 \longleftrightarrow j=7; \\ &i=7 \longleftrightarrow j=4;\ i=8 \longleftrightarrow j=8. \end{aligned} \tag{12}$$

Using the language of the knot theory, we can rewrite Eq. (11) in the following form:

$$\begin{aligned} &\{\mathrm{CNOT}_{w_2(c)}\}\{\mathrm{CNOT}_{w_1(c)}\}\{H_c\}\{|\,j\rangle\} \\ &= \{|\,\mathrm{GHZ}\rangle^i_{cw_1w_2}\}, \end{aligned} \tag{13}$$

where $\{\mathrm{CNOT}_{w_1(c)}\}$, $\{\mathrm{CNOT}_{w_2(c)}\}$, and $\{H_c\}$ are the knotted pictures of the $\mathrm{CNOT}_{w_1(c)}$ gate, $\mathrm{CNOT}_{w_2(c)}$ gate, and H_c gate, respectively, $\{|\,j\rangle\}$ and $\{|\,\mathrm{GHZ}\rangle^i_{cw_1w_2}\}$ are the knotted pictures of the

basic three-qubit states $|j\rangle$ and the GHZ states $|\,\mathrm{GHZ}\rangle^{i}_{cw_1w_2}$, respectively. We have already found $\{|\,\mathrm{GHZ}\rangle^{i}_{cw_1w_2}\}$ in Ref. [6], which are the eight oriented links of the linkage 12_1in the knot theory. Since the $\{\mathrm{CNOT}_{w(c)}\}$ depends on the controlled qubit, we must consider $\{\mathrm{CNOT}_{w(\Uparrow\rangle_c)}\}$ and $\{\mathrm{CNOT}_{w(\Downarrow\rangle_c)}\}$ for the cases of $|\uparrow\rangle_c$ and $|\downarrow\rangle_c$, separately. In Ref. [10], we have already obtained the explicit expressions of $\{H_c\}$, $\{\mathrm{CNOT}_{w(\Uparrow\rangle_c)}\}$, and $\{\mathrm{CNOT}_{w(\Downarrow\rangle_c)}\}$ in terms of surgical operations described in the knot theory. Since these expressions are somewhat complex, we do not repeat them here.

3. Torus knots

3.1. Primary cylinder method for constructing the trivial torus

A brief summary of the fundamentals of torus knots has already been given in our recent paper.[11] For convenience, here we only introduce the primary cylinder method to construct the trivial torus.[12] The trivial torus is a torus with a circular central axis and a circular cross section. The primary cylinder is a cylinder with an equal circular cross section everywhere, whose top and base are planes perpendicular to the central axis. The primary cylinder method for constructing the trivial torus is to take a primary cylinder with the unit circle C_1 as its base and the unit circle C_2 as its top (see Fig. 5(a)), then bend the primary cylinder (see Fig. 5(b)), and glue together C_1 and C_2 so that the central axis of the primary cylinder C becomes the trivial knot (see Fig. 5(c)).[11]

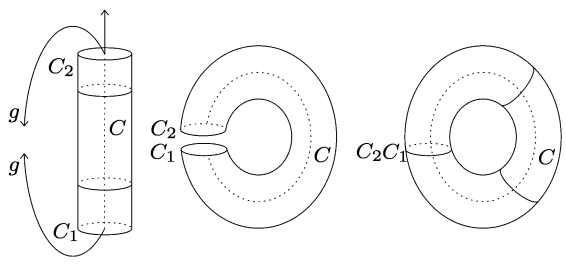


Fig. 5. Schematic drawing of the primary cylinder method for constructing the trivial torus.

3.2. Operations applied on the primary cylinder

Now we choose the above primary cylinder with a height of one unit and its base with a unit circle in xy-plane, then assign the base C_1 and the top C_2 with r points ($r = 0, 1, \ldots, r-1$). At the base C_1, there are points: $A_0 = (1, 0, 0)$, $A_1 = (\cos(2\pi/r), \sin(2\pi/r), 0), \ldots$, $A_{r-1} = (\cos(2(r-1)\pi/r), \sin(2(r-1)\pi/r), 0)$. At the top C_2, there are points: $B_0 = (1, 0, 1)$, $B_1 = (\cos(2\pi/r), \sin(2\pi/r), 1), \ldots$, $B_{r-1} = (\cos(2(r-1)\pi/r), \sin(2(r-1)\pi/r), 1)$. Connect the points A_k and B_k ($k = 0, 1, \ldots, r-1$) on the primary cylinder by the segments α_k. Hence, we have r parallel segments on the surface of the primary cylinder, as shown in Fig. 6(a).

