Nonlinearity and periodic solution of a standard-beam balance oscillation system*

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We present the motion equation of the standard-beam balance oscillation system, whose beam and suspensions, compared with the compound pendulum, are connected flexibly and vertically. The nonlinearity and the periodic solution of the equation are discussed by the phase-plane analysis. We find that this kind of oscillation can be equivalent to a standard-beam compound pendulum without suspensions; however, the equivalent mass centre of the standard beam is extended. The derived periodic solution shows that the oscillation period is tightly related to the initial pivot energy and several systemic parameters: beam length, masses of the beam, and suspensions, and the beam mass centre. A numerical example is calculated.

Keywords: balance oscillation, nonlinearity, limit cycle, phase plane

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1. Introduction

Standard-beam balance oscillation is a special kind of pendulum system.[1] It consists of a standard beam and two flexibly and vertically connected suspensions. Both the beam and suspensions are allowed to rotate freely around a horizontal axis. The difference between the standard-beam balance oscillation system and the compound pendulum is the "flexible connection of the beam and suspensions", which means that at any point in time or pivot angle, the forces produced due to the weight of the suspensions on two ends of the beam are constantly vertical. This difference would strengthen the nonlinearity of the standard-beam balance oscillation system as compared with a conventional pendulum. However, if we assume that there is no power loss in the oscillation, the mutual conversion between potential energy and the kinetic energy of the standard-beam balance makes the oscillation periodic.

Researching the periodic characteristics of such an oscillation system may lead to a series of useful applications. For example, the apparatus relates the mass to an atomic time standard with very high accuracy.[2,3] We will percept the mass by the change of oscillation period, and it can be used for measuring the air density,[4] supervising the drift of a test mass,[5] sensing, etc. The rest of this paper is organized as follows. We derive the differential motion equation of the standard-beam balance oscillation system in Section 2 and discuss the solution of the derived equation with its calculated periodic characteristics in Section 3. Based on the theoretical study, a numerical example is given in Section 4. In Section 5, we summarize the discussion and draw a conclusion.

2. Oscillation equation

The schematic structure of the standard-beam balance at the equilibrium position is shown in Fig. 1. The beam and the suspensions can be rotated around the horizontal axis at the same time, which could be realized in physics by a hard and sharp-edged knife. And the flexible connection could be the knife or a free-rolling axletree. Some symbols used in this paper are described here. The left and right beam beam lengths are L_1 and L_2, respectively. The total mass of the beam is M, which has a mass centre position of G with vertical distance δ from the rotating axis.

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The left suspension mass is m_1 while the right is m_2. At the equilibrium position, the moment of force is balanced, which can be expressed as

$$m_1 g L_1 = m_2 g L_2, \tag{1}$$

where g is the gravity acceleration. Without loss of generality, we analyse the oscillation system at an arbitrary pivot angle θ as shown in Fig. 2. For theoretical study, the pivot amplitude is assigned with very large pivot scale that $-\pi \leq \theta \leq \pi$ (for a knife-edged system, the actual pivot angle is in a range of $-\pi/2 \leq \theta \leq \pi/2$).

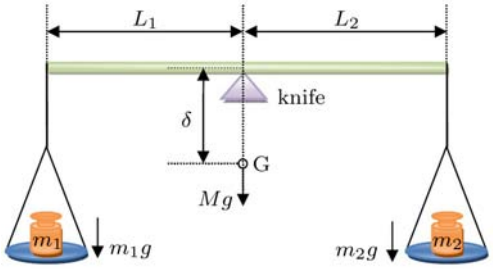


Fig. 1. (colour online) Standard-beam balance at the equilibrium position.

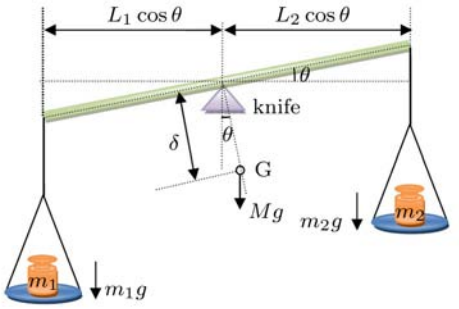


Fig. 2. (colour online) Standard-beam balance at an arbitrary pivot angle.

