Symmetry of Lagrangians of a holonomic variable mass system*

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The symmetry of Lagrangians of a holonomic variable mass system is studied. Firstly, the differential equations of motion of the system are established. Secondly, the definition and the criterion of the symmetry of the system are presented. Thirdly, the conditions under which there exists a conserved quantity deduced by the symmetry are obtained. The form of the conserved quantity is the same as that of the constant mass Lagrange system. Finally, an example is shown to illustrate the application of the result.

Keywords: Holonomic system, variable mass system, symmetry of Lagrangians, conserved quantity

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1. Introduction

"Undoubtedly, the study of symmetries of physical systems constitutes one of the most physically significant, mathematically instructive, and aesthetically appealing topics in physics."[1] Since the 1960s, people have studied the Noether symmetry,[2−8] the Lie symmetry,[9−13] and the form invariance.[14−18] The transformations in these symmetries are infinitesimal transformations of the coordinates and time. In 1966, Currie and Saletan studied the problem of equivalent Lagrangians for single degree of freedom mechanical systems and pointed out that there exist conserved quantities in this case.[19] In 1981, Hojman and Harleston extended the result to conservative multi-degree-of-freedom systems.[20] The transformations in this symmetry are ones which do not transform the coordinates and time variables. They are called fouling transformations by Currie and Saletan[19] and isotopic transformations by Santilli.[1]

The study of mechanical systems with variable mass has not only important theoretical significance, but also practical value. The systems, such as spacecraft, guided missiles, jet planes, rocket cars, fuel-consumed vehicles, and so on, are often those with variable mass. And great progress in the research of the systems has been made.[21−25]

In this paper, we attempt to extend the results on the constant mass system in Refs. [19] and [20] to the variable mass mechanical system.

2. Equations of motion of the variable mass system

Let us consider a mechanical system of N particles. At time t, the mass of the i-th particle is m_i, $(i = 1, \ldots, N)$, at time $t + \mathrm{d}t$, the mass of corpuscle abandoning from (or merging into) the particle is $\mathrm{d}m_i$. Let the position of the system be determined by the n generalized coordinates q_s $(s = 1, \ldots, n)$, and

$$m_i = m_i(t). \tag{1}$$

Then the equations of motion can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + P_s, \qquad (s = 1, \ldots, n), \tag{2}$$

where L is the Lagrangian of the system, Q_s the generalized nonpotential forces, P_s the generalized reverse thrusts, and we have

$$P_s = (\boldsymbol{R}_i + \dot{m}_i \dot{\boldsymbol{r}}_i) \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_s},$$
$$\boldsymbol{R}_i = \dot{m}_i \boldsymbol{u}_i, \tag{3}$$

where $\boldsymbol{r}_i$ is the position vector of the i-th particle, $\boldsymbol{u}_i$ the corpuscle's velocity relative to the particle.

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3. Definition and criterion of symmetry of Lagrangians

Let other Lagrangian, generalized nonpotential forces, and generalized reverse thrusts be respectively $\bar{L}$, $\bar{Q}_s$, and $\bar{P}_s$, and the differential equations of motion of the system expressed by these dynamical functions have the same solution set as that of the equations expressed by L, Q_s, and P_s. And let

$$\begin{aligned} L_r &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} - Q_r - P_r \\ &= W_{rs}\ddot{q}_s + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_s}\dot{q}_s + \frac{\partial^2 L}{\partial \dot{q}_r \partial t} \\ &\quad + \frac{\partial^2 L}{\partial \dot{q}_r \partial m_i}\dot{m}_i - \frac{\partial L}{\partial q_r} - Q_r - P_r, \qquad (4) \\ \bar{L}_r &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{q}_r} - \frac{\partial \bar{L}}{\partial q_r} - \bar{Q}_r - \bar{P}_r \\ &= \bar{W}_{rs}\ddot{q}_s + \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial q_s}\dot{q}_s + \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial t} \\ &\quad + \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial m_i}\dot{m}_i - \frac{\partial \bar{L}}{\partial q_r} - \bar{Q}_r - \bar{P}_r, \qquad (5) \end{aligned}$$

where

$$W_{rs} = \frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s}, \quad \bar{W}_{rs} = \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial \dot{q}_s}. \qquad (6)$$

Definition For the variable mass system (2), if one can obtain $\bar{L}_s = 0$, $(s = 1, \ldots, n)$ from $L_r = 0$, $(r = 1, \ldots, n)$, and *vice versa*, then the symmetry is called Lagrangian.