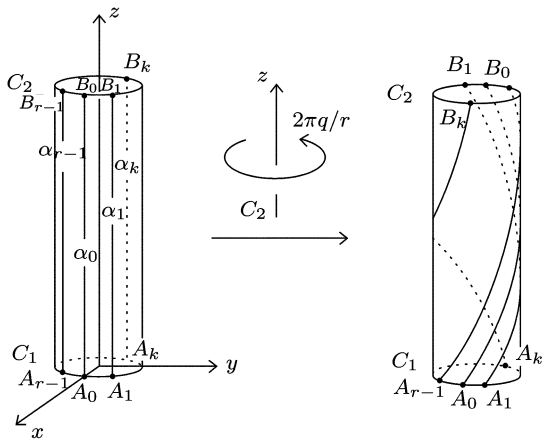


Fig. 6. Operations applied on the primary cylinder: (a) assigning to the base C_1 r points $A_0, A_1, \ldots, A_{r-1}$, and assigning to the top C_2 r points $B_0, B_1, \ldots, B_{r-1}$, (b) twisting the whole primary cylinder by rotating the top about the z axis by an angle of $(2\pi q)/r$ ($q > 0$).

Next, keeping the base C_1 fixed, we twist the whole primary cylinder by rotating the top C_2 about the z axis by an angle of $(2\pi q)/r$ (q is a nonzero integer), as shown in Fig. 6(b) for positive q. Then we use the glue operation shown in Fig. 5(b) to identify the point $(x, y, 0)$ of C_1 to the point $(x, y, 1)$ of C_2. Finally, we create a single trivial torus with r segments α_0, $\alpha_1, \ldots, \alpha_{r-1}$, which have been transformed into a knot (or link) on its surface.[12] This knot (or link) is called a (q, r)-torus knot (or link) and is denoted by $\mathrm{K}_{q,r}$. When $q = 6$ and $r = 3$, i.e., when there are initially $r = 3$ parallel segments on the surface of the primary cylinder and the twisting angle is $(2\pi q)/r = 4\pi$, the knotted picture is shown in Fig. 7.

In the particular case of $q = 0$, the twisting angle $(2\pi q)/r = 0$. Obviously, the resulted torus knot $\mathrm{K}_{0,3}$ is three disconnected circles, i.e., three trivial knots.

Since every segment can have two possible orientations: upward and downward, three orientated parallel segments have eight different kinds of orientation. Correspondingly, there are eight different kinds of orientation for the three trivial knots. The counterclockwise circle corresponds to the upward segment, whereas the clockwise circle corresponds to the downward segment. Clearly, $K_{0,3}$ is just the knotted pictures of the eight basic three qubit states. Figure 8 shows the knotted picture of $K_{0,3}$ for the case of $i = 1$.

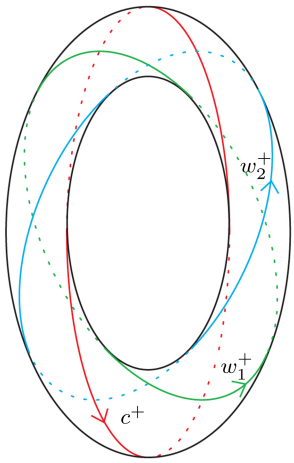


Fig. 7. (colour online) Knotted picture of $K_{6,3}$.

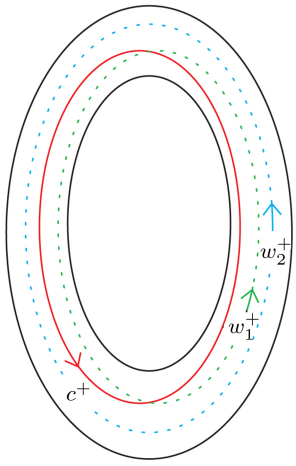


Fig. 8. (colour online) Knotted picture of $K_{0,3}$.

As mentioned above, when one applies the surgical operation and twist at an angle of $(2\pi q)/r = 4\pi$, we can easily obtain $K_{6,3}$ from $K_{0,3}$. It is very interesting that when we quit the trivial torus, we obtain the linkage 12_1. That is, when we support the linkage 12_1 on the trivial torus, we will obtain $K_{6,3}$. Thus, we conclude that the linkage 12_1 corresponds to $K_{6,3}$, which is just a special case of $K_{q,r}$.