As shown in Fig. 2, the restoring moment is contributed by three components: the left suspension τ_1, the right suspension τ_2, and the weight of the standard beam τ_0, which can be written respectively as

$$\tau_1 = -m_1 g L_1 \cos\theta, \tag{2}$$

$$\tau_2 = m_2 g L_2 \cos\theta, \tag{3}$$

$$\tau_0 = Mg\delta \sin\theta. \tag{4}$$

From the simultaneous Eqs. (1)–(4), the restoring moment τ is obtained as

$$\begin{aligned} \tau &= \tau_1 + \tau_2 + \tau_0 \\ &= -m_1 g L_1 \cos\theta + m_2 g L_2 \cos\theta + Mg\delta \sin\theta \\ &= Mg\delta \sin\theta. \end{aligned} \tag{5}$$

From Eq. (5), it can be seen that the weight of the left suspension $m_1 g$ generates a moment of resistance while the weights of the balance beam Mg and the right suspension $m_2 g$ produce kinetic moments. In addition, we notice that the combination of τ_1 and τ_2 is zero, which means that synthesized suspensions will not produce any restoring moment in the oscillation. Using the rotational motion law, the differential equation of the standard-beam balance oscillation system can be written as

$$J\ddot{\theta} + Mg\delta \sin\theta = 0, \tag{6}$$

where J is the rotational inertia of the oscillation system. Equation (6) shows the standard-beam balance oscillation equation and it is similar to the compound pendulum equation. However, the J in Eq. (6) can not be simply considered as the rotational inertia of the beam. Actually we should synthetically consider both components of the standard beam and the suspensions. For an easy way of calculating J, we can identify the oscillation system with a three-particle model: M at the beam mass centre, m_1 at the left end of the beam, and m_2 at the right end of the beam. As all three particles move circularly around the axis, the J has a constant value, which can be expressed as

$$J = M\delta^2 + m_1 L_1^2 + m_2 L_2^2. \tag{7}$$

Substituting Eq. (7) into Eq. (6), we obtain

$$(M\delta^2 + m_1 L_1^2 + m_2 L_2^2)\ddot{\theta} + Mg\delta \sin\theta = 0. \tag{8}$$

Equation (8) is the final mathematical form of the standard-beam balance oscillation system. If the oscillation is swinging at very small amplitude ($\theta \approx 0$), the standard-beam balance oscillation turns into the simple harmonic oscillation. Compared with the compound pendulum equation, the standard-beam balance oscillation has an additional coefficient of $(m_1 L_1^2 + m_2 L_2^2)$. When using the accuracy advantage in measuring the period, the nonlinearity caused by the first-degree residual term $(\sin\theta - \theta)$ should be considered and controlled in designing a standard-beam balance oscillation system.

3. Solution and discussion

Although the motion equation of a standard-beam balance oscillation system exhibits strong nonlinearity, if there is no power loss, the oscillation will be periodic with the mutual conversion between potential energy and kinetic energy. Here we solve the

above derived differential equation by the actual physical process.[6−8] In the oscillation, it is assumed that θ_0 presents the maximum pivot angle and $\omega = \mathrm{d}\theta/\mathrm{d}t$ is the angular velocity of the balance. On the top or at the bottom in the oscillation, the system energy W_1 is completely in the form of potential energy as

$$\begin{aligned} W_1 &= -m_1 g L_1 \sin\theta_0 + m_2 g L_2 \sin\theta_0 \\ &\quad + Mg\delta(1-\cos\theta_0) \\ &= Mg\delta(1-\cos\theta_0). \end{aligned} \tag{9}$$