From $\bar{L}_r = 0$, all of the generalized accelerations $\ddot{q}_s$ can be solved as

$$\begin{aligned} \ddot{q}_s = \bar{U}^{sk}\Big(&\bar{Q}_k + \bar{P}_k + \frac{\partial \bar{L}}{\partial q_k} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial q_l}\dot{q}_l \\ &- \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial t} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial m_i}\dot{m}_i\Big), \qquad (7) \end{aligned}$$

where

$$\bar{W}_{rs}\bar{U}^{sk} = \delta_r^k. \qquad (8)$$

Substituting Eq. (7) into $L_r = 0$, we obtain

$$\begin{aligned} &W_{rs}\bar{U}^{sk}\Big(\bar{Q}_k + \bar{P}_k + \frac{\partial \bar{L}}{\partial q_k} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial q_l}\dot{q}_l \\ &- \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial t} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial m_i}\dot{m}_i\Big) \\ &= Q_r + P_r + \frac{\partial L}{\partial q_r} - \frac{\partial^2 L}{\partial \dot{q}_r \partial q_l}\dot{q}_l \\ &- \frac{\partial^2 L}{\partial \dot{q}_r \partial t} - \frac{\partial^2 L}{\partial \dot{q}_r \partial m_i}\dot{m}_i. \qquad (9) \end{aligned}$$

Criterion If the two groups of dynamical functions L, Q_s, P_s, and $\bar{L}$, $\bar{Q}_s$, $\bar{P}_s$ all satisfy Eq. (9), then the symmetry is Lagrangian.

4. Conserved quantity deduced by symmetry of Lagrangians

We now use the symmetry of Lagrangians to deduce a kind of conserved quantities.

Substituting Eq. (9) into Eq. (4), we have

$$\begin{aligned} L_r &= W_{rs}\ddot{q}_s - W_{rs}\bar{U}^{sk}\Big(\bar{Q}_k + \bar{P}_k + \frac{\partial \bar{L}}{\partial q_k} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial q_l}\dot{q}_l \\ &\quad - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial t} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial m_i}\dot{m}_i\Big) \\ &= W_{rs}\bar{U}^{sk}\Big(\bar{W}_{kl}\ddot{q}_l + \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial q_l}\dot{q}_l + \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial t} \\ &\quad + \frac{\partial^2 \bar{L}}{\partial \dot{q}_k \partial m_i}\dot{m}_i - \frac{\partial \bar{L}}{\partial q_k} - \bar{Q}_k - \bar{P}_k\Big) \\ &= W_{rs}\bar{U}^{sk}\bar{L}_k. \qquad (10) \end{aligned}$$

Let

$$\bar{W}_{rs}U^{sk} = A_r^k, \qquad (11)$$

where

$$W_{rs}U^{sk} = \delta_r^k.$$

Then formula (10) can be written in the form

$$\bar{L}_k = A_k^r L_r. \qquad (12)$$

By means of Eq. (11), we have

$$\bar{W}_{rs} = A_r^k W_{ks}. \qquad (13)$$

Taking the partial derivative of Eq. (13) with respect to $\dot{q}_l$, we obtain

$$\begin{aligned} \frac{\partial A_r^k}{\partial \dot{q}_l}W_{ks} &= \frac{\partial^3 \bar{L}}{\partial \dot{q}_r \partial \dot{q}_s \partial \dot{q}_l} - A_r^k \frac{\partial^3 L}{\partial \dot{q}_k \partial \dot{q}_s \partial \dot{q}_l} \\ &= \frac{\partial \bar{W}_{rl}}{\partial \dot{q}_s} - A_r^k \frac{\partial W_{lk}}{\partial \dot{q}_s} = \frac{\partial A_r^k}{\partial \dot{q}_s}W_{kl}. \qquad (14) \end{aligned}$$

Taking the partial derivative of Eq. (13) with respect to q_l, multiplying the result by $\dot{q}_l$ and summing it over l, we have

$$A_r^k \frac{\partial W_{ks}}{\partial q_l}\dot{q}_l - \frac{\partial \bar{W}_{rs}}{\partial q_l}\dot{q}_l = -\frac{\partial A_r^k}{\partial q_l}W_{ks}\dot{q}_l. \qquad (15)$$