4. Complete process of obtaining the linkage 12_1 from the torus knot theory

In Section 3, we have described the process of obtaining $K_{6,3}$ from $K_{0,3}$. Since on the surface of the plane, the former corresponds to the linkage 12_1 and the latter corresponds to three trivial knots, i.e., the knotted pictures of the eight basic three-qubit states, we readily obtain the complete process of obtaining the linkage 12_1 from the torus knot theory by following five steps.

Step 1 Take the primary cylinder with a height of one unit, a unit circle C_1 in xy-plane base and the unit circle C_2 as its top, as shown in Fig. 5(a).

Step 2 Assign the three points A_c, A_{w_1}, and A_{w_2} to the base C_1, and the corresponding three points B_c, B_{w_1}, and B_{w_2} to the top C_2, where $A_c = (1,0,0)$, $A_{w_1} = (\cos(2\pi/3),\sin(2\pi/3),0)$, $A_{w_2} = (\cos(4\pi/3),\sin(4\pi/3),0)$, $B_c = (1,0,1)$, $B_{w_1} = (\cos(2\pi/r),\ \sin(2\pi/r),1)$, $B_{w_2} = (\cos(4\pi/3)$,$\sin(4\pi/3),1)$. Thus, A_cB_c, $A_{w_1}B_{w_1}$, and $A_{w_2}B_{w_2}$ are three parallel segments (parallel to the central axis) on the surface of the primary cylinder. There are eight different kinds of three orientated segments corresponding to eight basic three-qubit states. As an example, Fig. 9(a) shows the case of $j = 1$.

Step 3 Twist the whole primary cylinder by rotating the top about the z axis by an angle of 4π, as shown in Fig. 9(b).

Step 4 Using the glue operation shown in Fig. 5(b) to identify the point $(x, y, 0)$ to the point $(x, y, 1)$, we obtain a trivial torus, on the surface of which we have the torus link $K_{6,3}$, which is composed of three entangled orientated knots, as shown in Figs. 9(c) and 9(d).

Step 5 Quit the trivial torus and project the torus link $K_{6,3}$ on a plane. Finally, we obtain the linkage 12_1. For the case of $j = 1$, we obtain the knotted picture of the GHZ state $|\,\mathrm{GHZ}\rangle^1_{cw_1w_2}$, as shown in Figs. 9(d) and 9(e).

Consequently, we can easily obtain the knotted pictures of GHZ states. This method is much more vivid and simple than the method of using the two quantum gates: the Hadamard gate and the CNOT gate.