At an arbitrary pivot angle θ as shown in Fig. 2, potential energy W_{P} and kinetic energy W_{K} of the system can be calculated as

$$\begin{aligned} W_{\mathrm{P}} &= -m_1 g L_1 \sin\theta + m_2 g L_2 \sin\theta \\ &\quad + Mg\delta(1-\cos\theta) \\ &= Mg\delta(1-\cos\theta), \end{aligned} \tag{10}$$

$$\begin{aligned} W_{\mathrm{K}} &= \frac{1}{2}M(\omega\delta)^2 + \frac{1}{2}m_1(v_{1x}^2 + v_{1z}^2) \\ &\quad + \frac{1}{2}m_2(v_{2x}^2 + v_{2z}^2), \end{aligned} \tag{11}$$

where v_{1x} and v_{1z} are the horizontal and the vertical velocity components of the left suspension while v_{2x} and v_{2z} are those of the right. Mathematically, they can be expressed as

$$\begin{aligned} v_{1x} &= \frac{\partial(L_1 - L_1\cos\theta)}{\partial t} \\ &= L_1\frac{\partial(1-\cos\theta)}{\partial\theta}\frac{\partial\theta}{\partial t} \\ &= \omega L_1 \sin\theta, \end{aligned} \tag{12}$$

$$\begin{aligned} v_{2x} &= \frac{\partial(L_2\cos\theta - L_2)}{\partial t} \\ &= L_2\frac{\partial(\cos\theta-1)}{\partial\theta}\frac{\partial\theta}{\partial t} \\ &= -\omega L_2 \sin\theta, \end{aligned} \tag{13}$$

$$\begin{aligned} v_{1z} &= \frac{\partial(-L_1\sin\theta)}{\partial t} \\ &= L_1\frac{\partial(-\sin\theta)}{\partial\theta}\frac{\partial\theta}{\partial t} \\ &= -\omega L_1 \cos\theta, \end{aligned} \tag{14}$$

$$\begin{aligned} v_{2z} &= \frac{\partial(L_2\sin\theta)}{\partial t} \\ &= L_2\frac{\partial(\sin\theta)}{\partial\theta}\frac{\partial\theta}{\partial t} \\ &= \omega L_2 \cos\theta. \end{aligned} \tag{15}$$

According to the conservation of energy, we have

$$W_1 = W_{\mathrm{P}} + W_{\mathrm{K}}. \tag{16}$$

By solving the simultaneous Eqs. (9)–(16), we obtain

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \sqrt{\frac{2Mg\delta(\cos\theta - \cos\theta_0)}{M\delta^2 + m_1L_1^2 + m_2L_2^2}}. \tag{17}$$

Equation (17) can be used to verify the derived differential equation in Section 2, which has been listed in Appendix A. In the symmetrical oscillation, the moving period T can be calculated as four times the moving time when it pivots from $\theta = 0$ to $\theta = \theta_0$, thus

$$\begin{aligned} T &= 4\int_0^{\theta_0}\frac{\mathrm{d}t}{\mathrm{d}\theta}\mathrm{d}\theta \\ &= 4\sqrt{\frac{M\delta^2 + m_1L_1^2 + m_2L_2^2}{2Mg\delta}} \\ &\quad \times \int_0^{\theta_0}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\mathrm{d}\theta. \end{aligned} \tag{18}$$

If we make $\sin u = \sin\theta/\sin(\theta_0/2)$, the above integral can be rewritten in the elliptic function of the first kind as

$$T = 4\sqrt{\frac{M\delta^2 + m_1L_1^2 + m_2L_2^2}{Mg\delta}}\,\mathrm{F}\left(\sin\frac{\theta_0}{2}, \frac{\pi}{2}\right), \tag{19}$$

where $\mathrm{F}(k,\varphi)$ is Legendre's elliptic function of the first kind defined as

$$F(k,\varphi) = \int_0^{\varphi}\frac{1}{\sqrt{1-k^2\sin^2 u}}\mathrm{d}u. \tag{20}$$