Also, formula (9) can be written in the following form

$$\begin{aligned} &\frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial t} + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial m_i}\dot{m}_i - \frac{\partial \bar{L}}{\partial q_s} - \bar{Q}_s - \bar{P}_s \\ &= A_s^r\Big(\frac{\partial^2 L}{\partial \dot{q}_r \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial t} + \frac{\partial^2 L}{\partial \dot{q}_r \partial m_i}\dot{m}_i \end{aligned}$$

$$-\frac{\partial L}{\partial q_r}-Q_r-P_r\Big). \tag{16}$$

Taking the partial derivative of Eq. (16) with respect to $\dot{q}_l$, we obtain

$$\begin{aligned}&\frac{\partial^3\bar{L}}{\partial\dot{q}_s\partial q_k\partial\dot{q}_l}\dot{q}_k+\frac{\partial^2\bar{L}}{\partial\dot{q}_s\partial q_l}+\frac{\partial^3\bar{L}}{\partial\dot{q}_s\partial t\partial\dot{q}_l}+\frac{\partial^3\bar{L}}{\partial\dot{q}_s\partial m_i\partial\dot{q}_l}\dot{m}_i\\&-\frac{\partial^2\bar{L}}{\partial q_s\partial\dot{q}_l}-\frac{\partial\bar{Q}_s}{\partial\dot{q}_l}-\frac{\partial\bar{P}_s}{\partial\dot{q}_l}\\&=\frac{\partial A_s^r}{\partial\dot{q}_l}\Big(\frac{\partial^2L}{\partial\dot{q}_r\partial q_k}\dot{q}_k+\frac{\partial^2L}{\partial\dot{q}_r\partial t}+\frac{\partial^2L}{\partial\dot{q}_r\partial m_i}\dot{m}_i\\&-\frac{\partial L}{\partial q_r}-Q_r-P_r\Big)\\&+A_s^r\Big(\frac{\partial^3L}{\partial\dot{q}_r\partial q_k\partial\dot{q}_l}\dot{q}_k+\frac{\partial^2L}{\partial\dot{q}_r\partial q_l}+\frac{\partial^3L}{\partial\dot{q}_r\partial t\partial\dot{q}_l}\\&+\frac{\partial^3L}{\partial\dot{q}_r\partial m_i\partial\dot{q}_l}\dot{m}_i-\frac{\partial^2L}{\partial q_r\partial\dot{q}_l}-\frac{\partial Q_r}{\partial\dot{q}_l}-\frac{\partial P_r}{\partial\dot{q}_l}\Big).\end{aligned} \tag{17}$$

By virtue of $L_r=0$ and Eq. (4), we have

$$\begin{aligned}&\frac{\partial^2L}{\partial\dot{q}_s\partial q_k}\dot{q}_k+\frac{\partial^2L}{\partial\dot{q}_s\partial t}+\frac{\partial^2L}{\partial\dot{q}_s\partial m_i}\dot{m}_i-\frac{\partial L}{\partial q_s}-Q_s-P_s\\&=-W_{sr}\ddot{q}_r.\end{aligned} \tag{18}$$

Substituting (14), (15), and (18) into (17), we obtain

$$\begin{aligned}&\frac{\partial^2\bar{L}}{\partial\dot{q}_s\partial q_k}-\frac{\partial^2\bar{L}}{\partial q_s\partial\dot{q}_k}-\frac{\partial\bar{Q}_s}{\partial\dot{q}_k}-\frac{\partial\bar{P}_s}{\partial\dot{q}_k}\\&=-\frac{\mathrm{d}A_s^r}{\mathrm{d}t}W_{rk}+A_s^r\Big(\frac{\partial^2L}{\partial\dot{q}_r\partial q_k}-\frac{\partial^2L}{\partial q_r\partial\dot{q}_k}\\&-\frac{\partial Q_r}{\partial\dot{q}_k}-\frac{\partial P_r}{\partial\dot{q}_k}\Big).\end{aligned} \tag{19}$$

Introducing two matrices T and $\bar{T}$ whose elements are respectively

$$\begin{aligned}T_{rk}&=\frac{\partial^2L}{\partial\dot{q}_r\partial q_k}-\frac{\partial^2L}{\partial q_r\partial\dot{q}_k},\\\bar{T}_{rk}&=\frac{\partial^2\bar{L}}{\partial\dot{q}_r\partial q_k}-\frac{\partial^2\bar{L}}{\partial q_r\partial\dot{q}_k},\end{aligned} \tag{20}$$

and supposing that the following conditions:

$$\begin{aligned}&\frac{\partial}{\partial\dot{q}_k}\Big(\bar{Q}_s+\bar{P}_s\Big)=A_s^r\frac{\partial}{\partial\dot{q}_k}(Q_r+P_r),\\&(s,k,r=1,\ldots,n)\end{aligned} \tag{21}$$

are satisfied, we can express Eq. (19) in the following matrix form

$$\dot{A}=-\bar{T}U+ATU, \tag{22}$$

where $A=(A_s^r)$ and $U=(U^{sk})$ are $n\times n$ matrices. Noting the fact that T and $\bar{T}$ are antisymmetric, U and $\bar{W}$ are symmetric, and $A=\bar{W}U$, using the properties of the trace of matrices, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}[\,\mathrm{tr}\,(A^m)]=0 \tag{23}$$

for any positive integer m. We then find the conserved quantity

$$\mathrm{tr}\,(A^m)=\mathrm{const.} \tag{24}$$

Therefore, we have

Proposition For the variable mass system (2), if the conditions (21) are satisfied, then the symmetry of Lagrangians of the system will lead to the conserved quantity (24).