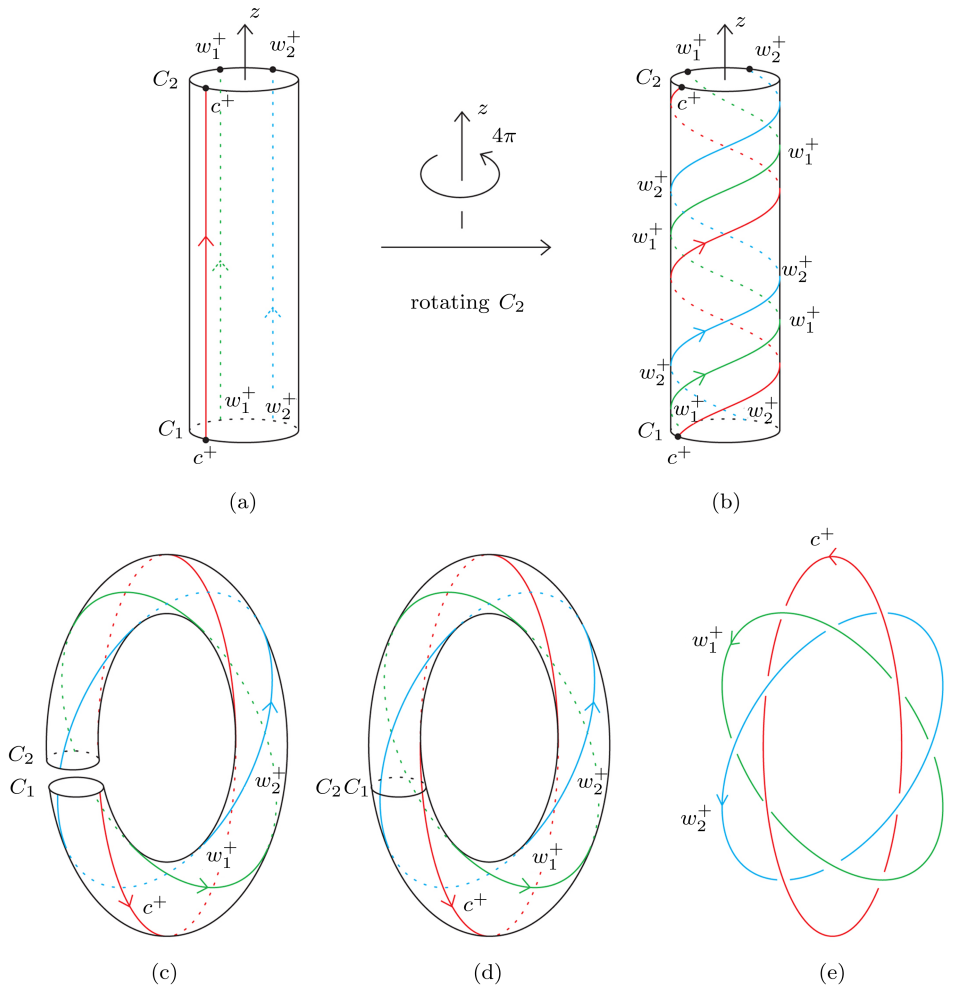


Fig. 9. (colour online) Complete process of obtaining the linkage 12_1: (a) for the case of $j = 1$ and $r = 3$, the lines c, w_1, and w_2 are oriented upward; (b) twisting at an angle of 4π, i.e., $q = 6$; (c) bending the cylinder; (d) completing the gluing operation; (e) quitting the trivial torus and projecting the torus link on a plane.

5. Conclusion

The knot theory is a powerful mathematical tool for researching physical phenomena.[1−3] We have found that there is an intimate relationship between quantum entanglement and the knot theory.[4−11] By comparing the covariance correlation tensor in the theory of quantum networks and the Alexander relation matrix in the theory of knot crystals, we have found that there is a one-to-one correspondence between four Bell bases and four oriented links of the linkage 4_1 in the knot theory. We also found a one-to-one correspondence between GHZ states and the oriented links.[6] Hence, we have used the classical language of the knot theory to describe the property of the algebraic structure of quantum entanglement, and revealed the interrelation of two seemingly different phenomena. Furthermore, we have given the knotted picture of the whole complete quantum measurement process of quantum teleportation.[10] Based on our previous works about the correspondence between the Bell bases and the linkage 4_1 in the knot theory, we have recently shown that there exists a very simple and vivid way of obtaining the linkage 4_1 by using the method of torus knot theory.[11] The significance of this work is that by means of this method we can easily study the complex problems about the correspondence between n-qubit ($n > 2$) quantum states and their corresponding knotted pictures. Thus, this method provides an alternative and effective way to study quantum entanglement. In this paper, we further use the method of torus knot theory[12] to represent the knotted pictures of the GHZ states on the surface of the trivial torus. The knotted pictures of eight basic qubit states are simply represented as eight kinds of three oriented parallel lines on the surface of the primary cylinder with circular cross section. Then, twisting the cylinder an angle of 4π, and using the method of con-

structing the trivial torus, we have easily obtained the eight corresponding oriented links of the eight GHZ states on the trivial torus, i.e., the torus links $K_{6,3}$ composed of three entangled orientated knots. When we quit the trivial torus and project the torus links $K_{6,3}$ on a plane, we obtain the linkage $12_1^3(c, w_1, w_2)$ for the knotted pictures of eight GHZ states on the ordinary plane. This method of constructing torus links involves five steps, as described in Section 4. Obviously, this method is much more vivid and simple than the method of using the two quantum gates: the Hadamard gate and the CNOT gate for constructing entangled states of n-qubit states ($n \geq 2$). Therefore, in the case of n-qubit GHZ states ($n = 2, 3$), we have already known that their corresponding knotted pictures on the surface of trivial torus are $K_{2n,n}$. Moreover, from the recurrence property of the network structure and the recognization of the correspondence between each quantum state and its knotted branch on the surface of the trivial torus, we can easily see that for n-qubit ($n \geq 4$) GHZ states $|\,\mathrm{GHZ}^{(n)}\rangle$, the corresponding torus link is $K_{2n,n}$. Furthermore, this method can predict some undiscovered entangled states due to the fact that rotating the top C_2 by any integral multiple of 2π about the base C_1 of the primary cylinder may create different linkages on the torus surface. From the one-to-one correspondence between the entangled state and the knotted picture, we can easily find some undiscovered entangled states, which deserve to be studied.

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