The period of the elliptic function can be computed using series

$$\begin{aligned} T &= 2\pi\sqrt{\frac{M\delta^2 + m_1L_1^2 + m_2L_2^2}{Mg\delta}} \\ &\quad \times \sum_{n=0}^{\infty}\left[\frac{(2n)!}{(2^n\cdot n!)^2}\right]^2 \sin^{2n}\left(\frac{\theta_0}{2}\right) \\ &= 2\pi\sqrt{\frac{M\delta^2 + m_1L_1^2 + m_2L_2^2}{Mg\delta}} \\ &\quad \times \Big(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 \\ &\quad + \frac{22931}{1321205760}\theta_0^8 + \cdots\Big). \end{aligned} \tag{21}$$

From the above expressions, we know that the oscillation period is a function sensitive to the initial pivot angle θ_0, beam mass M, mass center δ, beam lengths L_1 and L_2, and suspensions masses m_1 and m_2. In addition, the standard-beam balance oscillation system has a similar periodic solution to that of the compound pendulum, so it can be equivalent to a standard-beam compound pendulum without suspensions. However, the mass centre of the standard beam is extended as

$$\delta' = \delta + \frac{m_1L_1^2 + m_2L_2^2}{M\delta} = \delta + \Delta\delta, \tag{22}$$

where δ' is the equivalent length of the pendulum and $\Delta\delta$ is the extended component.

4. Numerical example

A numerical example is given to intuitively verify the discussion in Section 3. The parameters are set as follows: $M = 10$ kg, $g = 9.8$ m/s^2, $\delta = 0.01$ m, $m_1 = m_2 = 1$ kg, $L_1 = L_2 = 0.5$ m. In the phase plane, we have

$$0.501\omega^2 - 1.96(\cos\theta - \cos\theta_0) = 0. \tag{23}$$

From Eq. (23), characteristics of the above system are determined by θ_0. The numerically calculated limit cycles in the phase plan are shown in Fig. 3 with different values of θ_0. When θ_0 increases from 0 to π, the limit cycle evolutes from circles when $\theta_0 \approx 0$, to ellipses with median values, and finally tunes into eye-shaped circles when $\theta_0 = \pi$.

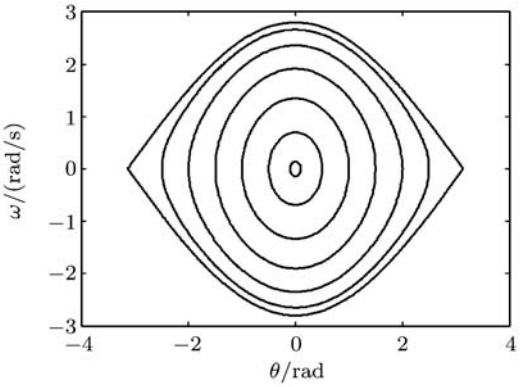


Fig. 3. Limit cycles in the phase plan with different values of θ_0. From inside to outside, values of θ_0 sequentially are 0.2, 0.5, 1.0, 1.5, 2.0, 2.5, 3.14.

Based on this example, assuming the beam to be symmetric with $m_1 = m_2 = m$ and $L_1 = L_2 = L$, we here discuss how the parameters influence the oscillation period. Four calculated oscillation periods with different values of parameters (m, δ, L, M) and initial energy (θ_0) are shown in Figs. 4–7 respectively. The set of parameters is described below in each figure.

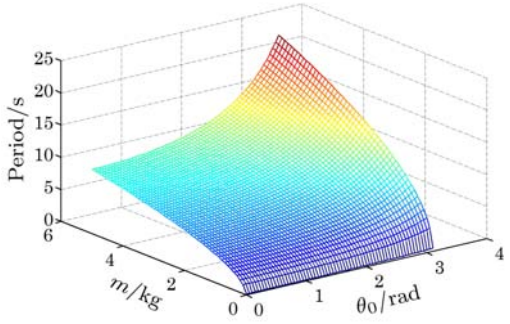


Fig. 4. (colour online) Calculated periods with different values of m and θ_0. The parameters are set as $L = 0.5$ m, $\delta = 0.01$ m, $M = 10$ kg, and $g = 9.8$ m/s^2.