Corollary 1 For a constant mass system, if $Q_r,\bar{Q}_s$ satisfy the following conditions:

$$\frac{\partial\bar{Q}_s}{\partial\dot{q}_k}=A_s^r\frac{\partial Q_r}{\partial\dot{q}_k}, \tag{25}$$

then the symmetry of Lagrangians of the system will lead to the conserved quantity (24).

Corollary 2 For a constant mass Lagrange system, the symmetry of Lagrangians of the system will lead to the conserved quantity (24).

Corollary 1 was obtained by Zhao and Mei,[26] and Corollary 2 was found by Hojman and Harleston.[20]

5. Illustrative example

In order to illustrate the above result, we give an example.

Suppose that a variable mass system is as follows:

$$\begin{aligned}&L=\frac{1}{2}m(t)\Big(\dot{q}_1^2+\dot{q}_2^2\Big)-q_1,\\&Q_1=0,\quad Q_2=-m\dot{q}_2,\\&P_1=\dot{m}\dot{q}_1,\quad P_2=\dot{m}\dot{q}_2,\end{aligned} \tag{26}$$

we study the symmetry of Lagrangians and the conserved quantity of the system.

Formula (4) gives

$$\begin{aligned}&L_1=\frac{\mathrm{d}}{\mathrm{d}t}\Big(m\dot{q}_1\Big)+1-\dot{m}\dot{q}_1=m\ddot{q}_1+1,\\&L_2=\frac{\mathrm{d}}{\mathrm{d}t}\Big(m\dot{q}_2\Big)+m\dot{q}_2-\dot{m}\dot{q}_2=m\ddot{q}_2+m\dot{q}_2.\end{aligned} \tag{27}$$

Choosing other Lagrangian $\bar{L}$, generalized nonpotential forces $\bar{Q}_s$, and generalized reverse thrusts $\bar{P}_s$ as

$$\begin{aligned}&\bar{L}=\frac{1}{6}m\Big(\dot{q}_1+\int\frac{\mathrm{d}t}{m}\Big)^3+\frac{1}{2}m\dot{q}_2^2,\\&\bar{Q}_1+\bar{P}_1=\frac{1}{2}\dot{m}\Big(\dot{q}_1+\int\frac{\mathrm{d}t}{m}\Big)^2,\\&\bar{Q}_2=Q_2,\quad\bar{P}_2=P_2,\end{aligned} \tag{28}$$

we have

$$\bar{L}_1 = \left(\dot{q}_1 + \int \frac{\mathrm{d}t}{m}\right) L_1,$$
$$\bar{L}_2 = L_2. \tag{29}$$

In this case, conditions (9) and (21) are verified. Then the symmetry is Lagrangian and the conserved quantity (24) gives

$$I_1 = \mathrm{tr}\, A = \dot{q}_1 + \int \frac{\mathrm{d}t}{m} + 1 = C_1, \tag{30}$$

i.e.

$$I_1 = \dot{q}_1 + \int \frac{\mathrm{d}t}{m} = D_1, \tag{31}$$

where $D_1 = C_1 - 1$.

Choosing

$$\bar{L} = \frac{1}{2} m \dot{q}_1^2 - q_1 + \frac{1}{6} m \left(\dot{q}_2 + q_2\right)^2,$$
$$\bar{Q}_1 + \bar{P}_1 = \dot{m} \dot{q}_1,$$
$$\bar{Q}_2 + \bar{P}_2 = -\frac{1}{2} m \left(\dot{q}_2 + q_2\right)^2 + \frac{1}{2} \dot{m} \left(\dot{q}_2 + q_2\right)^2, \tag{32}$$

we obtain

$$A_1^1 = 1, \quad A_1^2 = A_2^1 = 0, \quad A_2^2 = \dot{q}_2 + q_2,$$
$$I_2 = \mathrm{tr}\, A = \dot{q}_2 + q_2 + 1 = C_2, \tag{33}$$

namely

$$I_2 = \dot{q}_2 + q_2 = D_2, \tag{34}$$

where $D_2 = C_2 - 1$.

6. Conclusions

This paper presents a new symmetry of holonomic variable mass systems, i.e., the symmetry of Lagrangians, under which two group dynamical functions L, Q_s, P_s, and $\bar{L}$, $\bar{Q}_s$, $\bar{P}_s$ express the same differential equations of motion. The symmetry of Lagrangians is different from the Noether symmetry, the Lie symmetry, and the form invariance. The symmetry can also lead to the conserved quantity.

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