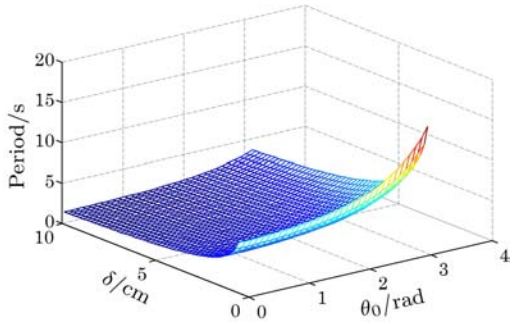


Fig. 5. (colour online) Calculated periods with different values of δ and θ_0. The parameters are set as $L = 0.5$ m, $m = 1$ kg, $M = 10$ kg, and $g = 9.8$ m/s^2.

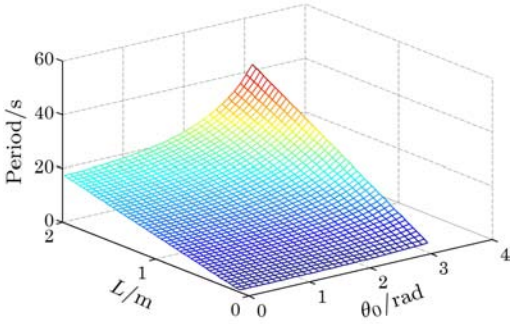


Fig. 6. (colour online) Calculated periods with different values of L and θ_0. The parameters are set as $m = 1$ kg, $\delta = 0.01$ m, $M = 10$ kg, and $g = 9.8$ m/s^2.

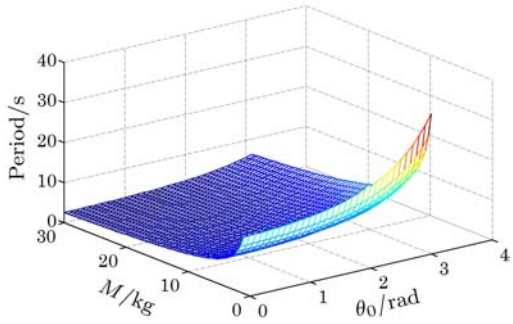


Fig. 7. (colour online) Calculated periods with different values of M and θ_0. The parameters are set as $L = 0.5$ m, $m = 1$ kg, $\delta = 0.01$ m, and $g = 9.8$ m/s^2.

These figures show that the period of the standard-beam balance oscillation system has different sensitivities to the sets of parameters, which can be calculated directly by solving the partial derivatives. For the application of such an oscillation system, the parameters should be strictly controlled if a high accuracy of period measurement is needed.

5. Conclusion

We derived the motion equation of the standard-beam balance oscillation system, discussed its nonlinearity and periodic solution by using phase-plane

analysis. It is concluded that the standard-beam oscillation can be equivalent to a standard-beam compound pendulum with the mass centre of the standard beam extended. The numerical example shows that the characteristics of the standard-beam balance system are sensitive to the values of the initial angle θ_0 and several systematic parameters, which should be controlled when the oscillation system is employed.

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Appendix A

This appendix is to verify the derived differential equation by using Eq. (17). The second derivative of θ with respect to time t can be calculated as

$$\begin{aligned}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) \\ &= \frac{1}{2}\sqrt{\frac{M\delta^2+m_1L_1^2+m_2L_2^2}{2Mg\delta(\cos\theta-\cos\theta_0)}} \\ &\quad\times\frac{-2Mg\delta\sin\theta}{M\delta^2+m_1L_1^2+m_2L_2^2}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= \frac{-Mg\delta\sin\theta}{M\delta^2+m_1L_1^2+m_2L_2^2}. \end{aligned} \tag{A1}$$

It is obviously correct if we substitute the above equation into Eq. (8).

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